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Universal Taylor series for non-simply connected domains

Séries universelles de Taylor pour les domaines non-simplement connexes

Stephen J. Gardiner and N. Tsirivas

Abstract

It is known that, for any simply connected proper subdomain $\Omega$ of the complex plane and any point $\zeta$ in $\Omega$, there are holomorphic functions on $\Omega$ that have “universal” Taylor series expansions about $\zeta$; that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in $\mathbb{C}\setminus\Omega$ that have connected complement. This note shows that this phenomenon can break down for non-simply connected domains $\Omega$, even when $\mathbb{C}\setminus\Omega$ is compact. This answers a question of Melas and disproves a conjecture of Müller, Vlachou and Yavrian.

Résumé

Il est connu que, pour un sous-domaine propre simplement connexe $\Omega$ du plan complexe et un point quelconque $\zeta$ de $\Omega$, il y a des fonctions holomorphes sur $\Omega$ qui possèdent des séries de Taylor «universelles» autour de $\zeta$; c’est-à-dire tout polynôme peut être approximé, sur tout compact de $\mathbb{C}\setminus\Omega$ ayant un complémentaire connexe, par les sommes partielles de la série de Taylor. Cette note montre que ce résultat n’est plus vrai en général pour les domaines non-simplement connexes $\Omega$, même lorsque $\mathbb{C}\setminus\Omega$ est compact. Cela répond à une question de Melas et réfute une conjecture de Müller, Vlachou et Yavrian.

1 Introduction

Let $\Omega$ be a proper subdomain of the complex plane $\mathbb{C}$ and let $\zeta \in \Omega$. A function $f$ on $\Omega$ is said to belong to the collection $U(\Omega, \zeta)$, of holomorphic

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functions on $\Omega$ with universal Taylor series expansions about $\zeta$, if the partial sums
\[
S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z - \zeta)^n
\]
of the Taylor series have the following property:

for every compact set $K \subset \mathbb{C}\backslash\Omega$ with connected complement and every function $g$ which is continuous on $K$ and holomorphic on $K^\circ$, there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to $g$ uniformly on $K$.

Nestoridis [17], [18] has shown that $U(\Omega, \zeta) \neq \emptyset$ for any simply connected domain $\Omega$ and any $\zeta \in \Omega$. (The corresponding result, where $K$ is required to be disjoint from $\Omega$, had previously been established by Luh [12] and Chui and Parnes [4].) In fact, Nestoridis showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains $\Omega$, in the sense that $U(\Omega, \zeta)$ is a dense $G_\delta$ subset of the space of all holomorphic functions on $\Omega$ endowed with the topology of local uniform convergence (see also Melas and Nestoridis [14] and the survey of Kahane [11]).

The situation when $\Omega$ is non-simply connected is much less well understood, despite much recent research: see, for example, [2], [3], [5], [6], [7], [9], [13], [15], [19], [22], [23], [24], [25]. Melas [13] (see also Costakis [5]) has shown that $U(\Omega, \zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C}\backslash\Omega$ is compact and connected, and has asked if $U(\Omega, \zeta)$ can be empty when $\mathbb{C}\backslash\Omega$ is compact but disconnected. On the other hand, Müller, Vlachou and Yavrian [15] have shown, for non-simply connected domains $\Omega$, that thinness of the set $\mathbb{C}\backslash\Omega$ at infinity is necessary for $U(\Omega, \zeta)$ to be non-empty, and have conjectured that this condition is also sufficient. There is clearly a large gap between the results of [13] and [15]. Also there has been no known example of a domain $\Omega$ and points $\zeta_1, \zeta_2 \in \Omega$ such that $U(\Omega, \zeta_1) \neq \emptyset$ and $U(\Omega, \zeta_2) = \emptyset$.

The purpose of this note is to establish the following result. We denote by $D(a, r)$ the open disc of centre $a$ and radius $r$, and write $\mathbb{D} = D(0, 1)$. By a non-degenerate continuum we mean a connected compact set containing more than one element.

**Theorem 1** Let $\Omega$ be a domain of the form $\mathbb{C}\backslash(L \cup \{1\})$, where $L$ is a non-degenerate continuum in $\mathbb{C}\backslash\mathbb{D}$. Then $U(\Omega, 0) = \emptyset$.

The conjecture of Müller, Vlachou and Yavrian is thus disproved. Also, if we take $L$ to be $\mathbb{D}(-5/3, 1/3)$, then $U(\Omega, 0) = \emptyset$ by Theorem 1 and yet a result of the second author [22] tells us that $U(\Omega, -1/2) \neq \emptyset$ (see also Costakis and Vlachou [7]). Thus we now have an example of a domain where the existence of functions with universal Taylor series depends on the chosen centre for expansion. The result of Melas, that $U(\Omega, 0) \neq \emptyset$ if $\mathbb{C}\backslash\Omega$
is compact and connected, is now seen to be sharp in the sense that, by Theorem 1, it can fail with the removal of one additional point from the domain. Theorem 1 fails if $L$ is allowed to be a singleton [13].

2 Proof

Let $\Omega$ be as in the statement of Theorem 1, and suppose, for the sake of contradiction, that there exists a function $f$ in $U(\Omega, 0)$. We can write $f = g + h$, where $g$ is the singular part of the Laurent expansion of $f$ associated with the singularity at 1, and $h$ is holomorphic on $\mathbb{C}\backslash L$. We denote the Taylor coefficients of $g$ and $h$ about 0 by $(a_n)$ and $(b_n)$, respectively. Since $(S_N(f, 0)(1))$ is dense in $\mathbb{C}$ and $(S_N(h, 0)(1))$ converges, we see that $g$ is non-zero.

Let $\rho = \inf\{|z| : z \in L\}$ and $0 < \delta < \varepsilon < \rho - 1$. The Taylor series for $g$ and $h$ about 0 converge absolutely in $\mathbb{D}$ and $D(0, \rho)$, respectively, so we can define the finite quantities

$$
\alpha_\delta = \sum_{n=0}^{\infty} \frac{|a_n|}{(1 + \delta)^n} \quad \text{and} \quad \beta_\delta = \sum_{n=0}^{\infty} |b_n| \left(\frac{\rho}{1 + \delta}\right)^n.
$$

Since $f \in U(\Omega, 0)$, we can choose a strictly increasing sequence $(N_k)$ of natural numbers such that

$$
S_{N_k}(g, 0)(z) + S_{N_k}(h, 0)(z) \to 0 \quad \text{as} \quad k \to \infty, \text{uniformly on } L. \quad (1)
$$

On $\overline{D}(0, \rho(1 + \varepsilon))$ we have

$$
|S_{N_k}(h, 0)(z)| \leq \sum_{n=0}^{N_k} |b_n| \rho^n (1 + \varepsilon)^n \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \beta_\delta,
$$

so by (1) we can choose $k_0$ such that

$$
|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} (\beta_\delta + 1) \quad (z \in L \cap \overline{D}(0, \rho(1 + \varepsilon)); k \geq k_0).
$$

We also have

$$
|S_{N_k}(g, 0)(z)| \leq \sum_{n=0}^{N_k} |a_n| (1 + \varepsilon)^n \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \alpha_\delta \quad (z \in \overline{D}(0, 1 + \varepsilon)),
$$

so

$$
|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \gamma_\delta \quad (z \in A_\varepsilon; k \geq k_0), \quad (2)
$$

where $\gamma_\delta = \max\{\alpha_\delta, \beta_\delta + 1\}$ and

$$
A_\varepsilon = \overline{D}(0, 1 + \varepsilon) \cup [L \cap \overline{D}(0, \rho(1 + \varepsilon))].
$$
Let $G_ε$ denote the Green function for the domain $D_ε = (C ∪ \{∞\}) \setminus A_ε$ with pole at infinity. Then

$$G_ε(z) - \log |z| → -\log C(A_ε) \quad (|z| → ∞),$$

where $C(A)$ denotes the logarithmic capacity of a set $A$ (see Section 5.8 of [1], or Section 5.2 of [21]). Thus we can choose $r_{δ,ε} > \max\{|z| : z ∈ L\}$ such that

$$G_ε(z) ≤ \log |z| - \log C(A_ε) + δ \quad (|z| ≥ r_{δ,ε}). \quad (3)$$

Bernstein’s lemma (Theorem 5.5.7 in [21]) tells us that any polynomial $q$ of degree $n ≥ 1$ satisfies

$$\left(\frac{|q(z)|}{\max_{A_ε}|q|}\right)^{1/n} ≤ e^{G_ε(z)} \quad (z ∈ D_ε \setminus \{∞\}).$$

Applying this inequality to the polynomial $S_{N_k}(g, 0)$, and using (2) and then (3), we obtain

$$|S_{N_k}(g, 0)(z)| \leq \left\{(1 + ε)(1 + δ)\right\}^{N_k} \cdot e^{N_k G_ε(z)}$$

$$\leq \left\{\left(1 + ε\right)\left(1 + δ\right)e^δ \cdot |z|\right\}^{N_k} \cdot \left(\frac{1}{C(A_ε)}\right)^{N_k} \cdot e^{N_k} \cdot e^{δ} \cdot r_{δ,ε}^{1/n} \cdot r_{δ,ε}^{1/n} \quad (n ≥ N_k; k ≥ k_0).$$

We next adapt an argument from pp.498,499 of Gehlen [8]. Let $ν ∈ (0, 1)$. Since

$$|a_n|^{1/n} = \left|\frac{1}{2πi} \int_{|z|=r_{δ,ε}} \frac{S_{N_k}(g, 0)(z)}{z^{n+1}} \, dz\right|^{1/n}$$

$$\leq \left\{\frac{(1 + ε)(1 + δ)e^δ}{C(A_ε)}\right\}^{N_k/n} \cdot \gamma_{δ}^{1/n} \cdot r_{δ,ε}^{1/n} \cdot e^{δ} \cdot r_{δ,ε}^{1/n} \quad (n ≤ N_k; k ≥ k_0),$$

we obtain

$$\limsup_{k→∞} \max_{νN_k ≤ n ≤ N_k} |a_n|^{1/n} ≤ \left\{\frac{(1 + ε)(1 + δ)e^δ}{C(A_ε)}\right\}^{1/ν} \cdot \gamma_{δ}^{1/ν} \cdot e^{δ} \cdot r_{δ,ε}^{1/ν} = \lambda, \quad \text{say.} \quad (4)$$

Since $L$ is a non-degenerate continuum that intersects $\{|z| = ρ\}$, we have

$$C(L ∩ \overline{D}(0, ρ(1 + ε))) > 0$$

and so

$$C(A_ε) > C(\overline{D}(0, 1 + ε)) = 1 + ε.$$

We can thus choose $δ$ sufficiently small that $(1 + ε)(1 + δ)e^δ < C(A_ε)$, and then choose $ν$ sufficiently close to 1 to ensure that $λ < 1$.

Finally, we will apply an observation of Müller (see Remark 2 in [16]). Since the function $g$ has its only singularity at 1 and vanishes at $∞$, Wigert’s
Theorem (Theorem 11.2.2 in Hille [10]) tells us that there is an entire function $F$ of exponential type $0$ such that $F(n) = a_n$ for all $n \geq 0$. However, Theorem V of Pólya [20] says that, for any $\mu > 0$, however small, such a function $F$ has the property that the sequence \{n \in \mathbb{N} : |F(n)| > e^{-\mu n}\} is of density 1. This contradicts (4) with $\lambda < 1$. Thus our original assumption, that there exists $f$ in $U(\Omega, 0)$, must be false, and the proof of the theorem is complete.

Remarks. 1) The assumption that $L$ is a continuum can be relaxed. It is enough to suppose that $L$ is a compact subset of $\mathbb{C}\setminus \mathbb{W}$ such that $\mathcal{C}(D(0, \rho^2) \cap L) > 0$ where $\rho = \inf \{|z| : z \in L\}$.

2) The proof actually shows that there is no holomorphic function $f$ on $\Omega$ such that $(S_N(f, 0))$ is divergent at $z = 1$ and has a subsequence that is uniformly bounded on $L$.

References


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