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A NOTE ON MILNOR-WITT K-THEORY AND A THEOREM OF SUSLIN

KEVIN HUTCHINSON, LIQUN TAO

Abstract. We give a simple presentation of the additive Milnor-Witt K-theory groups $K^{MW}_n(F)$ of the field $F$, for $n \geq 2$, in terms of the natural small set of generators. When $n = 2$, this specialises to a theorem of Suslin which essentially says that $K^{MW}_2(F) \cong H_2(\text{Sp}(F), \mathbb{Z})$.

1. Introduction

In [7], Suslin proved that for an infinite field $F$, $H_2(\text{Sl}(2, F), \mathbb{Z})$ is isomorphic to the fibre product $K^M_2(F) \times_{I_2/F} I_2^2(F)$, where $K^M_n(F)$ is the $n$-th Milnor $K$-group of $F$ and $I = I(F)$ is the ideal of even-dimensional forms in the Witt ring $W(F)$. The proof uses the Matsumoto-Moore presentation of the group $H_2(\text{Sp}(F), \mathbb{Z}) = H_2(\text{Sl}(2, F), \mathbb{Z})$ as well as the characterisation of the 2-torsion of $K^M_2(F)$ as the set of all elements of the form $\{-1, a\}$. (More recently, Mazzoleni, [3], has given an alternative proof of this theorem which by-passes the theorem of Matsumoto-Moore.)

More recently, F. Morel has introduced the Milnor-Witt $K$-theory, $K^{MW}_*(F)$ ([4], [5]). This is a graded algebra given by a simple presentation, due to Morel and M. Hopkins, from which the following properties are easily deduced: $K^{MW}_n(F) \cong W(F)$ for all $n < 0$; $K^{MW}_0(F) \cong GW(F)$, the Grothendieck-Witt ring of isometry classes of quadratic forms over $F$; there is an element $\eta$, of degree $-1$, such that $K^{MW}_*(F)/\langle \eta \rangle \cong K^M_*(F)$. The main result about Milnor-Witt $K$-theory is that it gives an exact description of certain operations in stable motivic homotopy theory; namely there is a natural isomorphism of graded rings

$$K^{MW}_*(F) \cong [S^0, (\mathbb{G}_m)^*]$$

where $S^0$ is the ‘motivic’ sphere spectrum, and $[ \ , \ ]$ denotes the group of morphisms in the stable $\mathbb{A}^1$-homotopy category ([4], section 6).

Morel has shown (see [5], for example) that, for all $n \geq 0$,

$$K^{MW}_n(F) \cong K^M_n(F) \times_{I^n/I^{n+1}} I^n(F).$$

In fact this result is essentially a reformulation of some of the main results of Arason and Elman, [1], on the powers of $I(F)$. Their work, in turn, relies heavily on the work of Voevodsky, Orlov and Vishik on the Milnor conjecture. In view of Morel’s result, Suslin’s theorem can be re-formulated as the statement that $H_2(\text{Sl}(2, F), \mathbb{Z}) \cong K^{MW}_2(F)$, at least when $F$ is infinite. Elsewhere ([6]), Morel has sketched a direct proof of this fact, using the machinery of $\mathbb{A}^1$-homotopy theory.

In this note, which is more elementary in nature than any of the references above, we prove that the Matsumoto-Moore relations give a simple presentation of the additive
group $K_n^{MW}(F)$, for all $n \geq 2$, in terms of the natural set of generators. When $n = 2$, this statement specializes to Suslin’s theorem, as re-formulated above.

As another application of our main theorem, we give an abstract additive presentation of the group $I^n(F)$ with $n$-fold Pfister forms as generators. (Corollary 2.16).

2. Milnor-Witt $K$-theory

Definition 2.1 (Hopkins-Morel, [4]). The Milnor-Witt $K$-theory of the field $F$ is the graded associative ring $K^{MW}_{*}(F)$ generated by the symbols $[u], u \in F^\times$, of degree $+1$ and one symbol $\eta$ of degree $-1$ subject to the following relations:

1. For each $a \in F^\times \setminus \{1\}$, $[a] \cdot [1 - a] = 0$.
2. For each $a, b \in F^\times$, $[ab] = [a] + [b] + [\eta]a[b]$.
3. For each $u \in F^\times$, $[u] \eta = \eta[u]$.
4. $\eta^2[-1] + 2\eta = 0$.

The following result is easily deduced ([6], Lemma 2.4):

Lemma 2.2. For all $n \in \mathbb{Z}$, $K^{MW}_n(F)$ has the following presentation as an additive group: It is generated by the elements $\eta^m[a_1] \cdots [a_r], m \geq 0$, $r = n + m \geq 0$ subject to the following relations:

1. $\eta^m[a_1] \cdots [a_r] = 0$ if $r \geq 2$ and $a_{i-1} + a_i = 1$ for some $i \geq 2$.
2. $\eta^m[a_1] \cdots [a_{i-1}][a_i][a_{i+1}] \cdots [a_r] = \eta^m[a_1] \cdots [a_{i-1}][a_i] \cdots [a_r] + \eta^{m+1}[a_1] \cdots [a_{i-1}][a_i] \cdots [a_r]$.
3. $\eta^{m+2}[a_1] \cdots [a_{i-1}][-1][a_{i+1}] \cdots [a_{r+2}][a_{i+1}] \cdots [a_{r+2}] = 0$.

However, in view of the relation $\eta[a_1][a_2] = [a_1a_2] - [a_1] - [a_2]$, it is clear that $K^{MW}_n(F)$ can be generated by the elements $[a_1] \cdots [a_n]$ whenever $n \geq 1$. Our main theorem is a presentation of $K^{MW}_n(F)$ in terms of these generators when $n \geq 2$.

The theorem of Matsumoto and Moore ([2]), for the case of the symplectic group $\text{Sp}(F)$, gives a presentation of the group $H_2(\text{Sp}(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_2 \rangle$, $a_i \in F^\times$, subject to the relations:

1. $\langle a_1, a_2 \rangle = 0$ if $a_i = 1$ for some $i$.
2. $\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle$.
3. $\langle a_1, a_2 a_2' \rangle + \langle a_2, a_2' \rangle = \langle a_1 a_2, a_2' \rangle + \langle a_1, a_2 \rangle$.
4. $\langle a_1, a_2 \rangle = \langle a_1, -a_1 a_2 \rangle$.
5. $\langle a_1, a_2 \rangle = \langle a_1, (1 - a_1)a_2 \rangle$.

This motivates the following (provisional) definition:

Definition 2.3. Let $n \geq 2$. For a field $F$, $K^{MM}_n(F)$ (MM is for ‘Matsumoto-Moore’) will denote the additive group which has the following presentation: the generators are $\langle a_1, \ldots, a_n \rangle$, $a_i \in F^\times$ subject to the following relations:

1. $\langle a_1, \ldots, a_n \rangle = 0$ if $a_i = 1$ for some $i$.
2. $\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_i^{-1}, a_{i-1}, \ldots, a_n \rangle$.
3. $\langle a_1, \ldots, a_{n-1}, a_n a_n' \rangle + \langle a_1, \ldots, a_{n-1}, a_n' \rangle = \langle a_1, \ldots, a_{n-1} a_n, a_n' \rangle + \langle a_1, \ldots, a_{n-1}, a_n \rangle$.
4. $\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, -a_{n-1} a_n \rangle$.
5. $\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, 1 - a_{n-1} a_n \rangle$.

Remark 2.4. In particular, $K^{MM}_2(F) \cong H_2(\text{Sp}(F), \mathbb{Z}) (= H_2(\text{Sl}(2, F), \mathbb{Z})$ if $F$ is infinite or at least sufficiently large) by the theorem of Matsumoto-Moore.
Observe that, using relation (ii) together with (iii), (iv) and (v), we easily deduce the following relations in \( K_n^{\text{MM}}(F) \):

\begin{align*}
(iii)' \quad & \langle a_1, \ldots, a_{i-1}, a_i a'_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i, a'_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}a_i, a'_i, \ldots, a_n \rangle + \\
(iv)' \quad & \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, -a_{i-1}a_i, \ldots, a_n \rangle \\
(v)' \quad & \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, (1-a_{i-1})a_i, \ldots, a_n \rangle \end{align*}

**Theorem 2.5.** \( K_n^{\text{MW}}(F) \cong K_n^{\text{MM}}(F) \) for all \( n \geq 2 \) via an isomorphism sending \([a_1] \cdots [a_n]\) to \( \langle a_1, \ldots, a_n \rangle \).

**Proof.** The theorem follows from Lemmas 2.8 and 2.15 below. \( \square \)

**Corollary 2.6.** For all infinite fields \( F \), \( K_2^{\text{MW}}(F) \cong H_2(\text{Sl}(2, F), \mathbb{Z}) \).

**Lemma 2.7.** The relations \([a][-a] = 0\) and \([a][b] = [b^{-1}][a]\) hold in \( K_\ast^{\text{MW}}(F) \) for all \( a, b \in F^\times \).

**Proof.** Using the identity

\[-a(1-a^{-1}) = 1-a\]

together with 2. gives

\([-a] = [1-a] - [1-a^{-1}] - \eta[-a][1-a^{-1}]\).

Hence

\[
[a][-a] = -[a](1+\eta[-a])[1-a^{-1}]
\quad \text{(using 1.)}
\]

\[
= -(1+\eta[-a])[a][1-a^{-1}]
\quad \text{(since \( \eta[u][v] = \eta[v][u] \) by 2.)}
\]

\[
= (1+\eta[-a])(1+\eta[a])[a^{-1}][1-a^{-1}]
\quad \text{(since \( [a] = (1+\eta[a])[a^{-1}] \) by 2.)}
\]

\[
= 0
\quad \text{(by 1.)}
\]

We will need the identity

\[
\eta([a] + [-a])[-1] = -2[-1]
\]

obtained by letting \( x = a \) and \( x = -a \) in the identity \([x] = [-1] + [-x] + \eta[-x][-1]\) and adding.

Observe also that \([a]^2 = [a][-a]\) for all \( a \in F^\times \) (let \( x = a \) above and use \([a][-a] = 0\)).

Now, for any \( a, b \in F^\times \) we have

\[
0 = [ab][-ab]
\]

\[
= ([a] + [b] + \eta[a][b])([-a] + [b] + \eta[-a][b])
\]

\[
= [a][b] + [b][-a] + \eta([a] + [-a])[b]^2 + [b]^2
\]

\[
= [a][b] + [b][-a] + \eta([a] + [-a])[-1][b] + [b][-1]
\]

\[
= [a][b] + [b][-a] - [b][-1]
\]

\[
= [a][b] + [b][(-a) - [-1])
\]

\[
= [a][b] + [b][a](1 + \eta[-1])
\]

and hence

\[
[a][b] = -[b][a](1 + \eta[-1]) = -(1 + \eta[-1])[b][a].
\]
Thus

\[ [a][b] - [b^{-1}][a] = - ([1 + \eta[-1]][b] + [b^{-1}])[a] \]
\[ = -([b] + [b^{-1}] + \eta[-1][b])[a] \]
\[ = -([b] + [b^{-1}] + \eta[b^{-1}][b])[a] = -[1][a] = 0 \]

(where we have used \([-1][b] = [b^{-1}][b]\) which follows from \([-1] = [-b] + [b^{-1}] + \eta[-b][b^{-1}]\)
and \([b][-b] = 0\)).

\[ \square \]

**Lemma 2.8.** Let \(n \geq 2\). The map \(\phi\) which sends the element \(\langle a_1, \ldots, a_n \rangle\) of \(K_n^{\text{MM}}(F)\) to \([a_1] \cdots [a_n]\) in \(K_n^{\text{MW}}(F)\) extends uniquely to a well-defined epimorphism of groups.

**Proof.** Well-definedness is the issue; we must prove that relations (i)–(v) are preserved by \(\phi\).

Relation (i): This follows from the identity \([1] = 0\) in \(K_n^{\text{MW}}(F)\) (since \([1] = 2[-1] + \eta[-1]^2\) by 2. and hence \(\eta[1] = 0\) by 4. and thus \([1] = 2[1]\)by 2. again).

Relation (ii): This follows immediately from the relation \([a_{i-1}][a_i] = [a_i^{-1}][a_{i-1}]\) (Lemma 2.7).

Relation (iii): We have

\[ [a_{n-1}][a_na'_n] + [a_n][a'_n] = [a_{n-1}][(a_n) + [a'_n] + \eta[a_n][a'_n]] + [a_n][a'_n] \]
\[ = ([a_{n-1}] + [a_n] + \eta[a_{n-1}][a_n])[a'_n] + [a_{n-1}][a_n] \]
\[ = [a_{n-1}a_n][a'_n] + [a_{n-1}][a_n] \]

Relation (iv): We have \([a_{n-1}][-a_{n-1}a_n] = [a_{n-1}]([-a_{n-1}] + [a_n] + \eta[-a_{n-1}][a_n]) = [a_{n-1}][a_n]\) by Lemma 2.7.

Relation (v): Similarly, \([a_{n-1}][(1 - a_{n-1})a_n] = [a_{n-1}][a_n]\) using 1. and 2.

\[ \square \]

**Lemma 2.9.** Let \(n \geq 2\). For \(a_1, \ldots, a_n, x \in F^\times\) let

\[ \rho_x(\langle a_1, \ldots, a_n \rangle) := \langle a_1, \ldots, a_n x \rangle - \langle a_1, \ldots, a_n \rangle - \langle a_1, \ldots, x \rangle. \]

Then \(\rho_x\) extends uniquely to an endomorphism of \(K_n^{\text{MM}}(F)\).

**Proof.** We must prove that \(\rho_x\) preserves defining relations (i)-(v).

Relation (i) is clear.

Relation (ii): When \(i < n\) in (ii), the result is clear. For the case \(i = n\), we find:

\[ \rho_x(\langle a_1, \ldots, a_{n-1}, a_n \rangle) = \langle a_1, \ldots, a_{n-1}, xa_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle \]
\[ = \langle a_1, \ldots, a_{n-1}, x a_n \rangle + \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, x, a_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle \]
\[ = \langle a_1, \ldots, a_{n-1}^{-1}, a_{n-1} x \rangle - \langle a_1, \ldots, a_{n-1}^{-1}, a_{n-1} \rangle - \langle a_1, \ldots, a_{n-1}^{-1}, x \rangle \]
\[ = \rho_x(\langle a_1, \ldots, a_{n-1}^{-1}, a_{n-1} \rangle). \]
Relation (iii):
\[
\rho_x((a_1, \ldots, a_{n-1}, a_n a'_n)) + \rho_x((a_1, \ldots, a_n, a'_n)) = \langle a_1, \ldots, a_{n-1}, a'_n x \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_{n-1}, a_n a'_n \rangle + (a_1, \ldots, a_n, a'_n) - \langle a_1, \ldots, a_{n-1}, a_n, a'_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_n, x \rangle - (\langle a_1, \ldots, a_{n-1} a_n, a'_n x \rangle + \langle a_1, \ldots, a_{n-1}, a'_n x \rangle) - (\langle a_1, \ldots, a_{n-1} a_n, a'_n \rangle + \langle a_1, \ldots, a_{n-1}, a'_n \rangle) - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_n, x \rangle - \langle a_1, \ldots, a_{n-1} a_n, a'_n x \rangle - \langle a_1, \ldots, a_{n-1} a_n, a'_n \rangle - \langle a_1, \ldots, a_{n-1}, a'_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n x \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_n, a'_n \rangle + \rho_x((a_1, \ldots, a_n, a'_n))
\]
Relations (iv) and (v) are immediate.

For \(n \geq 2\), we will denote the element \(\rho_x((a_1, \ldots, a_n))\) in \(K_n^{\text{MM}}(F)\) by \([a_1, \ldots, a_n, x]\).

Lemma 2.10. Let \(n \geq 2\). For any permutation, \(\sigma\), of \(\{1, \ldots, n+1\}\), \([a_1, \ldots, a_{n+1}] = [a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}]\).

Proof. It is immediate from the definition that \([a_1, \ldots, a_n, a_{n+1}] = [a_1, \ldots, a_{n+1}, a_n]\); i.e. the result is true when \(\sigma\) is the transposition \((n \ n+1)\). From this it follows that
\[
[a_1, \ldots, a_{n-1}, a_n, x] = \rho_x((a_1, \ldots, a_n)) = \rho_{a_n}((a_1, \ldots, a_{n-1}, x)) = \rho_{a_n}((a_1, \ldots, x^{-1}, a_{n-1})) = [a_1, \ldots, x^{-1}, a_{n-1}, a_n] = [a_1, \ldots, x^{-1}, a_n, a_{n-1}] = \rho_x((a_1, \ldots, a_n, a_{n-1})) = [a_1, \ldots, a_n, a_{n-1}, x]
\]
This fact, together with relation (ii), now implies that
\[
[a_1, \ldots, a_{i-1}, a_i, \ldots, a_n, x] = \rho_x((a_1, \ldots, a_{i-1}, a_i, \ldots, a_n)) = \rho_x((a_1, \ldots, a_i, a_{i-1}, \ldots, a_n)) = [a_1, \ldots, a_{i-1}, a_i, \ldots, a_n, x]
\]
proving the lemma.

Lemma 2.11. For all \(x, y \in F^\times\), \(\rho_x \rho_y = \rho_y \rho_x\).

Proof. Let \(a_1, \ldots, a_n \in F^\times\). Then
\[
\rho_y((a_1, \ldots, a_n)) = [a_1, \ldots, a_n, y] = [y, a_1, \ldots, a_n]
\]
and hence
\[ \rho_x(\rho_y(\langle a_1, \ldots, a_n \rangle)) = \rho_x([y, a_1, \ldots, a_n]) \]
\[ = \rho_x([y, a_1, \ldots, a_{n-1}a_n]) - \rho_x(\langle y, a_1, \ldots, a_{n-1} \rangle) - \rho_x(\langle y, a_1, \ldots, a_{n-2}, a_n \rangle) \]
\[ = [y, \ldots, a_{n-1}a_n, x] - [y, a_1, \ldots, a_{n-1}, x] - [y, a_1, \ldots, a_n, x] \]
\[ = [x, \ldots, a_{n-1}a_n, y] - [x, a_1, \ldots, a_{n-1}, y] - [x, a_1, \ldots, a_n, y] \]
\[ = \rho_y(\rho_x(\langle a_1, \ldots, a_n \rangle)). \]

\[ \square \]

More generally, we define elements \([a_1, \ldots, a_r] \in K_{n}^{MM}(F)\) \((r > n)\) recursively by the formula
\[ [a_1, \ldots, a_r] := \rho_{a_{r+1}}([a_1, \ldots, a_r]). \]
We will also use the notation \([a_1, \ldots, a_n] := \langle a_1, \ldots, a_n \rangle\) \((i.e., \text{ when } r = n)\).

**Corollary 2.12.** Fix \(n \geq 2\). For all \(r > n\) and for all permutations, \(\sigma\), of \(\{1, \ldots, r\}\)
\[ [a_1, \ldots, a_r] = [a_{\sigma(1)}, \ldots, a_{\sigma(r)}]. \]

**Proof.** We use induction on \(r\). The case \(r = n + 1\) has already been proved.
For permutations of \(\{1, \ldots, r + 1\}\) which fix \(r + 1\), the result holds by induction since
\[ [a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}([a_1, \ldots, a_r]). \]
On the other hand, when \(r > n\), the result holds for the permutation \(\sigma = (r + 1)\) since
\[ [a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}(\rho_{a_r}([a_1, \ldots, a_{r-1}])) = [a_1, \ldots, a_{r+1}, a_r]. \]

\[ \square \]

**Remark 2.13.** Observe that it follows that the relations (i)-(v) extend to the symbols \([a_1, \ldots, a_r]\) \((r \geq n)\) since we can always permute the key entries to before the \(n\)-th position and then use the fact that \([a_1, \ldots, a_r] = \phi(\langle a_1, \ldots, a_n \rangle)\) for an appropriate endomorphism \(\phi\). Furthermore, property (ii) and symmetry (Corollary 2.12) imply that
\[ [a_1, \ldots, a_i, \ldots, a_r] = [a_1, \ldots, a_i^{-1}, \ldots, a_r] \text{ for any } i. \]

**Corollary 2.14.** Let \(n \geq 2\). If \(r > n\) and if \(a_1, \ldots, a_r \in F^\times\) then \([a_1, \ldots, a_r] = 0\) if \(a_i\) is a square in \(F^\times\) for some \(i \leq r\).

**Proof.** By symmetry we can suppose that \(i > 1\). Suppose that \(a_i = b_i^2\). We thus have
\[ [a_1, \ldots, a_{i-1}, b_i^2, \ldots] = [a_1, \ldots, a_{i-1}b_i, b_i, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, b_i, b_i, \ldots] \]
\[ = [a_1, \ldots, a_{i-1}b_i, b_i^{-1}, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, b_i, b_i^{-1}, \ldots] \]
\[ = [a_1, \ldots, a_{i-1}, b_i \cdot b_i^{-1}, \ldots] = [a_1, \ldots, a_{i-1}, 1, \ldots] = 0. \]

\[ \square \]

**Lemma 2.15.** Let \(n \geq 2\). There is a unique epimorphism \(\lambda : K_{n}^{MW}(F) \to K_{n}^{MM}(F)\) satisfying
\[ \lambda(\eta^m[a_1] \cdots [a_r]) = [a_1, \ldots, a_r] \quad (r = n + m) \]
Proof. We must show that \( \lambda \) preserves relations (1)-(3) of Lemma 2.2.

Relation (1) follows from (i) and (v) (see Remark 2.13).

Relation (2): We must prove that, for \( r \geq n \) and \( i \leq r \),

\[
[a_1, \ldots, a_i a_i', \ldots, a_r] = [a_1, \ldots, a_i, \ldots, a_r] + [a_1, \ldots, a_i', \ldots, a_r] + [a_1, \ldots, a_i, a_i', \ldots, a_r].
\]

By symmetry we can assume \( i \leq n \) and thus we reduce to the key case \( r = n \); i.e. we must prove

\[
\langle a_1, \ldots, a_i a_i', \ldots, a_n \rangle = \langle a_1, \ldots, a_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i', \ldots, a_n \rangle + [a_1, \ldots, a_i, a_i', \ldots, a_n].
\]

By property (ii), and by symmetry, we can assume \( i = n \). The identity is now just the definition of \([a_1, \ldots, a_n, a_n']\).

Relation (3): We must prove that for \( r \geq n \)

\[
[a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_{r+2}] = -2[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{r+2}].
\]

By symmetry, we can suppose that \( r = n \) and \( i = n + 1 \). So we must show that

\[
[a_1, \ldots, a_n, -1, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}].
\]

Now

\[
[a_1, \ldots, a_n, -1] = [a_1, \ldots, a_n, a_n] \quad \text{(by (iv))}
\]

and thus

\[
[a_1, \ldots, a_n, -1, a_{n+2}] = [a_1, \ldots, a_n^2, a_{n+2}] - 2[a_1, \ldots, a_n, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}]
\]

by Corollary 2.14.

As an application, we derive a simple additive presentation of the ideals \( I^n(F) \), \( n \geq 2 \), in the Witt Ring of a field \( F \):

**Corollary 2.16.** For any field \( F \), let \( I(F) \) be the ideal of even-dimensional forms in the Witt Ring, \( W(F) \), of the field \( F \). As an additive group, \( I^n(F) = I(F)^n \) has the following abstract presentation:

It is generated by the classes of Pfister forms \( <<< a_1, \ldots, a_n >>> \), \( a_i \in F^\times \) subject to the following relations:

1. \( <<< a_1, \ldots, a_n >>> = 0 \) if \( a_i \) is a square for some \( i \)
2. \( <<< a_1, \ldots, a_i, a_i', \ldots, a_n >>> = <<< a_1, \ldots, a_i, a_i', \ldots, a_n >>> + <<< a_1, \ldots, a_i, a_i', \ldots, a_n >>> \)
3. \( <<< a_1, \ldots, a_{n-2}, a_n^2 >>> + <<< a_1, \ldots, a_n >>> = <<< a_1, \ldots, a_{n-2}, a_n^2 >>> + <<< a_1, \ldots, a_{n-2}, a_n >>>
4. \( <<< a_1, \ldots, a_{n-2}, a_n >>> = <<< a_1, \ldots, a_{n-2}, (1 - a_{n-1})a_n >>>

**Proof.** Morel’s theorem ([5], Théorème 5.3), shows that there is an exact sequence

\[
0 \to K_n^M(F)[2] \to K_n^{MW}(F) \to I^n(F) \to 0
\]

where the first (nontrivial) homomorphism maps \( \{a_1, \ldots, a_n\}^2 = \{a_1, a_2^2, \ldots, a_n^2\} \) to \([a_1] \cdots [a_i^2] \cdots [a_n] \) (for any \( i \)) and the next homomorphism sends \([a_1] \cdots [a_n] \) to the class of the Pfister form \( <<< a_1, \ldots, a_n >>> \). Combining this with Theorem 2.5 give the result, since the identity \( -a = (1 - a)/(1 - a^{-1}) \) shows that (i) and (iv) imply the identity \( <<< a_1, \ldots, a_{n-1}, a_n >>> = <<< a_1, \ldots, a_{n-1}, -a_{n-1}a_n >>> \).

**Remark 2.17.** Compare this with the presentation of \( I^n(F) \) given by Arason and Elman ([1], Theorem 3.1). Of course, Corollary 2.16 – like the result of Arason and Elman – requires the proof of the Milnor conjecture (since it is needed for Morel’s theorem), and conversely easily implies the Milnor conjecture.
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References


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