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Feedback Stabilization of a Class of Diagonal Infinite-Dimensional Systems with Delay Boundary Control

Hugo Lhachemi and Christophe Prieur

***Abstract*—This paper studies the boundary feedback stabilization of a class of diagonal infinite-dimensional boundary control systems. In the studied setting, the boundary control input is subject to a constant delay while the open loop system might exhibit a finite number of unstable modes. The proposed control design strategy consists in two main steps. First, a finite-dimensional subsystem is obtained by truncation of the original Infinite-Dimensional System (IDS) via modal decomposition. It includes the unstable components of the infinite-dimensional system and allows the design of a finite-dimensional delay controller by means of the Artstein transformation and the pole-shifting theorem. Second, it is shown via the selection of an adequate Lyapunov function that 1) the finite-dimensional delay controller successfully stabilizes the original infinite-dimensional system ; 2) the closed-loop system is exponentially Input-to-State Stable (ISS) with respect to distributed disturbances. Finally, the obtained ISS property is used to derive a small gain condition ensuring the stability of an IDS-ODE interconnection.**

***Index Terms*—Distributed parameter systems, Delay boundary control, Lyapunov function, PDE-ODE interconnection.**

I. Introduction

Feedback control of finite-dimensional systems in the presence of input delays has been extensively investigated [1], [20]. The extension of this topic to Infinite-Dimensional Systems (IDSs), and in particular to Partial Differential Equations (PDEs), has attracted much attention in the recent years.

There exist essentially two types of control inputs for infinite-dimensional systems: bounded and unbounded control operators. The stability of linear and semilinear infinite-dimensional system under time-varying delayed feedback acting via a bounded linear control operator has been studied, e.g., in [9], [22]. In this paper, we are interested in the second type of control, i.e., when the control input acts on the system via an unbounded operator. For PDEs, such a setting takes the form of a control acting in the boundary conditions.

Unbounded control operators have been considered in the stability study of various PDEs. The cases of the heat [17] and wave [15]–[17] equations were studied via Lyapunov methods for slow time varying delays. The cases of a parabolic PDE and a second-order evolution equation were reported in [25] and [8], respectively. The extension to a delayed ODE–heat cascade under actuator saturation was reported in [10].

In this paper, we are interested in the boundary feedback stabilization of a class of diagonal infinite-dimensional boundary control systems in the presence of a constant input delay. Specifically, we consider the case of a boundary control system [7] for which the associated disturbance-free operator is a Riesz-spectral operator admitting a finite number of unstable eigenvalues. The control design objective consists in the feedback stabilization of the system by means of a delay boundary control.

One of the very first contributions on input delayed unstable PDEs deals with a reaction-diffusion equation [12] where the controller was designed by resorting to the backstepping technique. The approach adopted in this paper differs. It relies on the following three steps procedure initially reported in [21]: 1) obtaining a finite-dimensional subsystem capturing the unstable modes by truncation of the original infinite-dimensional via a modal decomposition ; 2) design of a finite-dimensional control law that stabilizes the finite-dimensional unstable part of the system ; 3) use of an adequate Lyapunov function to assess that the designed control law stabilizes the original infinite-dimensional system. Such a control design strategy was successfully applied to the stabilization of semilinear heat [5] and wave [6] equations via (undelayed) boundary feedback control. The extension of this design procedure to the delay feedback control of a linear reaction-diffusion equation was reported in [19], [24]. The delayed finite-dimensional model was obtained via spectral reduction. Then, the control law was computed by applying the Artstein transformation [1], [20] and by resorting to the classical pole-shifting theorem. A distinguished feature is that, under the knowledge of the constant delay $D \geq 0$, the obtained finite-dimensional control law amounts stabilizing the closed-loop system, whatever the value of the time-delay D may be.

In this context, the contributions of this paper is fourfold.

1) We generalize the approach developed in [19] for the delay feedback control of a linear reaction-diffusion equation with one-dimensional control input to the general case of the delay boundary feedback stabilization of a class of diagonal infinite-dimensional boundary control systems with finitely many unstable modes and finite-dimensional input. The study of this problem is motivated by the fact that many applications, including reaction-diffusion phenomena [5], [19], phase turbulence phenomena [3] and certain models of structural

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vibrations phenomena such as wave [6], [7] and beam equations [7], [14] in the presence of a damping term, exhibit such a structure. The control design strategy relies on the design of a state-feedback control law based on a finite-dimensional truncated part of the original system. The truncation is performed via a spectral decomposition used to capture the unstable modes of the system. The control law is then obtained based on this finite-dimensional subsystem with delay control input by means of the Artstein transformation and the pole-shifting theorem. The exponential stability of the resulting closed-loop infinite-dimensional system is assessed via the introduction of a suitable Lyapunov function. This is worth noting that the control design strategy is presented in a constructive manner, taking the form of a predictor feedback as the ones classically used for the control of finite-dimensional LTI systems.

2) In [5], [6], [19] the control design was performed on the time derivative $v = \dot{u}$ of the actual input signal u. Thus, the application of the control law required an *a posteriori* integration of v to obtain the actual control input u. In this paper, we simply the control law by avoiding such an *a posteriori* integration. Such a simplification is allowed by an adequate spectral decomposition that only involves the value of the control input while avoiding the occurrence of its time derivative.
3) We show that the resulting closed-loop system is exponentially Input-to-State Stable (ISS) [23] with respect to distributed disturbances acting via a bounded operator.
4) Taking advantage of the ISS property of the closed-loop infinite-dimensional system, we derive a small gain condition ensuring the stability of an IDS-ODE interconnection. We follow here the methodology presented in [11] that relies on the conversion of the ISS estimates satisfied by each component of the interconnection into fading memory estimates [11, Lemma 7.1]. However, such a conversion does not apply to the studied closed-loop infinite-dimensional system due to the time-varying nature of the control strategy. This pitfall is avoided by working directly with the Lyapunov function instead of the trajectories of the system.

The remainder of this paper is organized as follows. Both problem setting and control objectives are introduced in Section II. The comprehensive construction of the control strategy is presented in Section III. The study of the ISS property of the resulting closed-loop infinite-dimensional system is carried out via the introduction of an adequate Lyapunov function in Section IV. We take advantage of these results to derive in Section V a small gain condition ensuring the stability of an IDS-ODE interconnection. In Section VI, we check the assumptions on a IDS-ODE system and in particular the small gain condition. The obtained numerical results are compliant with the theoretical predictions. Finally, concluding remarks are provided in Section VII.

II. Problem setting and control objective

The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. Throughout the paper, we assume that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a separable Hilbert space over the field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. All the finite-dimensional spaces $\mathbb{K}^p$ are endowed with the usual euclidean inner product $\langle x, y \rangle = x^* y$ and the associated 2-norm $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^* x}$, where $x^* = \overline{x}^\top$. For any matrix $M \in \mathbb{K}^{p \times q}$, $\|M\|$ stands for the induced norm of M associated with the above 2-norms.

A. Problem setting

We consider the abstract boundary control systems [7] with delayed boundary control

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + d(t), \qquad t \geq 0 \tag{1a}$$

$$\mathcal{B}X(t) = u_D(t) \triangleq u(t-D), \qquad t \geq 0 \tag{1b}$$

$$X(0) = X_0 \tag{1c}$$

with

- $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ a linear (unbounded) operator;
- $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathbb{K}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ a linear boundary operator;
- $d : \mathbb{R}_+ \to \mathcal{H}$ a distributed disturbance;
- $u : [-D, +\infty) \to \mathbb{K}^m$, with a known constant delay $D > 0$ and $u|_{[-D,0)} = 0$, the boundary control.

We assume that $(\mathcal{A}, \mathcal{B})$ is a boundary control system, i.e.,

1) the disturbance-free operator $\mathcal{A}_0$, defined on the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$, is the generator of a C_0-semigroup S on $\mathcal{H}$;
2) there exists a bounded operator $B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, called a lifting operator, such that $\mathrm{R}(B) \subset D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, and $\mathcal{B}B = I_{\mathbb{K}^m}$.

It is recalled that $\ker(\mathcal{B})$ is the kernel of $\mathcal{B}$ while $\mathrm{R}(B)$ stands for the range of B. We make the following assumptions.

Assumption 2.1: The disturbance-free operator $\mathcal{A}_0$ is a Riesz spectral operator [7], i.e., is a linear and closed operator with simple eigenvalues λ_n and corresponding eigenvectors $\phi_n \in D(\mathcal{A}_0)$, $n \in \mathbb{N}^*$, that satisfy:

1) $\{\phi_n,\ n \in \mathbb{N}^*\}$ is a Riesz basis [4]:
   a) $\overline{\operatorname{span}_{\mathbb{K}}}_{n \in \mathbb{N}^*} \phi_n = \mathcal{H}$;
   b) there exist constants $m_R, M_R \in \mathbb{R}_+^*$ such that for all $N \in \mathbb{N}^*$ and all $\alpha_1, \ldots, \alpha_N \in \mathbb{K}$,

$$m_R \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|_{\mathcal{H}}^2 \leq M_R \sum_{n=1}^{N} |\alpha_n|^2. \tag{2}$$

2) The closure of $\{\lambda_n,\ n \in \mathbb{N}^*\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$, $[a, b] \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$.

Assumption 2.2: There exist $N_0 \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}_+^*$ such that $\operatorname{Re} \lambda_n \leq -\alpha$ for all $n \geq N_0 + 1$.

Remark 2.3: Note that Assumption 2.2 is equivalent to:

- the number of unstable eigenvalues is finite, i.e., $\operatorname{Card}(\{\lambda_n : \operatorname{Re} \lambda_n \geq 0\}) < \infty$;
- the set composed of the real part of the stable eigenvalues is not accumulating at 0, i.e., $\sup\{\operatorname{Re} \lambda_n : n \geq 1, \operatorname{Re} \lambda_n < 0\} < 0$.

Physically meaningful problems that take the form of (1) and such that Assumptions 2.1 and 2.2 hold include reaction-diffusion phenomena [5], [19], phase turbulence phenomena [3] and certain models of structural vibrations phenomena such as wave [6], [7] and beam equations [7], [14].

From the well-known properties of Riesz bases [4], we introduce $\{\psi_n,\ n \in \mathbb{N}^*\}$ the biorthogonal sequence associated with the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e., $\langle \phi_k, \psi_l \rangle_{\mathcal{H}} = \delta_{k,l}$ with $\delta_{k,l} = 1$ if $k = l$ while $\delta_{k,l} = 0$ if $k \neq l$. Then, the series expansion $x = \sum_{n \geq 1} \langle x, \psi_n \rangle_{\mathcal{H}} \phi_n$ holds for all $x \in \mathcal{H}$. Furthermore, as $\mathcal{A}_0$ is assumed to be a Riesz-spectral operator, then ψ_n is an eigenvector of the adjoint operator $\mathcal{A}_0^*$ associated with the eigenvalue $\overline{\lambda_n}$.

B. Control objective

The control objective is twofold. First, in the absence of distributed disturbance (i.e. $d = 0$), the objective is to design a control law u that ensure the exponential stability of the closed-loop system. Second, the control law must ensure the ISS property of the closed-loop system with respect to the distributed disturbance d.

Because we are only concerned in controlling the system from the starting time $t = 0$, we assume that the system is uncontrolled for $t < 0$. This is why it is imposed $u|_{[-D,0)} = 0$. Therefore, due to the delay D in the control input of (1), the system remains open-loop for $t < D$ while the effect of the control input has an impact on the system only at times $t \geq D$.

Note that the $N_0 \in \mathbb{N}^*$ and $\alpha > 0$ provided by Assumption 2.2 are not unique. For instance, one could select $N_0 \in \mathbb{N}^*$ such that $\lambda_1, \ldots, \lambda_{N_0}$ are all with non negative real part. In this case, the control design reduces to stabilize the unstable part of the system. Nevertheless, one could also want to improve the decay rate or the damping of certain stable modes. In this case, $\lambda_1, \ldots, \lambda_{N_0}$ would include all the unstable eigenvalues and certain stable eigenvalues of the open-loop system.

III. Construction of the feedback control strategy

In order to derive the control law, we make in this section the *a priori* assumption that $u \in \mathcal{C}^2([-D, +\infty); \mathbb{K}^m)$. This assumption is necessary to ensure the existence of classical solutions of (1), and thus to proceed to the upcoming computations (see [7]). Consequently, the construction of the control law must ensure that such a regularity property holds. For the proposed control law, this regularity property will be assessed in the next section in Lemma 4.2. This result will ensure the validity of the computations reported in this section.

A. Spectral decomposition

Assuming that $u_D \in \mathcal{C}^2([0, +\infty); \mathbb{K}^m)$, $X_0 \in D(\mathcal{A})$ such that $\mathcal{B}X_0 = u_D(0) = 0$ (i.e., $X_0 \in D(\mathcal{A}_0)$), and $d \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$, we denote by $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ the unique classical solution of (1). Introducing $c_n(t) \triangleq \langle X(t), \psi_n \rangle_{\mathcal{H}}$ the coefficients of the projection of $X(t)$ into the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$, we have the series expansion:

$$X(t) = \sum_{n \geq 1} \langle X(t), \psi_n \rangle_{\mathcal{H}} \phi_n = \sum_{n \geq 1} c_n(t) \phi_n. \tag{3}$$

Then $c_n \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K})$ and, following [13] and introducing $d_n(t) \triangleq \langle d(t), \psi_n \rangle_{\mathcal{H}}$, we infer from (1) that, for all $t \geq 0$ and all $n \geq 1$,

$$\begin{aligned} \dot{c}_n(t) &= \langle \mathcal{A}X(t), \psi_n \rangle_{\mathcal{H}} + \langle d(t), \psi_n \rangle_{\mathcal{H}} \\ &= \langle \mathcal{A}_0 \{X(t) - Bu_D(t)\}, \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Bu_D(t), \psi_n \rangle_{\mathcal{H}} + d_n(t) \\ &= \langle X(t) - Bu_D(t), \mathcal{A}_0^* \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Bu_D(t), \psi_n \rangle_{\mathcal{H}} + d_n(t) \\ &= \left\langle X(t) - Bu_D(t), \overline{\lambda_n} \psi_n \right\rangle_{\mathcal{H}} + \langle \mathcal{A}Bu_D(t), \psi_n \rangle_{\mathcal{H}} + d_n(t) \\ &= \lambda_n c_n(t) - \lambda_n \langle Bu_D(t), \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Bu_D(t), \psi_n \rangle_{\mathcal{H}} + d_n(t), \end{aligned} \tag{4}$$

where we used that $\mathcal{B}\{X(t) - Bu_D(t)\} = u_D(t) - u_D(t) = 0$, showing that $X(t) - Bu_D(t) \in D(\mathcal{A}) \cap \ker(\mathcal{B}) = D(\mathcal{A}_0)$.

Remark 3.1: The ODE (4) describing the time evolution of the coefficient $c_n(t) = \langle X(t), \psi_n \rangle_{\mathcal{H}}$ only involves the delayed control input $u_D(t)$ while avoiding the occurrence of its time derivative $\dot{u}_D(t)$. Therefore, whereas it was necessary in [5], [6], [19], due to the presence of the term $\dot{u}_D(t)$ in the ODEs resulting from the spectral decomposition, to augment the state of the finite-dimensional subsystem and to use $\dot{u}_D(t)$ as a control input, we avoid here such a procedure. This yields a simplification of the control law by avoiding an *a posteriori* integration of $\dot{u}$ to obtain the actual control law u.

Let $\mathcal{E} = (e_1, e_2, \ldots, e_m)$ be the canonical basis of $\mathbb{K}^m$ and consider the projections $u_1, u_2, \ldots, u_m \in \mathcal{C}^2([-D, +\infty); \mathbb{K})$ such that:

$$u = \sum_{k=1}^{m} u_k e_k = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

Introducing $b_{n,k} \triangleq -\lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Be_k, \psi_n \rangle_{\mathcal{H}}$, we obtain from (4) that

$$\dot{c}_n(t) = \lambda_n c_n(t) + \sum_{k=1}^{m} b_{n,k} u_k(t - D) + \langle d(t), \psi_n \rangle_{\mathcal{H}}.$$

Then, the following ODE with delay input holds for all $t \geq 0$

$$\dot{Y}(t) = A_{N_0} Y(t) + B_{N_0} u_D(t) + D_{N_0}(t), \tag{5}$$

where $A_{N_0} = \operatorname{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{K}^{N_0 \times N_0}$, $B_{N_0} = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq m} \in \mathbb{K}^{N_0 \times m}$,

$$Y(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_{N_0}(t) \end{bmatrix} = \begin{bmatrix} \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix} \in \mathbb{K}^{N_0}, \tag{6}$$

and

$$D_{N_0}(t) = \begin{bmatrix} d_1(t) \\ \vdots \\ d_{N_0}(t) \end{bmatrix} = \begin{bmatrix} \langle d(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle d(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix} \in \mathbb{K}^{N_0}. \tag{7}$$

Note that the norm of $D_{N_0}(t)$ can be bounded above in function of the norm of the full distributed disturbance $d(t)$ as follows. For all $t \geq 0$, we have

$$\begin{aligned} \|D_{N_0}(t)\|^2 = \sum_{k=1}^{N_0} |\langle d(t), \psi_k \rangle_{\mathcal{H}}|^2 &\leq \sum_{k \geq 1} |\langle d(t), \psi_k \rangle_{\mathcal{H}}|^2 \\ &\overset{(2)}{\leq} \frac{1}{m_R} \|d(t)\|_{\mathcal{H}}^2. \end{aligned} \tag{8}$$

The finite-dimensional linear ODE (5) captures the part of the dynamics of (1) that must be stabilized/controlled by the feedback control u. The idea consists in first designing a control law that exponentially stabilizes the linear ODE (5). Then, we assess that the proposed control law amounts stabilizing the original infinite-dimensional system (1) by means of an adequate Lyapunov function.

B. Stabilization of the finite-dimensional subsystem

At this point, we need to design a control law that stabilizes the linear ODE with input delay (5). First, we resort to the Artstein model reduction [1], [20] to obtain an equivalent linear ODE that is free of delay. Specifically, we introduce for all $t \geq 0$

$$\begin{aligned} Z(t) &= Y(t) + \int_{t-D}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} u(s) \,\mathrm{d}s \\ &= Y(t) + \int_{0}^{D} e^{-\tau A_{N_0}} B_{N_0} u(t-D+\tau) \,\mathrm{d}\tau. \end{aligned}$$

Straightforward computations show that we have, for all $t \geq 0$,

$$\dot{Z}(t) = A_{N_0} Z(t) + e^{-DA_{N_0}} B_{N_0} u(t) + D_{N_0}(t).$$

As $e^{-DA_{N_0}}$ is invertible and commutes with A_{N_0}, the pair $(A_{N_0}, e^{-DA_{N_0}} B_{N_0})$ satisfies the Kalman condition if and only if the pair (A_{N_0}, B_{N_0}) satisfies the Kalman condition. Consequently, in order to be able to apply the pole-shifting theorem, we make the following assumption.

Assumption 3.2: (A_{N_0}, B_{N_0}) satisfies the Kalman condition.

Remark 3.3: In the case of a one-dimensional control input, i.e., $m = 1$, we have that

$$\begin{aligned} &\det(B_{N_0}, A_{N_0} B_{N_0}, \ldots, A_{N_0}^{N_0-1} B_{N_0}) \\ &\qquad = \prod_{n=1}^{N_0} b_{n,1} \times \mathrm{VdM}(\lambda_1, \ldots, \lambda_{N_0}), \end{aligned}$$

where $\mathrm{VdM}(\lambda_1, \ldots, \lambda_{N_0})$ is the Vandermonde determinant associated with $\lambda_1, \ldots, \lambda_{N_0}$. Therefore, Assumption 3.2 is fulfilled if and only if $\lambda_1, \ldots, \lambda_{N_0}$ are all distinct and $b_{n,1} \neq 0$ for all $1 \leq n \leq N_0$. In the general case $m \geq 1$, we can easily apply the PBH test [26] due to the diagonal nature of the matrix A_{N_0}. Assume without loss of generality that $\lambda_1, \ldots, \lambda_{N_0}$ are ordered such that there exist $n_1, \ldots, n_p \in \mathbb{N}^*$ with $n_1 + \ldots + n_p = N_0$ such that 1) for all $1 \leq l \leq p$, $\lambda_{s_{l-1}+1} = \lambda_{s_{l-1}+2} = \ldots = \lambda_{s_l}$; 2) $l_1 \neq l_2$ implies $\lambda_{s_{l_1}} \neq \lambda_{s_{l_2}}$, where $s_l = n_1 + n_2 + \ldots + n_l$. Then, Assumption 3.2 is fulfilled if and only if $\mathrm{rank}[(b_{n,k})_{s_{l-1}+1 \leq n \leq s_l, 1 \leq k \leq m}] = n_l$ for all $1 \leq l \leq p$. In particular, it requires the necessary condition that $n_l \leq m$ for all $1 \leq l \leq p$.

Remark 3.4: Note that $b_{n,k}$ is computed based on the selection of a given lifting operator B. Even if such a lifting operator is not unique, the quantity $b_{n,k}$ is actually independent of the particularly selected lifting operator. Indeed, let B and $\tilde{B}$ be two distinct lifting operators associated with $(\mathcal{A}, \mathcal{B})$. Then, introducing $\hat{B} = B - \tilde{B}$, one has $\mathcal{B}\hat{B} = \mathcal{B}B - \mathcal{B}\tilde{B} = I_{\mathbb{K}^m} - I_{\mathbb{K}^m} = 0$. Thus, $R(\hat{B}) \subset D(\mathcal{A}) \cap \ker(\mathcal{B}) = D(\mathcal{A}_0)$ and we obtain that

$$\begin{aligned} \left\langle \mathcal{A}\hat{B}e_k, \psi_n \right\rangle_{\mathcal{H}} &= \left\langle \mathcal{A}_0 \hat{B}e_k, \psi_n \right\rangle_{\mathcal{H}} = \left\langle \hat{B}e_k, \mathcal{A}_0^* \psi_n \right\rangle_{\mathcal{H}} \\ &= \left\langle \hat{B}e_k, \overline{\lambda_n} \psi_n \right\rangle_{\mathcal{H}} = \lambda_n \left\langle \hat{B}e_k, \psi_n \right\rangle_{\mathcal{H}}. \end{aligned}$$

We deduce the claimed result, i.e.,

$$\begin{aligned} &- \lambda_n \left\langle Be_k, \psi_n \right\rangle_{\mathcal{H}} + \left\langle \mathcal{A}Be_k, \psi_n \right\rangle_{\mathcal{H}} \\ &\qquad = -\lambda_n \left\langle \tilde{B}e_k, \psi_n \right\rangle_{\mathcal{H}} + \left\langle \mathcal{A}\tilde{B}e_k, \psi_n \right\rangle_{\mathcal{H}}. \end{aligned}$$

Therefore, the commandability property of the pair (A_{N_0}, B_{N_0}) is an intrinsic property of the boundary control system $(\mathcal{A}, \mathcal{B})$ in the sense that it does not depend on the selection of a particular lifting operator B.

Assuming that Assumption 3.2 holds, we can find a feedback gain $K \in \mathbb{K}^{m \times N_0}$ and $P \in \mathcal{H}_{N_0}^{+*}$ a positive definite Hermitian matrix such that $A_{\mathrm{cl}} \triangleq A_{N_0} + e^{-DA_{N_0}} B_{N_0} K$ is Hurwitz with desired pole placement and

$$A_{\mathrm{cl}}^* P + P A_{\mathrm{cl}} = -I_{N_0}.$$

Then, a natural choice for the control input would be $u(t) = \chi_{[0,+\infty)}(t) K Z(t)$. However, the resulting $u_D(t) = u(t-D) = \chi_{[D,+\infty)}(t) K Z(t-D)$ is discontinuous at $t = D$ while u_D must be of class $\mathcal{C}^2$ over $\mathbb{R}_+$ to ensure the existence of a classical solution of (1). Let $t_0 > 0$ be given. We consider a transition signal (from open loop to closed loop) $\varphi \in \mathcal{C}^2([-D, +\infty); \mathbb{R})$ which is such that $0 \leq \varphi \leq 1$, $\varphi|_{[-D,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$. We define the state-feedback control input $u = \varphi K Z$. It satisfies $u|_{[-D,0]} = 0$ and, for all $t \geq 0$,

$$\begin{aligned} u(t) &= \varphi(t) K Z(t) \\ &= \varphi(t) K Y(t) \\ &\quad + \varphi(t) K \int_{\max(t-D,0)}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} u(s) \,\mathrm{d}s, \end{aligned} \tag{9}$$

where it has been used that the system is uncontrolled for $t \leq 0$. In particular, the control law is such that $u_D(t) = u(t-D) = \varphi(t-D) K Z(t-D)$ with $u_D(t) = 0$ for $t \leq D$ and $u_D(t) = K Z(t-D)$ for $t \geq D + t_0$.

Remark 3.5: This is worth noting that the proposed control law (9) is presented in a constructive manner and takes the form of a predictor feedback as the ones classically designed for the control of finite-dimensional LTI systems in the presence of a delayed control input.

C. Characterization of the control law

In practice, it is convenient to use the control law expressed under the form (9) since it allows its computation at time t based on the measure of Y at time t and the past history of the control law u. To do so, we must show that (9) fully characterizes u, i.e., the uniqueness of the function u satisfying the implicit equation (9). In other words, it requires to invert the Artstein transformation [2] when weighted by the transition signal φ. For any locally integrable function $f : I \to \mathbb{K}^m$ with either $I = \mathbb{R}_+$ or $I = [0, T]$ for some $T > 0$, we define $T_D f : I \to \mathbb{K}^m$ as follows:

$$(T_D f)(t) = \varphi(t) K \int_{\max(t-D,0)}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} f(s) \,\mathrm{d}s.$$

In particular $T_D f \in \mathcal{C}^0(I;\mathbb{K}^m)$, and thus we can consider the iterations $T_D^k f$ for any $k \in \mathbb{N}$.

Lemma 3.6: Let I be either $I = \mathbb{R}_+$ or $I = [0,T]$ for some $T > 0$. Let $D > 0$, $g \in \mathcal{C}^0(I;\mathbb{K}^{N_0})$, and $\varphi \in \mathcal{C}^0(I;\mathbb{R})$ such that $0 \leq \varphi \leq 1$ be given. Then, there exists a unique locally integrable function v defined on I such that, for all $t \in I$,

$$\begin{aligned} v(t) &= \varphi(t)Kg(t) \\ &\quad + \varphi(t)K\int_{\max(t-D,0)}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} v(s)\,\mathrm{d}s. \end{aligned}$$

Furthermore $v \in \mathcal{C}^0(I;\mathbb{K}^m)$ and is given by the series expansion $v(t) = \sum_{k\geq 0} (T_D^k(\varphi K g))(t)$ where the series converge uniformly over any time interval of finite length.

The inversion of the Artstein transformation in the case $\varphi = 1$ has been investigated in [2]. The proof of Lemma 3.6 is a straightfoward extension of theorem 1 in [2] by noting that $v = \varphi K g + T_D v$, φ is a continuous function, and $0 \leq \varphi \leq 1$.

IV. Study of the closed-loop infinite-dimensional system

Throughout this section, we assume that Assumptions 2.1, 2.2, and 3.2 hold. Under these conditions, it has been proposed in Section III to resort to the control law given by (9) to stabilize the infinite-dimensional system (1). As the control law has been derived on a finite-dimensional part of the original infinite-dimensional system, we must guarantee that the proposed control strategy successfully stabilizes the full system. Furthermore, in order to make valid the computations performed in the previous section, we must ensure that the *a priori* regularity assumption on the control input u is indeed satisfied, i.e., u provided by (9) is of class $\mathcal{C}^2$.

A. Dynamics of the closed-loop system

Let $D, t_0 > 0$ be given. We consider a given transition signal $\varphi \in \mathcal{C}^2([-D,+\infty);\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi|_{[-D,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$. The closed-loop system dynamics takes the following form:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + d(t), \tag{10a}$$

$$\mathcal{B}X(t) = u_D(t) = u(t-D), \tag{10b}$$

$$u|_{[-D,0]} = 0 \tag{10c}$$

$$u(t) = \varphi(t)KY(t) \tag{10d}$$

$$+ \varphi(t)K\int_{\max(t-D,0)}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s, \tag{10e}$$

$$X(0) = X_0 \tag{10f}$$

for any $t \geq 0$. The adopted control strategy takes the form of a state-feedback in which the signal $Y(t)$ is computed based on the state $X(t)$ via (6). The feedback gain $K \in \mathbb{K}^{m\times N_0}$ is selected such that $A_{\mathrm{cl}} \triangleq A_{N_0} + e^{-DA_{N_0}} B_{N_0} K$ is Hurwitz. Function $d : \mathbb{R}_+ \to \mathcal{H}$ represents a distributed disturbance.

Remark 4.1: We assume that the initial control input is identically zero, i.e., $u_0 \triangleq u|_{[-D_0-\delta,0]} = 0$. This can be obtained in practice by initially applying a zero control input. This avoids the necessity of regularity assumptions on u_0 and the introduction of compatibility conditions restricting the admissible initial conditions X_0.

B. Well-posedness in terms of classical solutions

The following lemma ensures both the well-posedness of the closed-loop system in terms of classical solutions and the sufficient regularity of the control input. The proof is placed in Annex A.

Lemma 4.2: Let $(\mathcal{A},\mathcal{B})$ be an abstract boundary control system such that Assumptions 2.1, 2.2, and 3.2 hold. For any $X_0 \in D(\mathcal{A}_0)$ and $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$, the closed-loop system (10) admits a unique classical solution $X \in \mathcal{C}^0(\mathbb{R}_+;D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$. The associated control law u is uniquely defined and is of class $\mathcal{C}^2([-D,+\infty);\mathbb{K}^m)$. It can be written under the form $u = \varphi K Z$ with, for all $t \geq 0$,

$$Z(t) \triangleq Y(t) + \int_{t-D}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s, \tag{11}$$

which is such that $Z \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{K}^{N_0})$ and satisfies, for all $t \geq 0$,

$$\dot{Z}(t) = (A_{N_0} + \varphi(t)e^{-DA_{N_0}} B_{N_0} K)Z(t) + D_{N_0}(t), \tag{12}$$

where $D_{N_0}(t)$ is defined by (7). In particular, for all $t \geq t_0$,

$$\dot{Z}(t) = A_{\mathrm{cl}} Z(t) + D_{N_0}(t). \tag{13}$$

C. Exponential ISS property of the closed-loop system

This section is devoted to the demonstration of the following stability result.

Theorem 4.3: Let $(\mathcal{A},\mathcal{B})$ be an abstract boundary control system such that Assumptions 2.1, 2.2, and 3.2 hold. Then the closed-loop system (10) is exponential ISS in the sense that there exist constants $\kappa_0 > 0$ and $\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4 \in \mathbb{R}_+$ such that, for any $X_0 \in D(\mathcal{A}_0)$ and $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$, the classical solution X of (10) associated with the initial condition X_0 and the distributed disturbance d satisfies, for all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}} \leq \overline{C}_1 e^{-\kappa_0 t}\|X_0\|_{\mathcal{H}} + \overline{C}_2 \sup_{\tau\in[0,t]} \|d(\tau)\|_{\mathcal{H}}, \tag{14}$$

and the control law is such that

$$\|u(t)\| \leq \overline{C}_3 e^{-\kappa_0 t}\|X_0\|_{\mathcal{H}} + \overline{C}_4 \sup_{\tau\in[0,t]} \|d(\tau)\|_{\mathcal{H}}. \tag{15}$$

Remark 4.4: Theorem 4.3 ensures the stability of the closed-loop system whatever the value of the delay $D > 0$ may be. In particular, the number of modes N_0 to be considered in the control design is only constrained by the number of open-loop unstable modes via Assumption 2.2, regardless of the value of the delay $D > 0$.

To prove the theorem, we consider throughout this section $X_0 \in D(\mathcal{A}_0)$ and $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ arbitrarily given. Let $X \in \mathcal{C}^0(\mathbb{R}_+;D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ be the classical solution of the closed-loop system (10) associated with the initial condition $X_0 \in D(\mathcal{A}_0)$ and the distributed disturbance $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$. We denote by Z the function defined by (11).

1) Definition of the Lyapunov function candidate: The proof of the theorem relies on the following Lyapunov function candidate, defined for all $t \geq 0$ by

$$\begin{aligned} V(t) = \gamma_1 \left\{ Z(t)^* P Z(t) + \int_{t-D}^{t} \varphi(s) Z(s)^* P Z(s) \,\mathrm{d}s \right\} \\ + \gamma_2 \varphi(t-D) Z(t-D)^* P Z(t-D) \\ + \frac{1}{2} \sum_{k \geq N_0+1} \left| \langle X(t) - B u_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 , \end{aligned} \tag{16}$$

where, because $A_{\mathrm{cl}} = A_{N_0} + e^{-DA_{N_0}} B_{N_0} K$ is Hurwitz, $P \in \mathcal{H}_{N_0}^{+*}$ is a positive definite Hermitian matrix such that

$$A_{\mathrm{cl}}^* P + P A_{\mathrm{cl}} = -I_{N_0}. \tag{17}$$

Constant $\gamma_1, \gamma_2 \in \mathbb{R}_+^*$ are sufficiently large parameters to be selected latter, independently of the initial condition X_0 and the distributed disturbance d. Note that, as a direct consequence of the definition, one has $V(t) \geq 0$ for all $t \geq 0$.

Remark 4.5: Function V is well-defined and belongs to $\mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$. This follows from Annex B and the fact that functions φ, X, Z, and u are of class $\mathcal{C}^1$.

Remark 4.6: At this point, it is relevant to discuss the motivation behind the choice of the different terms of the Lyapunov function candidate (16).

1) Assuming a zero distributed disturbance ($d = 0$), the term $Z(t)^* P Z(t)$ provides, based on (17), a Lyapunov function for the finite-dimensional system $\dot{Z}(t) = A_{\mathrm{cl}} Z(t)$. It aims at ensuring the exponential convergence to zero of the N_0 first coefficients $\langle X(t), \psi_n \rangle_{\mathcal{H}}$ corresponding to the projection of the system trajectory $X(t)$ into the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$ (see (3)).
2) In order to ensure the stability of the full infinite-dimensional system, the Lyapunov function candidate V must ensure the convergence of all the modes of the plant. This includes the coefficients $c_n(t) = \langle X(t), \psi_n \rangle_{\mathcal{H}}$, $n \geq N_0 + 1$, which were not considered in the synthesis of the control law, but which are still impacted by the control input u according to the dynamics (4). A natural choice to capture these coefficients would consist in the use of the term $\frac{1}{2} \sum_{k \geq N_0+1} |\langle X(t), \psi_k \rangle_{\mathcal{H}}|^2$. However, the ODE describing the time domain evolution of $\langle X(t), \psi_n \rangle_{\mathcal{H}}$ given by (4) shows that the eigenvalue λ_n appears via the following term: $\lambda_n \langle X(t) - B u_D(t), \psi_n \rangle_{\mathcal{H}}$. Therefore, in order to be able to absorb all the occurrences of the eigenvalue λ_n, $n \geq N_0 + 1$, via the inequality $\operatorname{Re} \lambda_n \leq -\alpha$ of Assumption 2.2, we consider the term $\frac{1}{2} \sum_{k \geq N_0+1} |\langle X(t) - B u_D(t), \psi_k \rangle_{\mathcal{H}}|^2$ (see (35) for details).
3) As $u = \varphi K Z$, the introduction of the term $u_D(t)$ in the Lyapunov function candidate V yields the occurrence of the term $Z(t-D)$. It requires the introduction of the term $\varphi(t-D) Z(t-D)^* P Z(t-D)$ for compensation purposes. The switching signal φ is used to materialize the fact that the contribution of this term is relevant only for $t \geq D$.
4) Finally, the contribution of the term $\int_{t-D}^{t} \varphi(s) Z(s)^* P Z(s) \,\mathrm{d}s$ is to provide an upper bound on the norm of the system trajectory $X(t)$ which only depends on $V(t)$ (see Lemma 4.7).

The detailed properties of the Lyapunov function candidate V are detailed in the next lemmas.

2) Upper bound on the norm of X: First, we establish a connection between the norm of the system trajectory $X(t)$ and the value of the Lyapunov function candidate $V(t)$. To do so, we define the constant $C_1 > 0$ by

$$C_1 \triangleq 2 \max \left(1, D e^{2D \|A_{N_0}\|} \|B_{N_0} K\|^2 \right). \tag{18}$$

We denote by $\lambda_m(P) > 0$ the smallest eigenvalue of P.

Lemma 4.7: Under the assumptions of Theorem 4.3 and for $\gamma_1 > C_1/\lambda_m(P)$ and $\gamma_2 > \|BK\|^2/(m_R \lambda_m(P))$ arbitrarily given, we have, for all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}} \leq C_4 \sqrt{V(t)}, \tag{19}$$

where $C_4 = \sqrt{2M_R} + \dfrac{\sqrt{m_R}\|BK\|}{\sqrt{\gamma_2 m_R \lambda_m(P) - \|BK\|^2}} > 0$.

The proof of Lemma 4.7 can be found in Annex C.

3) Exponential convergence of the closed-loop system trajectories: In order to study the exponential decay of V, we consider the time interval over which the infinite-dimensional system is fully placed in closed loop, i.e., for $t > D + t_0$ which corresponds to $\varphi(t) = 1$. For $t > D + t_0$, one has

$$\begin{aligned} V(t) = \gamma_1 \left\{ Z(t)^* P Z(t) + \int_{t-D}^{t} Z(s)^* P Z(s) \,\mathrm{d}s \right\} \\ + \gamma_2 Z(t-D)^* P Z(t-D) \\ + \frac{1}{2} \sum_{k \geq N_0+1} \left| \langle X(t) - B u_D(t), \psi_k \rangle \right|^2 , \end{aligned}$$

with $u_D(t) = u(t-D) = KZ(t-D)$. We also introduce the positive constant

$$C_5 \triangleq \frac{2m}{\alpha m_R} \sum_{i=1}^{m} \left\{ \|\mathcal{A} B e_i\|_{\mathcal{H}}^2 \|K_i\|^2 + \|B e_i\|_{\mathcal{H}}^2 \|K_i A_{\mathrm{cl}}\|^2 \right\}, \tag{20}$$

where K_i is the i-th line of the matrix of feedback gain K. The study of the time derivative $\dot{V}$ of V for $t > D + t_0$ is reported in Annex D and yields the following result.

Lemma 4.8: Let $\beta \in (0,1)$ be arbitrarily given. Under the assumptions of Theorem 4.3, and for any arbitrarily given $\gamma_1 > C_1/\lambda_m(P)$ and $\gamma_2 > \max\left(\|BK\|^2/(m_R \lambda_m(P)), C_5/(1-\beta) \right)$, there exist constants[1] $\kappa_0 = \kappa_0(\beta, \gamma_2) > 0$ and $C_6 = C_6(\beta, \gamma_1, \gamma_2) > 0$, independent of X_0 and d, such that we have for all $t \geq D + t_0$

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} \leq C_4 e^{-\kappa_0 (t-D-t_0)} \sqrt{V(D+t_0)} \\ + C_4 \sqrt{\frac{C_6}{2\kappa_0}} \sup_{\tau \in [0,t]} \|d(\tau)\|_{\mathcal{H}}, \end{aligned} \tag{21}$$

with a control input such that

$$\|u(t)\| \leq \frac{\|K\|}{\sqrt{C_2(\gamma_1)}} e^{-\kappa_0 (t-D-t_0)} \sqrt{V(D+t_0)}$$

[1]These constants are explicitly given in Annex D by (36-37)

$$+\frac{\|K\|}{\sqrt{C_2(\gamma_1)}}\sqrt{\frac{C_6}{2\kappa_0}}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}. \tag{22}$$

Remark 4.9: Coefficient $\beta \in (0,1)$ represents a trade-off between the guaranteed decay rate κ_0 and the coefficient $C_6/(2\kappa_0)$ that reflects the impact of the external disturbance on the system trajectory. In particular (see (36-37) in Annex D for details), taking $\beta \to 0^+$ will result in an increase of the decay rate κ_0 but also $C_6/(2\kappa_0) \to +\infty$.

4) ISS estimate: In order to complete the proof of Theorem 4.3, we resort to the following lemma that provides an estimate of $V(t)$ over the time interval $[0, D+t_0]$.

Lemma 4.10: Under the assumptions of Theorem 4.3, there exist constants $C_9 = C_9(\gamma_1,\gamma_2) > 0$ and $C_{10} = C_{10}(\gamma_1,\gamma_2) > 0$, independent of X_0 and d, such that for all $t \in [0, D+t_0]$,

$$V(t) \leq C_9\|X_0\|^2 + C_{10}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}^2. \tag{23}$$

The proof of Lemma 4.10 is in Annex E. We can now complete the proof of Theorem 4.3. Indeed, for a given arbitrary $\beta \in (0,1)$ and by selecting $\gamma_1 > C_1/\lambda_m(P)$ and $\gamma_2 > \max\left(\|BK\|^2/(m_R\lambda_m(P)), C_5/(1-\beta)\right)$, we obtain from (23) and (39) that the following estimate holds

$$V(t) \leq C_9 e^{-2\kappa_0(t-D-t_0)}\|X_0\|_{\mathcal{H}}^2 + \left(\frac{C_6}{2\kappa_0} + C_{10}\right)\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}^2,$$

for all $t \geq 0$. From (19), we obtain that, for all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}} \leq \left\{C_4\sqrt{C_9}e^{\kappa_0(D+t_0)}\right\}e^{-\kappa_0 t}\|X_0\|_{\mathcal{H}} + C_4\sqrt{\frac{C_6}{2\kappa_0} + C_{10}}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}.$$

This shows that the claimed ISS estimate (14) holds. The estimate of the control input (15) follows from (40), which concludes the proof of Theorem 4.3.

V. Application to the stability analysis of a closed-loop interconected IDS-ODE system

As an application of the ISS property of the closed-loop system (10), we propose to study the stability of a related IDS-ODE interconnection. Specifically, we consider the case where the external input d depends on the state of an ODE satisfying a certain ISS estimate. This study is motivated by the fact that certain physical systems such as chemical reactors and water tanks are naturally modeled by a coupled PDE-ODE system as the one studied in this section [11].

A. Dynamics of the closed-loop interconnected IDS-ODE system and well-posedness

Let $D, t_0 > 0$ be given. We consider a given transition signal $\varphi \in \mathcal{C}^2([-D,+\infty);\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi|_{[-D,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$. Let $f_1 \in \mathcal{C}^1(\mathbb{K}^n \times \mathcal{H} \times \mathbb{K}^{m_v};\mathbb{K}^n)$ and $f_2 \in \mathcal{C}^1(\mathbb{K}^n \times \mathcal{H} \times \mathbb{K}^{m_v},\mathcal{H})$ be two vector fields. We make the following assumption.

Assumption 5.1:

1) Vector fields $f_1(x,X,v)$ and $f_2(x,X,v)$ are (globally) Lipschitz continuous in (x,X) on $\mathbb{K}^n \times \mathcal{H}$, uniformly in v over any compact subset of $\mathbb{K}^{m_v}$.
2) There exist constants $D_1, D_2, D_3 \geq 0$ such that, for all $x \in \mathbb{K}^n$, $X \in \mathcal{H}$, and $v \in \mathbb{K}^{m_v}$,
$$\|f_2(x,X,v)\|_{\mathcal{H}} \leq D_1\|x\| + D_2\|X\|_{\mathcal{H}} + D_3\|v\|. \tag{24}$$
3) The ODE $\dot{x} = f_1(x,X,v)$ is such that there exist $\tilde{\kappa}_0 > 0$ and $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2 \in \mathbb{R}_+$ such that, for any given initial condition $x(0) = x_0 \in \mathbb{K}^n$ and functions $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ and $v \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{K}^{m_v})$, the following ISS estimate holds for all $t \geq 0$
$$\|x(t)\|^2 \leq \tilde{C}_0^2 e^{-2\tilde{\kappa}_0 t}\|x_0\|^2 + \sup_{\tau\in[0,t]}\left\{\tilde{C}_1^2\|X(\tau)\|_{\mathcal{H}}^2 + \tilde{C}_2^2\|v(\tau)\|^2\right\}. \tag{25}$$

Note that the above assumption implies (by taking $t = 0$, $x_0 \neq 0$ and null X and v) that $\tilde{C}_0 \geq 1$. The considered closed-loop system takes the following form:

$$\dot{x}(t) = f_1(x(t), X(t), v(t)), \tag{26a}$$
$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + f_2(x(t), X(t), v(t)), \tag{26b}$$
$$\mathcal{B}X(t) = u_D(t) = u(t-D), \tag{26c}$$
$$u|_{[-D,0]} = 0 \tag{26d}$$
$$u(t) = \varphi(t)KY(t) + \varphi(t)K\int_{\max(t-D,0)}^{t} e^{(t-s-D)A_{N_0}}B_{N_0}u(s)\,\mathrm{d}s, \tag{26e}$$
$$x(0) = x_0, \tag{26f}$$
$$X(0) = X_0 \tag{26g}$$

for $t \geq 0$ with Y defined by (6). The feedback gain $K \in \mathbb{K}^{m\times N_0}$ is such that $A_{\mathrm{cl}} \triangleq A_{N_0} + e^{-DA_{N_0}}B_{N_0}K$ is Hurwitz. Function u still represents the control input while function $v : \mathbb{R}_+ \to \mathbb{K}^{m_v}$ represents an external disturbance.

The well-posedness of the closed-loop system (26) is assessed via the following result.

Lemma 5.2: Let $(\mathcal{A},\mathcal{B})$ be an abstract boundary control system and $f_1 \in \mathcal{C}^1(\mathbb{K}^n \times \mathcal{H} \times \mathbb{K}^{m_v};\mathbb{K}^n)$ and $f_2 \in \mathcal{C}^1(\mathbb{K}^n \times \mathcal{H} \times \mathbb{K}^{m_v},\mathcal{H})$ be vector fields such that Assumptions 2.1, 2.2, 3.2, and 5.1 hold. For any $(x_0,X_0) \in \mathbb{K}^n \times D(\mathcal{A}_0)$ and $v \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{K}^{m_v})$, the closed-loop system (26) has a unique classical solution $(x,X) \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{K}^n) \times \left(\mathcal{C}^0(\mathbb{R}_+;D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})\right)$. Introducing $d(t) \triangleq f_2(x(t),X(t),v(t))$, we have $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$. Thus, X is the classical solution of (10) associated with the initial condition X_0 and the distributed disturbance d. Consequently, both Lemma 4.2 and Theorem 4.3 apply to X.

The proof of Lemma 5.2 follows from the same arguments as the one used in the proof of Lemma 4.2 and from classical theorems on the existence and uniqueness of classical solutions for lipschitz perturbations of linear evolution equations, see, e.g., [18, Th. 1.2 and Th. 1.5, Chap. 6].

Remark 5.3: The first point of Assumption 5.1 is used to ensure the existence of solutions defined on $\mathbb{R}_+$. If this assumption is replaced by a Lipschitz condition in (x,X) on any bounded subset of the state-space, uniformly in v on compact sets, the existence of the classical solution is *a priori* only guaranteed over a time interval $[0, t_{\max})$ with $0 < t_{\max} \leq +\infty$. Furthermore, if $t_{\max} < +\infty$, we have the blow up of the solution in finite time, i.e., $\|x(t)\| + \|X(t)\|_{\mathcal{H}} \underset{t \to (t_{\max})^-}{\longrightarrow} +\infty$, see, e.g., [18, Th. 1.4 and Th. 1.5, Chap. 6]. In this case, the reasoning presented next still applies over the time interval $[0, t_{\max})$ at the condition that no blow up occurs over the time interval $[0, D + t_0]$, i.e., $t_{\max} > D + t_0$. This can be ensured by assuming that the following small gain condition holds:

$$(D_1 \tilde{C}_1 + D_2) C_4 \sqrt{C_{10}} < 1. \tag{27}$$

Indeed, from (19), (23), and (24-25), we obtain that, for all $t \in [0, D + t_0] \cap [0, t_{\max})$,

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} \leq & D_1 \tilde{C}_0 C_4 \sqrt{C_{10}} \|x_0\| + C_4 \sqrt{C_9} \|X_0\|_{\mathcal{H}} \\ & + (D_1 \tilde{C}_1 + D_2) C_4 \sqrt{C_{10}} \sup_{\tau \in [0,t]} \|X(\tau)\|_{\mathcal{H}} \\ & + (D_1 \tilde{C}_2 + D_3) C_4 \sqrt{C_{10}} \sup_{\tau \in [0,t]} \|v(\tau)\|. \end{aligned}$$

Under the small gain assumption (27), we can introduce $\Gamma \triangleq \left(1 - (D_1 \tilde{C}_1 + D_2) C_4 \sqrt{C_{10}}\right)^{-1} > 0$, which yields

$$\begin{aligned} & \sup_{\tau \in [0, D+t_0] \cap [0, t_{\max})} \|X(\tau)\|_{\mathcal{H}} \\ & \leq \Gamma D_1 \tilde{C}_0 C_4 \sqrt{C_{10}} \|x_0\| + \Gamma C_4 \sqrt{C_9} \|X_0\|_{\mathcal{H}} \\ & \quad + \Gamma (D_1 \tilde{C}_2 + D_3) C_4 \sqrt{C_{10}} \sup_{\tau \in [0, D+t_0]} \|v(\tau)\| \\ & < \infty. \end{aligned}$$

From (25) we infer that

$$\sup_{\tau \in [0, D+t_0] \cap [0, t_{\max})} \{ \|x(\tau)\| + \|X(\tau)\|_{\mathcal{H}} \} < \infty,$$

and, consequently, $t_{\max} > D + t_0$.

B. Small gain condition ensuring the stability of the IDS-ODE interconnection

The main result of this section is the following result.

Theorem 5.4: Let $(\mathcal{A}, \mathcal{B})$ be an abstract boundary control system and $f_1 \in \mathcal{C}^1(\mathbb{K}^n \times \mathcal{H} \times \mathbb{K}^{m_v}; \mathbb{K}^n)$ and $f_2 \in \mathcal{C}^1(\mathbb{K}^n \times \mathcal{H} \times \mathbb{K}^{m_v}, \mathcal{H})$ be vector fields such that Assumptions 2.1, 2.2, 3.2, and 5.1 hold. We assume that the small gain condition

$$(D_1 \tilde{C}_1 + D_2) C_4 \sqrt{\frac{C_6}{2\kappa_0}} < 1 \tag{28}$$

is satisfied. Then, there exist constants $\delta_\epsilon \in (0, \kappa_0)$ and $G_i, H_i \in \mathbb{R}_+$, $0 \leq i \leq 3$, such that, for any $(x_0, X_0) \in \mathbb{K}^n \times D(\mathcal{A}_0)$ and $v \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K}^{m_v})$, the classical solution (x, X) of (26) associated with the initial condition (x_0, X_0) and the disturbance v satisfies for all $t \geq D + t_0$ the following fading memory estimate:

$$\begin{aligned} \|x(t)\| + \|X(t)\|_{\mathcal{H}} \leq & G_0 e^{-\delta_\epsilon t} (\|x_0\| + \|X_0\|_{\mathcal{H}}) \\ & + G_1 e^{-\delta_\epsilon t} \sup_{\tau \in [0, D+t_0]} \|x(\tau)\| \\ & + G_2 e^{-\delta_\epsilon t} \sup_{\tau \in [0, D+t_0]} \|X(\tau)\|_{\mathcal{H}} \\ & + G_3 \sup_{\tau \in [0,t]} e^{-\delta_\epsilon (t-\tau)} \|v(\tau)\|, \end{aligned} \tag{29}$$

and the control law satisfies

$$\begin{aligned} \|u(t)\| \leq & H_0 e^{-\delta_\epsilon t} (\|x_0\| + \|X_0\|_{\mathcal{H}}) \\ & + H_1 e^{-\delta_\epsilon t} \sup_{\tau \in [0, D+t_0]} \|x(\tau)\| \\ & + H_2 e^{-\delta_\epsilon t} \sup_{\tau \in [0, D+t_0]} \|X(\tau)\|_{\mathcal{H}} \\ & + H_3 \sup_{\tau \in [0,t]} e^{-\delta_\epsilon (t-\tau)} \|v(\tau)\| \end{aligned} \tag{30}$$

for all $t \geq D + t_0$.

The proof of Theorem 5.4 is placed in Annex F. This consists in an adaptation of the approach presented in [11] for the study of the stability of IDS-ODE or PDE-PDE interconnections via a small gain approach.

Remark 5.5: As the system is in open loop over the time interval $[0, D]$ and then the time interval $[D, D + t_0]$ is employed to switch from open loop to closed loop, we can interpret $x|_{[0, D+t_0]}$ and $X|_{[0, D+t_0]}$ as initial perturbations. In this case, (29) can be seen as an ISS estimate with fading memory with respect to the initial perturbations $x|_{[0, D+t_0]}$ and $X|_{[0, D+t_0]}$ and the disturbance v.

Remark 5.6: In the context of Remark 5.3, the reasoning reported in Annex F still applies over the time interval $[0, t_{\max})$ because $t_{\max} > D + t_0$. In this case, estimate (29) holds for all $t \in [D + t_0, t_{\max})$. As the supremum of the right-hand side of (29) over any time interval $[D + t_0, T]$ of finite length is finite, we deduce that $t_{\max} = +\infty$. Therefore, the conclusions of Theorem 5.4 still apply.

VI. CASE STUDY

In this section, $\mathcal{H}$ denotes the real Hilbert space of square-integrable functions $L^2(0, L)$ endowed with the inner product $\langle f, g \rangle_{\mathcal{H}} = \int_0^L fg \, \mathrm{d}\xi$. We consider the following academic system composed of a one-dimensional ODE and a one-dimensional reaction-diffusion equation on $(0, L)$ with delayed Dirichlet boundary controls located at both ends of the domain

$$\begin{cases} \dot{x}(t) = f_1(x(t), y(t, \cdot), v(t)) \\ y_t(t, \xi) = a y_{\xi\xi}(t, \xi) + c y(t, \xi) + f_2(x(t), y(t, \cdot), v(t)) \\ \begin{bmatrix} y(t, 0) \\ y(t, 1) \end{bmatrix} = u(t - D) \end{cases}$$

where $(t, \xi) \in \mathbb{R}_+ \times (0, L)$, $X(t) = y(t, \cdot) \in \mathcal{H}$, $x(t), v(t) \in \mathbb{R}$, and $u(t) \in \mathbb{R}^2$. The considered coupling functions are given by $f_1(x, X, v) = -a_1 x + \dfrac{b_1}{L} \int_0^L \eta_1 X \, \mathrm{d}\xi + c_1 v$ and $f_2(x, X, v) = a_2 x \theta_1 + b_2 \arctan\left(\dfrac{d_2}{L} \int_0^L \eta_2 X \, \mathrm{d}\xi\right) \theta_2 + c_2 v \theta_3$ with $a, c, a_i, b_i, c_i, d_i \in \mathbb{R}$, $a, a_1 > 0$, and $\eta_i, \theta_i \in \mathcal{H}$ such that $\|\eta_i\|_{\mathcal{H}} = \|\theta_i\|_{\mathcal{H}} = 1$.

We define the operator $\mathcal{A}f = af'' + cf$ over the domain $D(\mathcal{A}) = H^2(0, L)$ and the boundary operator $\mathcal{B}f = (f(0), f(L))$ over the domain $D(\mathcal{B}) = H^1(0, L)$. We introduce the lifting operator B defined for any $(u_1, u_2) \in \mathbb{R}^2$ by $\{B(u_1, u_2)\}(\xi) = u_1 + (u_2 - u_1)\xi / L$ with $\xi \in (0, L)$. We have that the disturbance-free operator $\mathcal{A}_0$: 1) generates a C_0-semigroup ; 2) is a Riesz-spectral operator with $\lambda_n = c - an^2\pi^2/L^2$ and $\phi_n(\xi) = \psi_n(\xi) = \sqrt{2/L} \sin(n\pi\xi/L)$,

$n \geq 1$. Thus, $(\mathcal{A}, \mathcal{B})$ is a boundary control system satisfying Assumptions 2.1 and 2.2. Furthermore, straightforward computations show that $b_{n,1} = an\pi\sqrt{2/L^3}$ and $b_{n,2} = (-1)^{n+1}an\pi\sqrt{2/L^3}$. Thus, based on Remark 3.3, Assumption 3.2 about the Kalman condition is satisfied.

Finally, with the considered coupling functions f_1 and f_2, Assumption 5.1 holds with $\tilde{C}_0 = \sqrt{2}$, $\tilde{C}_1 = 2|b_1|/(a_1 L)$, $\tilde{C}_2 = 2|c_1|/a_1$, $D_1 = |a_2|$, $D_2 = |b_2 d_2|/L$, and $D_3 = |c_2|$.

For numerical computations, we take $L = 2\pi$, $D = 0.1\,\mathrm{s}$, $a = 5$ and $c = 2.5$. Thus, we have one unstable mode with $\lambda_1 = 1.25$ while $\lambda_2 = -2.5$ and $\lambda_3 = -8.75$. For design purposes, we consider a second order truncated model, i.e., $N_0 = 2$ and $\alpha = 8.75$. Then, the feedback gain matrix $K \in \mathbb{R}^{2\times 2}$ is computed based on this truncated model such that the two poles are both placed at -3. Following the developments of Section IV, the degrees of freedom available in the choice of the parameters $\beta \in (0,1)$, $\gamma_1 > C_1/\lambda_m(P)$, and $\gamma_2 > \max\left(\|BK\|^2/(m_R\lambda_m(P)), C_5/(1-\beta)\right)$ are used to minimize the value of the constant $C_4\sqrt{C_6/(2\kappa_0)}$ involved in the small gain condition (28). With the MATLAB function `fminsearch`, we obtain with $\beta = 0.4131$, $\gamma_1 = 106.3290$, and $\gamma_2 = 337.1938$ the value $C_4\sqrt{C_6/(2\kappa_0)} \approx 8.6260$. Thus, Theorem 5.4 applies when the vector fields f_1 and f_2 are such that $2|b_1 a_2|/a_1 + |b_2 d_2| < L/8.6260 \approx 0.7284$.

Consequently, we select for numerical simulations $a_1 = 1.5$, $b_1 = 0.5$, $c_1 = 0.2$, $a_2 = 0.7$, $b_2 = 0.55$, $c_2 = 10$, $d_2 = 0.45$, $\eta_1 = \eta_2 = \theta_2 = \sqrt{6\xi(L-\xi)}/L^{3/2}$, $\theta_1 = \sqrt{2\xi}/L$, and $\theta_3 = \sqrt{2(L-\xi)}/L$. The transition time t_0 is set to $t_0 = 0.2\,\mathrm{s}$ while the switching function $\varphi|_{[0,t_0]}$ is selected as the restriction over $[0, t_0]$ of the unique quintic polynomial function f satisfying $f(0) = f'(0) = f''(0) = f'(t_0) = f''(t_0) = 0$ and $f(t_0) = 1$. The adopted numerical scheme consists in the discretization of the reaction-diffusion equation using its first 10 modes. The evolution of the closed-loop system is depicted in Figs. 1-3 for the initial condition $x_0 = -2$ and $X_0(\xi) = -5\xi(L/2-\xi)(L-\xi)$, and with the external disturbance $v(t) = \sin(2t)\sin(5t)$. The obtained numerical results are compliant with the theoretical predictions.

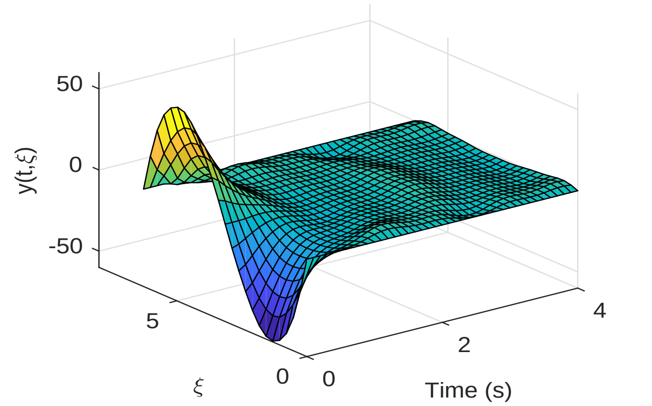

Fig. 1. Time evolution of the reaction-diffusion part of the closed-loop system

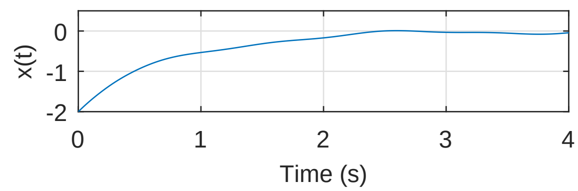

Fig. 2. Time evolution of the ODE part of the closed-loop system

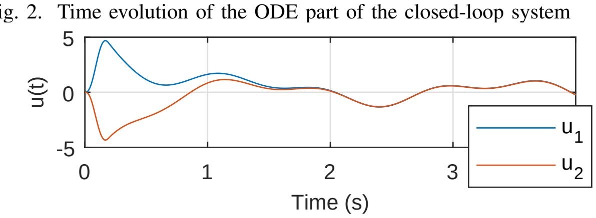

Fig. 3. Command effort of the closed-loop system

VII. Conclusion

This paper discussed the feedback stabilization of a class of diagonal Infinite-Dimensional Systems (IDS) with delay boundary control. The proposed approach generalizes a design method formerly reported for a reaction-diffusion equation while proposing a simplification of the boundary control law. The method consists, via a spectral decomposition, in the synthesis of a state-feedback for a finite-dimensional subsystem capturing the unstable dynamics of the plant. Due to the input delay, the design of the control law on the truncated subsystem has been carried out by means of the Artstein transformation. Then, an adequate Lyapunov function has been introduced to assess that the control law designed on the truncated subsystem also ensures the stabilization of the original IDS. Furthermore, it has been shown that this Lyapunov function also allows the assessment of the Input-to-State Stability (ISS) of the closed-loop system with respect to distributed disturbances. Finally, this ISS property has been used to study the stability of the closed-loop IDS when interconnected with an Ordinary Differential Equation (ODE) that also satisfies an ISS property. It has been shown that the satisfaction of a certain small gain condition ensures the stability of the IDS-ODE loop for the proposed delayed boundary control law. Future research directions include the evaluation of the robustness of the control strategy w.r.t modeling uncertainties such as mismatches in the computation of the vectors of the Riesz-basis.

Appendix A
Proof of Lemma 4.2

We first note that, as $u|_{[-D,0]} = 0$, the two first lines of (10), along with the initial condition, are equivalent over the time interval $[0, D]$ to the following evolution problem:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}_0 X(t) + d(t), & t \in [0, D] \\ X(0) = X_0 \end{cases}$$

As $\mathcal{A}_0$ generates a C_0-semigroup and d is of class $\mathcal{C}^1$, we deduce (see, e.g., [7]) the existence and the uniqueness of a classical solution $X \in \mathcal{C}^0([0,D]; D(\mathcal{A})) \cap \mathcal{C}^1([0,D];\mathcal{H})$ such that (10) holds over the time interval $[0, D]$ with associated control input $u = 0 \in \mathcal{C}^2([-D,0];\mathbb{K}^m)$.

We now proceed by induction. Assume that, for a given $n \in \mathbb{N}^*$, there exists a unique classical solution $X \in$

$\mathcal{C}^0([0,nD];D(\mathcal{A}))\cap\mathcal{C}^1([0,nD];\mathcal{H})$ of (10) over the time interval $[0,nD]$ with associated control input $u\in\mathcal{C}^2([-D,(n-1)D];\mathbb{K}^m)$ satisfying $u|_{[-D,0]}=0$ and

$$u(t)=\varphi(t)KY(t)+(T_D u)(t) \tag{31}$$

for all $0\leq t\leq (n-1)D$. We show that there exists a unique classical solution of (10) over the time interval $[0,(n+1)D]$, denoted by $\tilde{X}\in\mathcal{C}^0([0,(n+1)D];D(\mathcal{A}))\cap\mathcal{C}^1([0,(n+1)D];\mathcal{H})$, with a uniquely defined associated control input $\tilde{u}\in\mathcal{C}^2([-D,nD];\mathbb{K}^m)$. In particular, such a solution must satisfy (10) over the restricted time interval $[0,nD]$. Thus, by induction hypothesis, we must have $\tilde{X}\Big|_{[0,nD]}=X$. Furthermore, $\tilde{X}$ must satisfy

$$\begin{cases}\dfrac{\mathrm{d}\tilde{X}}{\mathrm{d}t}(t)=\mathcal{A}\tilde{X}(t)+d(t), & t\in[nD,(n+1)D]\\ \mathcal{B}\tilde{X}(t)=\tilde{u}_D(t)=\tilde{u}(t-D), & t\in[nD,(n+1)D]\\ \tilde{u}|_{[-D,0]}=0 & \\ \quad\tilde{u}(t)=\varphi(t)KY(t)+(T_D\tilde{u})(t), & t\in[0,nD]\\ \quad Y(t)=\begin{bmatrix}\langle X(t),\psi_1\rangle_{\mathcal{H}}\\ \vdots\\ \langle X(t),\psi_{N_0}\rangle_{\mathcal{H}}\end{bmatrix}, & t\in[0,nD]\\ \tilde{X}(nD)=X(nD) & \end{cases} \tag{32}$$

Note that, due to the delay $D>0$, the control input $\tilde{u}$ is only defined by X over the time interval $[0,nD]$ and does not depend on $\tilde{X}$ over $[nD,(n+1)D]$. As $X\in\mathcal{C}^1([0,nD];\mathcal{H})$, we have that $Y\in\mathcal{C}^1([0,nD];\mathbb{K}^{N_0})$. Then, according to the Lemma 3.6, 1) the control $\tilde{u}$ is well and uniquely defined on $[-D,nD]$; 2) $\tilde{u}$ is continuous over $[-D,nD]$; 3) as both u and $\tilde{u}|_{[-D,(n-1)D]}$ satisfy (31) for all $t\in[0,(n-1)D]$, we have by uniqueness that $\tilde{u}|_{[-D,(n-1)D]}=u$. Now, we can write $\tilde{u}(t)=\varphi(t)KZ(t)$ with, for all $t\in[0,nD]$,

$$Z(t)=Y(t)+\int_{t-D}^{t}e^{(t-s-D)A_{N_0}}B_{N_0}\tilde{u}(s)\,\mathrm{d}s.$$

Thus, we infer that $Z\in\mathcal{C}^1([0,nD];\mathbb{K}^{N_0})$. As X is a classical solution of (10) over the time interval $[0,nD]$, we obtain with the same approach used to derive (5) that Y satisfies over the time interval $[0,nD]$ the following ODE:

$$\dot{Y}(t)=A_{N_0}Y(t)+B_{N_0}\tilde{u}(t-D)+D_{N_0}(t),$$

where $D_{N_0}(t)$ is defined by (7). We have for all $t\in[0,nD]$,

$$\begin{aligned}\dot{Z}(t)&=A_{N_0}Z(t)+e^{-DA_{N_0}}B_{N_0}\tilde{u}(t)+D_{N_0}(t)\\ &=(A_{N_0}+\varphi(t)e^{-DA_{N_0}}B_{N_0}K)Z(t)+D_{N_0}(t).\end{aligned}$$

As $d\in\mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$, we have $D_{N_0}\in\mathcal{C}^1(\mathbb{R}_+;\mathbb{K}^{N_0})$. We deduce that Z is of class $\mathcal{C}^2$ over $[0,nD]$. Thus, the control law satisfies $\tilde{u}=\varphi KZ\in\mathcal{C}^2([-D,nD];\mathbb{K}^m)$, showing that $\tilde{u}_D\in\mathcal{C}^2([0,(n+1)D];\mathbb{K}^m)$. Furthermore, the distributed disturbance is such that $d\in\mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ while the initial condition of (32) given at $t=nD$ is such that $X(nD)\in D(\mathcal{A})$ and $\mathcal{B}X(nD)=u_D(nD)=\tilde{u}_D(nD)$. This yields (see, e.g., [7, Th. 3.3.3]) the existence and uniqueness of a classical solution $\tilde{X}\Big|_{[nD,(n+1)D]}\in\mathcal{C}^0([nD,(n+1)D];D(\mathcal{A}))\cap C^1([nD,(n+1)D];\mathcal{H})$ associated with (32). As $\tilde{X}(nD)=X(nD)$ and $(\mathrm{d}\tilde{X}/\mathrm{d}t)(nD)=\mathcal{A}\tilde{X}(nD)+d(nD)=\mathcal{A}X(nD)+d(nD)=(\mathrm{d}X/\mathrm{d}t)(nD)$, it shows that the obtained $\tilde{X}$ is such that $\tilde{X}\in\mathcal{C}^0([0,(n+1)D];D(\mathcal{A}))\cap\mathcal{C}^1([0,(n+1)D];\mathcal{H})$ and is the unique classical solution of (10) over $[0,(n+1)D]$ with associated control input $\tilde{u}\in\mathcal{C}^2([-D,nD];\mathbb{K}^m)$. Furthermore, the obtained $\tilde{X}$ and $\tilde{u}$ are extensions of X and u, respectively.

By induction, it shows the existence and uniqueness of both the classical solution $X\in\mathcal{C}^0(\mathbb{R}_+;D(\mathcal{A}))\cap\mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ and the associated control input $u\in\mathcal{C}^2([-D,+\infty);\mathbb{K}^m)$ for the closed-loop system (10) associated with $X_0\in D(\mathcal{A}_0)$ and $d\in\mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$. The claimed properties for u follow from the above developments and the application of Lemma 3.6. □

Appendix B
Regularity and time derivative of an infinite sum

Let $\{e_n,\,n\in\mathbb{N}^*\}$ be a Hilbert basis of $\mathcal{H}$. Then, as $\{\phi_n,\,n\in\mathbb{N}^*\}$ is a Riesz basis with associated biorthogonal set $\{\psi_n,\,n\in\mathbb{N}^*\}$, there exists $T\in\mathcal{L}(\mathcal{H})$ such that $T^{-1}\in\mathcal{L}(\mathcal{H})$ and, for all $n\geq 1$, $\phi_n=Te_n$ and $\psi_n=(T^{-1})^*e_n$. Let $A\in\mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ be given. We obtain that, for all $t\geq 0$, $\sum_{k\geq 1}|\langle A(t),\psi_k\rangle|^2=\sum_{k\geq 1}|\langle A(t),(T^{-1})^*e_k\rangle|^2=\sum_{k\geq 1}|\langle T^{-1}A(t),e_k\rangle|^2=\|T^{-1}A(t)\|_{\mathcal{H}}^2=\langle T^{-1}A(t),T^{-1}A(t)\rangle_{\mathcal{H}}$. Thus $\sum_{k\geq 1}|\langle A,\psi_k\rangle|^2\in\mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ and we have for all $t\geq 0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\sum_{k\geq 1}|\langle A(t),\psi_k\rangle_{\mathcal{H}}|^2\right]\\ &=\operatorname{Re}\left\langle T^{-1}\sum_{k\geq 1}\left\langle\frac{\mathrm{d}A}{\mathrm{d}t}(t),\psi_k\right\rangle_{\mathcal{H}}\phi_k, T^{-1}\sum_{l\geq 1}\langle A(t),\psi_l\rangle_{\mathcal{H}}\phi_l\right\rangle_{\mathcal{H}}\\ &=\sum_{k,l\geq 1}\operatorname{Re}\left\{\left\langle\frac{\mathrm{d}A}{\mathrm{d}t}(t),\psi_k\right\rangle_{\mathcal{H}}\overline{\langle A(t),\psi_l\rangle_{\mathcal{H}}}\langle e_k,e_l\rangle_{\mathcal{H}}\right\}\\ &=\sum_{k\geq 1}\operatorname{Re}\left\{\left\langle\frac{\mathrm{d}A}{\mathrm{d}t}(t),\psi_k\right\rangle_{\mathcal{H}}\overline{\langle A(t),\psi_k\rangle_{\mathcal{H}}}\right\}.\end{aligned}$$

Noting that, for all $k\geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|\langle A(t),\psi_k\rangle_{\mathcal{H}}|^2\right]=\operatorname{Re}\left\{\left\langle\frac{\mathrm{d}A}{\mathrm{d}t}(t),\psi_k\right\rangle_{\mathcal{H}}\overline{\langle A(t),\psi_k\rangle_{\mathcal{H}}}\right\},$$

we deduce that $\sum_{k\geq N_0+1}|\langle A,\psi_k\rangle|^2\in\mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$.

Appendix C
Proof of Lemma 4.7

From (11) and using the identity $u=\varphi KZ$, we have $Y(t)=Z(t)-\int_{t-D}^{t}\varphi(s)e^{(t-s-D)A_{N_0}}B_{N_0}KZ(s)\,\mathrm{d}s$ for all $t\geq 0$. Using the Cauchy-Schwarz (C.S.) inequality and the fact that $0\leq\varphi\leq 1$, we deduce that, for all $t\geq 0$,

$$\|Y(t)\|^2\leq C_1\left\{\|Z(t)\|^2+\int_{t-D}^{t}\varphi(s)\,\|Z(s)\|^2\,\mathrm{d}s\right\}, \tag{33}$$

where C_1 is defined by (18). Now, from the definition of V given by (16) and using (2), we have for all $t \geq 0$,

$$\begin{aligned} V(t) \geq \gamma_1 \lambda_m(P) & \left\{ \|Z(t)\|^2 + \int_{t-D}^{t} \varphi(s) \|Z(s)\|^2 \, \mathrm{d}s \right\} \\ & + \gamma_2 \lambda_m(P) \varphi(t-D) \|Z(t-D)\|^2 \\ & + \frac{1}{2M_R} \|X(t) - Bu_D(t)\|_{\mathcal{H}}^2 \\ & - \frac{1}{2} \sum_{k=1}^{N_0} |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 . \end{aligned}$$

Recalling that $u_D(t) = u(t-D) = \varphi(t-D)KZ(t-D)$ and $0 \leq \varphi \leq 1$ which gives $\varphi^2 \leq \varphi$, we have

$$\begin{aligned} & \sum_{k=1}^{N_0} |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 \\ & \qquad \leq 2 \sum_{k=1}^{N_0} \left\{ |\langle X(t), \psi_k \rangle_{\mathcal{H}}|^2 + |\langle Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 \right\} \\ & \qquad \overset{(2)}{\leq} 2\|Y(t)\|^2 + \frac{2}{m_R} \|Bu_D(t)\|_{\mathcal{H}}^2 \\ & \qquad \leq 2\|Y(t)\|^2 + \frac{2\|BK\|^2}{m_R} \varphi(t-D) \|Z(t-D)\|^2 . \end{aligned}$$

We deduce that

$$\begin{aligned} V(t) \geq \gamma_1 \lambda_m(P) & \left\{ \|Z(t)\|^2 + \int_{t-D}^{t} \varphi(s) \|Z(s)\|^2 \, \mathrm{d}s \right\} \\ & + \left\{ \gamma_2 \lambda_m(P) - \frac{\|BK\|^2}{m_R} \right\} \varphi(t-D) \|Z(t-D)\|^2 \\ & + \frac{1}{2M_R} \|X(t) - Bu_D(t)\|_{\mathcal{H}}^2 - \|Y(t)\|^2 . \end{aligned}$$

Using (33), this yields for all $t \geq 0$,

$$\begin{aligned} V(t) \geq \{\gamma_1 \lambda_m(P) - C_1\} & \left\{ \|Z(t)\|^2 + \int_{t-D}^{t} \varphi(s) \|Z(s)\|^2 \, \mathrm{d}s \right\} \\ & + \left\{ \gamma_2 \lambda_m(P) - \frac{\|BK\|^2}{m_R} \right\} \varphi(t-D) \|Z(t-D)\|^2 \\ & + \frac{1}{2M_R} \|X(t) - Bu_D(t)\|_{\mathcal{H}}^2 . \end{aligned}$$

As $\gamma_1, \gamma_2 \in \mathbb{R}_+^*$ are such that $\gamma_1 > C_1/\lambda_m(P)$ and $\gamma_2 > \|BK\|^2/(m_R \lambda_m(P))$, we have $C_2(\gamma_1) \triangleq \gamma_1 \lambda_m(P) - C_1 > 0$ and $C_3(\gamma_2) \triangleq \gamma_2 \lambda_m(P) - \dfrac{\|BK\|^2}{m_R} > 0$ are such that

$$\begin{aligned} V(t) \geq C_2(\gamma_1) & \left\{ \|Z(t)\|^2 + \int_{t-D}^{t} \varphi(s) \|Z(s)\|^2 \, \mathrm{d}s \right\} \\ & + C_3(\gamma_2) \varphi(t-D) \|Z(t-D)\|^2 \\ & + \frac{1}{2M_R} \|X(t) - Bu_D(t)\|_{\mathcal{H}}^2 . \end{aligned} \tag{34}$$

In particular, this yields for all $t \geq 0$,

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} & \leq \|X(t) - Bu_D(t)\|_{\mathcal{H}} + \|Bu_D(t)\|_{\mathcal{H}} \\ & \leq \sqrt{2M_R V(t)} + \|BK\| \times \sqrt{\varphi(t-D)} \|Z(t-D)\| \\ & \leq \left\{ \sqrt{2M_R} + \frac{\|BK\|}{\sqrt{C_3(\gamma_2)}} \right\} \sqrt{V(t)} . \end{aligned}$$

Introducing $C_4 \triangleq \sqrt{2M_R} + \dfrac{\|BK\|}{\sqrt{C_3(\gamma_2)}} > 0$, the claimed inequality (19) holds. □

APPENDIX D
PROOF OF LEMMA 4.8

From the definition of P, we have that for all $t > t_0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} [Z^* P Z](t) & \overset{(13)}{=} Z(t)^* [A_{\mathrm{cl}}^* P + P A_{\mathrm{cl}}] Z(t) \\ & \quad + D_{N_0}(t)^* P Z(t) + Z(t)^* P D_{N_0}(t) \\ & \overset{(17)}{=} -\|Z(t)\|^2 + D_{N_0}(t)^* P Z(t) + Z(t)^* P D_{N_0}(t) . \end{aligned}$$

Thus, for all $t > D + t_0$,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{t-D}^{t} Z(s)^* P Z(s) \, \mathrm{d}s \right] (t) \\ & \quad = Z(t)^* P Z(t) - Z(t-D)^* P Z(t-D) \\ & \quad = - \int_{t-D}^{t} \|Z(s)\|^2 \, \mathrm{d}s \\ & \qquad + \int_{t-D}^{t} D_{N_0}(s)^* P Z(s) + Z(s)^* P D_{N_0}(s) \, \mathrm{d}s . \end{aligned}$$

Let $\beta \in (0,1)$ be arbitrarily given. We infer from the Young inequality (Y.I.) that, for all $t > t_0$,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} [Z^* P Z](t) \\ & \quad \leq -\|Z(t)\|^2 + 2\|P\| \|D_{N_0}(t)\| \|Z(t)\| \\ & \quad \overset{\text{Y.I.}}{\leq} -\|Z(t)\|^2 + 2 \left(\frac{\beta}{2} \|Z(t)\|^2 + \frac{1}{2\beta} \|P\|^2 \|D_{N_0}(t)\|^2 \right) \\ & \quad \overset{(8)}{\leq} -(1-\beta) \|Z(t)\|^2 + \frac{\|P\|^2}{\beta m_R} \|d(t)\|_{\mathcal{H}}^2 , \end{aligned}$$

and for all $t > D + t_0$,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{t-D}^{t} Z(s)^* P Z(s) \, \mathrm{d}s \right] (t) \\ & \leq - \int_{t-D}^{t} \|Z(s)\|^2 \, \mathrm{d}s + 2 \int_{t-D}^{t} \|P\| \|D_{N_0}(s)\| \|Z(s)\| \, \mathrm{d}s \\ & \overset{\text{Y.I.}}{\leq} - \int_{t-D}^{t} \|Z(s)\|^2 \, \mathrm{d}s \\ & \quad + 2 \int_{t-D}^{t} \frac{\beta}{2} \|Z(s)\|^2 + \frac{1}{2\beta} \|P\|^2 \|D_{N_0}(s)\|^2 \, \mathrm{d}s \\ & \leq -(1-\beta) \int_{t-D}^{t} \|Z(s)\|^2 \, \mathrm{d}s + \frac{D\|P\|^2}{\beta m_R} \sup_{\tau \in [t-D, t]} \|d(\tau)\|_{\mathcal{H}}^2 . \end{aligned}$$

Finally, we have (see Annex B)

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{2} \sum_{k \geq N_0+1} |\langle X - Bu_D, \psi_k \rangle_{\mathcal{H}}|^2 \right] (t) \\ & \quad = \sum_{k \geq N_0+1} \operatorname{Re} \left\{ \left\langle \frac{\mathrm{d}X}{\mathrm{d}t}(t) - B\dot{u}_D(t), \psi_k \right\rangle_{\mathcal{H}} \right. \\ & \qquad \qquad \left. \times \overline{\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}} \right\} . \end{aligned}$$

As X is a classical solution of the abstract Cauchy problem, using (4), Assumption 2.2, and the Young inequality, we have for $k \geq N_0 + 1$ that

$$\begin{aligned}
&\mathrm{Re}\left\{ \left\langle \frac{\mathrm{d}X}{\mathrm{d}t}(t) - B\dot{u}_D(t), \psi_k \right\rangle_{\mathcal{H}} \overline{\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}} \right\} \\
&\overset{(4)}{=} \mathrm{Re}(\lambda_k) \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \qquad (35) \\
&\quad + \mathrm{Re}\left\{ (\langle \mathcal{A}Bu_D(t), \psi_k \rangle_{\mathcal{H}} + d_k(t) - \langle B\dot{u}_D(t), \psi_k \rangle_{\mathcal{H}}) \right. \\
&\qquad\qquad \left. \times \overline{\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}} \right\} \\
&\leq -\alpha \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \{ |\langle \mathcal{A}Bu_D(t), \psi_k \rangle_{\mathcal{H}}| + |d_k(t)| + |\langle B\dot{u}_D(t), \psi_k \rangle_{\mathcal{H}}| \} \\
&\qquad \times |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}| \\
&\overset{\text{Y.I.}}{\leq} -\frac{\alpha}{2} \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \frac{1}{2\alpha} \{ |\langle \mathcal{A}Bu_D(t), \psi_k \rangle_{\mathcal{H}}| + |d_k(t)| + |\langle B\dot{u}_D(t), \psi_k \rangle_{\mathcal{H}}| \}^2 .
\end{aligned}$$

Introducing K_i the i-th line of the matrix of feedback gain K, one has, for all $t > D + t_0$,

$$u_D(t) = u(t-D) = KZ(t-D) = \sum_{i=1}^{m} \{ K_i Z(t-D) \} e_i$$

and

$$\begin{aligned}
\dot{u}_D(t) &\overset{(13)}{=} K(A_{\mathrm{cl}} Z(t-D) + D_{N_0}(t-D)) \\
&= \sum_{i=1}^{m} \{ K_i A_{\mathrm{cl}} Z(t-D) \} e_i + K D_{N_0}(t-D).
\end{aligned}$$

This yields

$$\begin{aligned}
&\mathrm{Re}\left\{ \left\langle \frac{\mathrm{d}X}{\mathrm{d}t}(t) - B\dot{u}_D(t), \psi_k \right\rangle_{\mathcal{H}} \overline{\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}} \right\} \\
&\leq -\frac{\alpha}{2} \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \frac{1}{2\alpha} \{ |\langle \mathcal{A}BKZ(t-D), \psi_k \rangle_{\mathcal{H}}| + |\langle BKA_{\mathrm{cl}} Z(t-D), \psi_k \rangle_{\mathcal{H}}| \\
&\qquad\quad + |d_k(t)| + |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}| \}^2 \\
&\leq -\frac{\alpha}{2} \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \frac{2}{\alpha} \left\{ |\langle \mathcal{A}BKZ(t-D), \psi_k \rangle_{\mathcal{H}}|^2 + |\langle BKA_{\mathrm{cl}} Z(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \right. \\
&\qquad\quad \left. + |d_k(t)|^2 + |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \right\} \\
&\leq -\frac{\alpha}{2} \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \frac{2}{\alpha} \left| \sum_{i=1}^{m} \langle \mathcal{A}Be_i, \psi_k \rangle_{\mathcal{H}} K_i Z(t-D) \right|^2 \\
&\quad + \frac{2}{\alpha} \left| \sum_{i=1}^{m} \langle Be_i, \psi_k \rangle_{\mathcal{H}} K_i A_{\mathrm{cl}} Z(t-D) \right|^2 \\
&\quad + \frac{2}{\alpha} \left\{ |d_k(t)|^2 + |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \right\} \\
&\leq -\frac{\alpha}{2} \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \frac{2m}{\alpha} \sum_{i=1}^{m} |\langle \mathcal{A}Be_i, \psi_k \rangle_{\mathcal{H}} K_i Z(t-D)|^2 \\
&\quad + \frac{2m}{\alpha} \sum_{i=1}^{m} |\langle Be_i, \psi_k \rangle_{\mathcal{H}} K_i A_{\mathrm{cl}} Z(t-D)|^2 \\
&\quad + \frac{2}{\alpha} \left\{ |d_k(t)|^2 + |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \right\} \\
&\leq -\frac{\alpha}{2} \left| \langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}} \right|^2 \\
&\quad + \frac{2m}{\alpha} \left\{ \sum_{i=1}^{m} |\langle \mathcal{A}Be_i, \psi_k \rangle_{\mathcal{H}}|^2 \|K_i\|^2 \right. \\
&\qquad\qquad \left. + \sum_{i=1}^{m} |\langle Be_i, \psi_k \rangle_{\mathcal{H}}|^2 \|K_i A_{\mathrm{cl}}\|^2 \right\} \|Z(t-D)\|^2 \\
&\quad + \frac{2}{\alpha} \left\{ |d_k(t)|^2 + |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \right\} .
\end{aligned}$$

We deduce that, for $t > D + t_0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{2} \sum_{k \geq N_0+1} |\langle X - Bu_D, \psi_k \rangle_{\mathcal{H}}|^2 \right] (t) \\
&\leq -\frac{\alpha}{2} \sum_{k \geq N_0+1} |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 \\
&\quad + \frac{2m}{\alpha} \sum_{k \geq N_0+1} \left\{ \sum_{i=1}^{m} |\langle \mathcal{A}Be_i, \psi_k \rangle_{\mathcal{H}}|^2 \|K_i\|^2 \right. \\
&\qquad\qquad \left. + \sum_{i=1}^{m} |\langle Be_i, \psi_k \rangle_{\mathcal{H}}|^2 \|K_i A_{\mathrm{cl}}\|^2 \right\} \|Z(t-D)\|^2 \\
&\quad + \frac{2}{\alpha} \sum_{k \geq N_0+1} \left\{ |d_k(t)|^2 + |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \right\} \\
&\leq -\frac{\alpha}{2} \sum_{k \geq N_0+1} |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 \\
&\quad + \frac{2m}{\alpha} \left\{ \sum_{i=1}^{m} \sum_{k \geq 1} |\langle \mathcal{A}Be_i, \psi_k \rangle_{\mathcal{H}}|^2 \|K_i\|^2 \right. \\
&\qquad\qquad \left. + \sum_{i=1}^{m} \sum_{k \geq 1} |\langle Be_i, \psi_k \rangle_{\mathcal{H}}|^2 \|K_i A_{\mathrm{cl}}\|^2 \right\} \|Z(t-D)\|^2 \\
&\quad + \frac{2}{\alpha} \sum_{k \geq 1} |d_k(t)|^2 + \frac{2}{\alpha} \sum_{k \geq 1} |\langle BKD_{N_0}(t-D), \psi_k \rangle_{\mathcal{H}}|^2 \\
&\overset{(2)}{\leq} -\frac{\alpha}{2} \sum_{k \geq N_0+1} |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 + C_5 \|Z(t-D)\|^2 \\
&\quad + \frac{2}{\alpha m_R} \|d(t)\|_{\mathcal{H}}^2 + \frac{2\|BK\|^2}{\alpha m_R^2} \|d(t-D)\|_{\mathcal{H}}^2
\end{aligned}$$

with constant C_5 given by (20). As $\gamma_2 > C_5/(1-\beta)$, we deduce that, for all $t > D + t_0$,

$$\begin{aligned}
\dot{V}(t) \leq &-\gamma_1(1-\beta) \left\{ \|Z(t)\|^2 + \int_{t-D}^{t} \|Z(s)\|^2 \,\mathrm{d}s \right\} \\
&- (\gamma_2(1-\beta) - C_5) \|Z(t-D)\|^2 \\
&- \frac{\alpha}{2} \sum_{k \geq N_0+1} |\langle X(t) - Bu_D(t), \psi_k \rangle_{\mathcal{H}}|^2 \\
&+ \frac{1}{m_R} \left(\frac{2}{\alpha} + \frac{\gamma_1 \|P\|^2}{\beta} \right) \|d(t)\|_{\mathcal{H}}^2 \\
&+ \frac{1}{m_R} \left(\frac{2\|BK\|^2}{\alpha m_R} + \frac{\gamma_2 \|P\|^2}{\beta} \right) \|d(t-D)\|_{\mathcal{H}}^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma_1 D\|P\|^2}{\beta m_R}\sup_{\tau\in[t-D,t]}\|d(\tau)\|_{\mathcal{H}}^2\\
\leq &-\frac{\gamma_1(1-\beta)}{\lambda_M(P)}\left\{Z(t)^*PZ(t)+\int_{t-D}^{t}Z(s)^*PZ(s)\,\mathrm{d}s\right\}\\
&-\frac{\gamma_2(1-\beta)-C_5}{\lambda_M(P)}Z(t-D)^*PZ(t-D)\\
&-\frac{\alpha}{2}\sum_{k\geq N_0+1}|\langle X(t)-Bu_D(t),\psi_k\rangle_{\mathcal{H}}|^2\\
&+\frac{1}{m_R}\left(\frac{2(m_R+\|BK\|^2)}{\alpha m_R}+\frac{(\gamma_1(1+D)+\gamma_2)\|P\|^2}{\beta}\right)\\
&\times\sup_{\tau\in[t-D,t]}\|d(\tau)\|_{\mathcal{H}}^2\\
\leq &-2\kappa_0 V(t)+C_6\sup_{\tau\in[t-D,t]}\|d(\tau)\|_{\mathcal{H}}^2,
\end{aligned}
$$

where $\lambda_M(P)>0$ stands for the largest eigenvalue of P,

$$\kappa_0 \triangleq \frac{1}{2}\min\left(\frac{1-\beta}{\lambda_M(P)},\frac{1-\beta-C_5/\gamma_2}{\lambda_M(P)},\alpha\right)>0, \tag{36}$$

and

$$C_6 \triangleq \frac{2(m_R+\|BK\|^2)}{\alpha m_R^2}+\frac{(\gamma_1(1+D)+\gamma_2)\|P\|^2}{\beta m_R}. \tag{37}$$

Then, for all $t>D+t_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2\kappa_0(\cdot)}V\right](t)\leq C_6 e^{2\kappa_0 t}\sup_{\tau\in[t-D,t]}\|d(\tau)\|_{\mathcal{H}}^2. \tag{38}$$

As $V\in\mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$, we infer that, for all $t\geq D+t_0$,

$$V(t)\leq e^{-2\kappa_0(t-D-t_0)}V(D+t_0)+\frac{C_6}{2\kappa_0}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}^2, \tag{39}$$

and thus, from (19) and using the inequality $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ for all $a,b\geq 0$, we obtain that the claimed estimate (21) holds for all $t\geq D+t_0$. Finally, from (34), the control input is such that, for all $t\geq 0$,

$$\|u(t)\|\leq\|K\|\|Z(t)\|\leq\frac{\|K\|}{\sqrt{C_2(\gamma_1)}}\sqrt{V(t)}, \tag{40}$$

from which we can deduce that the estimate (22) is also satisfied for all $t\geq D+t_0$. □

Appendix E
Proof of Lemma 4.10

With $W(t)=\frac{1}{2}\|Z(t)\|^2$, the use of Cauchy-Schwarz inequality, Young's inequality, (8), and (12) yields

$$\dot{W}(t)\leq 2C_7W(t)+\frac{1}{2m_R}\|d(t)\|_{\mathcal{H}}^2$$

for all $t\geq 0$ with $C_7\triangleq\|A_{N_0}\|+\|e^{-DA_{N_0}}B_{N_0}K\|+1/2>0$. Then, for all $t\geq 0$,

$$W(t)\leq e^{2C_7t}W(0)+\frac{1}{4m_RC_7}e^{2C_7t}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}^2.$$

Using (2), and, from (11), $Z(0)=Y(0)$, we have $\|Z(0)\|=\|Y(0)\|\leq\|X_0\|_{\mathcal{H}}/\sqrt{m_R}$. We deduce that, for all $t\geq 0$,

$$\|Z(t)\|^2\leq\frac{e^{2C_7t}}{m_R}\|X_0\|^2+\frac{1}{2m_RC_7}e^{2C_7t}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}^2. \tag{41}$$

From $u_D(t)=u(t-D)=\varphi(t-D)KZ(t-D)$, we infer that, for all $t\in[0,D+t_0]$,

$$\|u_D(t)\|\leq\frac{\|K\|e^{C_7t_0}}{\sqrt{m_R}}\|X_0\|_{\mathcal{H}}+\frac{\|K\|e^{C_7t_0}}{\sqrt{2m_RC_7}}\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}} \tag{42}$$

and, based on (12), we obtain that, for all $t\in[0,D+t_0]$,

$$
\begin{aligned}
\|\dot{u}_D(t)\|\leq&\frac{C_8e^{C_7t_0}}{\sqrt{m_R}}\|X_0\|_{\mathcal{H}}\\
&+\frac{1}{\sqrt{m_R}}\left(\|K\|+\frac{C_8}{\sqrt{2C_7}}e^{C_7t_0}\right)\sup_{\tau\in[0,t]}\|d(\tau)\|_{\mathcal{H}}
\end{aligned} \tag{43}
$$

with $C_8\triangleq\|\dot{\varphi}\|_\infty\|K\|+\|K\|\left(\|A_{N_0}\|+\|e^{-DA_{N_0}}B_{N_0}K\|\right)$.

To conclude, it is sufficient to note that from (16), we have for all $t\geq 0$,

$$
\begin{aligned}
V(t)\leq&\gamma_1\lambda_M(P)\left\{\|Z(t)\|^2+\int_{t-D}^{t}\varphi(s)\|Z(s)\|^2\,\mathrm{d}s\right\}\\
&+\gamma_2\lambda_M(P)\varphi(t-D)\|Z(t-D)\|^2\\
&+\frac{1}{m_R}\|X(t)\|_{\mathcal{H}}^2+\frac{\|B\|^2}{m_R}\|u_D(t)\|_{\mathcal{H}}^2,
\end{aligned}
$$

where, as X is a classical solution of (10) and noting that $u_D(0)=u(-D)=0$, we have

$$
\begin{aligned}
X(t)=&S(t)X_0+Bu_D(t)\\
&+\int_0^t S(t-\tau)\left\{-B\dot{u}_D(\tau)+\mathcal{A}Bu_D(\tau)+d(\tau)\right\}\mathrm{d}\tau.
\end{aligned}
$$

By direct estimation and using (41-43), we deduce that the conclusion of the lemma holds true. □

Appendix F
Proof of Theorem 5.4

In order to be able to apply the results of Section IV, V is still defined by (16) with γ_1,γ_2 large enough[2].

A. Conversion of the ISS estimates into fading memory estimates

Following the methodology presented in [11] for studying the stability of IDS-ODE or PDE-PDE interconnections, the key step relies in the conversion of the ISS estimates satisfied by each component of the interconnections into fading memory estimates via the following lemma [11, Lemma 7.1].

Lemma F.1 (Conversion Lemma): For every $\sigma>0$, $M\geq 1$, and $\epsilon>0$, there exists a constant $\delta\in(0,\sigma)$ such that for any continuous functions $\phi:\mathbb{R}_+\to\mathbb{R}_+$ and $y:\mathbb{R}_+\to\mathbb{R}_+$ for which there exists a constant $\gamma\geq 0$ such that the following inequality holds for all $t_0\geq 0$ and $t\geq t_0$,

$$\phi(t)\leq Me^{-\sigma(t-t_0)}\phi(t_0)+\gamma\sup_{s\in[t_0,t]}y(s), \tag{44}$$

then the following inequality holds for all $t\geq 0$:

$$\phi(t)\leq Me^{-\delta t}\phi(0)+\gamma(1+\epsilon)\sup_{s\in[0,t]}e^{-\delta(t-s)}y(s).$$

[2]More precisely, they are selected such that $\gamma_1>C_1/\lambda_m(P)$ and $\gamma_2>\max\left(\|BK\|^2/(m_R\lambda_m(P)),C_5/(1-\beta)\right)$.

Even if the trajectories X of (10) satisfy the ISS estimate (14) provided by Theorem 4.3, we cannot directly apply the Conversion Lemma because the semigroup property does not hold. This is due to the time-varying nature of (10) induced by the transition from open loop to closed loop via φ, yielding $u_D|_{[0,D)} = 0$. Therefore, we cannot directly deduce from the ISS estimate (14) that an estimate similar to (44) holds for all $t \geq t_0 \geq 0$. In order to avoid this pitfall, we are not going to apply the Conversion Lemma to the system trajectories X but to the Lyapunov function V. Indeed, with $d(t) = f_2(x(t), X(t), v(t))$, we know from Lemma 5.2 that X is solution of (10) associated with the initial condition X_0 and the distributed disturbance d. Consequently, we deduce from (38) that, for all $t_2 \geq t_1 \geq D + t_0$,

$$\begin{aligned} &e^{2\kappa_0 t_2} V(t_2) - e^{2\kappa_0 t_1} V(t_1) \\ &\leq C_6 \int_{t_1}^{t_2} e^{2\kappa_0 s} \sup_{\tau \in [s-D,s]} \|d(\tau)\|_{\mathcal{H}}^2 \,\mathrm{d}s \\ &\leq \frac{C_6}{2\kappa_0} e^{2\kappa_0 t_2} \sup_{s \in [t_1,t_2]} \sup_{\tau \in [s-D,s]} \|d(\tau)\|_{\mathcal{H}}^2 . \end{aligned}$$

This yields, for all $t_2 \geq t_1 \geq D + t_0$,

$$V(t_2) \leq e^{-2\kappa_0 (t_2 - t_1)} V(t_1) + \frac{C_6}{2\kappa_0} \sup_{s \in [t_1,t_2]} \sup_{\tau \in [s-D,s]} \|d(\tau)\|_{\mathcal{H}}^2 .$$

Introducing $\hat{\kappa}_0 = \min(\kappa_0, \tilde{\kappa}_0) > 0$ and noting that $\tilde{C}_0 \geq 1$, then we have for all $t_2 \geq t_1 \geq 0$,

$$\begin{aligned} V(t_2 + (D+t_0)) \leq \tilde{C}_0^2 e^{-2\hat{\kappa}_0 (t_2 - t_1)} V(t_1 + (D+t_0)) \quad (45) \\ + \frac{C_6}{2\kappa_0} \sup_{s \in [t_1,t_2]} \sup_{\tau \in [s+t_0, s+(D+t_0)]} \|d(\tau)\|_{\mathcal{H}}^2 . \end{aligned}$$

Furthermore, as the trajectories of the ODE $\dot{x} = f_1(x, X, v)$ satisfy the semigroup property, we also have from (25) that[3] for all $t_2 \geq t_1 \geq 0$,

$$\begin{aligned} \|x(t_2)\|^2 \leq \tilde{C}_0^2 e^{-2\hat{\kappa}_0 (t_2 - t_1)} \|x(t_1)\|^2 \quad (46) \\ + \sup_{\tau \in [t_1,t_2]} \left\{ \tilde{C}_1^2 \|X(\tau)\|_{\mathcal{H}}^2 + \tilde{C}_2^2 \|v(\tau)\|^2 \right\} . \end{aligned}$$

Remark F.2: The introduction of the constant $\tilde{C}_0^2 \geq 1$ in (45) is motivated by the will to apply the Conversion Lemma simultaneously to both (45-46). Even if this yields some conservatism is the estimate with respect to the value of V at the lower bound of the interval of integration, such an introduction will have no impact on the conservatism of the small gain condition (28).

We now apply the Conversion Lemma. For $\sigma = 2\hat{\kappa}_0$ and $M = \tilde{C}_0^2 \geq 1$, we denote by $2\delta_\epsilon \in (0, 2\hat{\kappa}_0)$ the constant "δ" provided by the Conversion Lemma (which is independent of x_0, X_0, and v) for any given $\epsilon > 0$. From the proof of the Conversion Lemma in [11, Lemma 7.1], we can select δ_ϵ such that $\delta_\epsilon \underset{\epsilon \to 0^+}{\longrightarrow} 0^+$.

Applying the Conversion Lemma to (45) with $\phi(t) = V(t + (D+t_0))$, $y(t) = \sup_{\tau \in [t+t_0, t+(D+t_0)]} \|d(\tau)\|_{\mathcal{H}}^2$, and $\gamma = C_6/(2\kappa_0)$, we infer that, for all $t \geq 0$,

$$\begin{aligned} &e^{2\delta_\epsilon t} V(t + (D+t_0)) \leq \tilde{C}_0^2 V(D + t_0) \\ &+ \frac{C_6}{2\kappa_0} (1+\epsilon) \sup_{s \in [0,t]} \left\{ e^{2\delta_\epsilon s} \sup_{\tau \in [s+t_0, s+(D+t_0)]} \|d(\tau)\|_{\mathcal{H}}^2 \right\} . \end{aligned}$$

Noting that $s + t_0 \leq \tau$ implies $s \leq \tau - t_0$ and thus $e^{2\delta_\epsilon s} \leq e^{2\delta_\epsilon \tau} e^{-2\delta_\epsilon t_0}$, we obtain for all $t \geq 0$,

$$\begin{aligned} &e^{2\delta_\epsilon t} V(t + (D+t_0)) \leq \tilde{C}_0^2 V(D + t_0) \\ &+ \frac{C_6}{2\kappa_0} (1+\epsilon) e^{-2\delta_\epsilon t_0} \sup_{\tau \in [t_0, t+(D+t_0)]} e^{2\delta_\epsilon \tau} \|d(\tau)\|_{\mathcal{H}}^2 . \quad (47) \end{aligned}$$

Using (19), we obtain that, for all $t \geq 0$,

$$\begin{aligned} &e^{\delta_\epsilon t} \|X(t + (D+t_0))\|_{\mathcal{H}} \leq C_4 \tilde{C}_0 \sqrt{V(D+t_0)} \\ &+ C_4 \sqrt{\frac{C_6}{2\kappa_0} (1+\epsilon)} e^{-\delta_\epsilon t_0} \sup_{\tau \in [t_0, t+(D+t_0)]} e^{\delta_\epsilon \tau} \|d(\tau)\|_{\mathcal{H}} . \quad (48) \end{aligned}$$

From the application of the Conversion Lemma to (46) with $\phi(t) = \|x(t)\|^2$, $y(t) = \tilde{C}_1^2 \|X(t)\|_{\mathcal{H}}^2 + \tilde{C}_2^2 \|v(t)\|^2$, and $\gamma = 1$, we infer that, for all $t \geq 0$,

$$\begin{aligned} &e^{2\delta_\epsilon t} \|x(t)\|^2 \leq \tilde{C}_0^2 \|x_0\|^2 \\ &\quad + (1+\epsilon) \sup_{\tau \in [0,t]} e^{2\delta_\epsilon \tau} \left\{ \tilde{C}_1^2 \|X(\tau)\|_{\mathcal{H}}^2 + \tilde{C}_2^2 \|v(\tau)\|^2 \right\} . \end{aligned}$$

This yields, for all $t \geq 0$,

$$\begin{aligned} e^{\delta_\epsilon t} \|x(t)\| \leq \tilde{C}_0 \|x_0\| + \tilde{C}_1 \sqrt{1+\epsilon} \sup_{\tau \in [0,t]} e^{\delta_\epsilon \tau} \|X(\tau)\|_{\mathcal{H}} \quad (49) \\ + \tilde{C}_2 \sqrt{1+\epsilon} \sup_{\tau \in [0,t]} e^{\delta_\epsilon \tau} \|v(\tau)\| . \end{aligned}$$

B. Stability of the interconnected IDS-ODE

We can now proceed to the proof of Theorem 5.4. From (24) and (49) we obtain that, for all $t \geq 0$,

$$\begin{aligned} &e^{\delta_\epsilon t} \|d(t)\|_{\mathcal{H}} \\ &= e^{\delta_\epsilon t} \|f_2(x(t), X(t), v(t))\|_{\mathcal{H}} \\ &\leq D_1 e^{\delta_\epsilon t} \|x(t)\| + D_2 e^{\delta_\epsilon t} \|X(t)\|_{\mathcal{H}} + D_3 e^{\delta_\epsilon t} \|v(t)\| \\ &\leq D_1 \tilde{C}_0 \|x_0\| + (D_1 \tilde{C}_1 \sqrt{1+\epsilon} + D_2) \sup_{\tau \in [0,t]} e^{\delta_\epsilon \tau} \|X(\tau)\|_{\mathcal{H}} \\ &\quad + (D_1 \tilde{C}_2 \sqrt{1+\epsilon} + D_3) \sup_{\tau \in [0,t]} e^{\delta_\epsilon \tau} \|v(\tau)\| . \quad (50) \end{aligned}$$

This yields, for all $t \geq 0$,

$$\begin{aligned} &\sup_{\tau \in [t_0, t+(D+t_0)]} e^{\delta_\epsilon \tau} \|d(\tau)\|_{\mathcal{H}} \\ &\leq D_1 \tilde{C}_0 \|x_0\| + (D_1 \tilde{C}_1 \sqrt{1+\epsilon} + D_2) \sup_{\tau \in [0, D+t_0]} e^{\delta_\epsilon \tau} \|X(\tau)\|_{\mathcal{H}} \\ &\quad + (D_1 \tilde{C}_1 \sqrt{1+\epsilon} + D_2) \sup_{\tau \in [D+t_0, t+(D+t_0)]} e^{\delta_\epsilon \tau} \|X(\tau)\|_{\mathcal{H}} \\ &\quad + (D_1 \tilde{C}_2 \sqrt{1+\epsilon} + D_3) \sup_{\tau \in [0, t+(D+t_0)]} e^{\delta_\epsilon \tau} \|v(\tau)\| . \end{aligned}$$

Therefore, we deduce from (48) that, for all $t \geq 0$,

$$\begin{aligned} &\sup_{\tau \in [D+t_0, t+(D+t_0)]} e^{\delta_\epsilon \tau} \|X(\tau)\|_{\mathcal{H}} \\ &\leq C_4 \tilde{C}_0 e^{\delta_\epsilon (D+t_0)} \sqrt{V(D+t_0)} \\ &\quad + D_1 \tilde{C}_0 C_4 \sqrt{\frac{C_6}{2\kappa_0} (1+\epsilon)} e^{\delta_\epsilon D} \|x_0\| \\ &\quad + (D_1 \tilde{C}_1 \sqrt{1+\epsilon} + D_2) C_4 \sqrt{\frac{C_6}{2\kappa_0} (1+\epsilon)} e^{\delta_\epsilon D} \end{aligned}$$

[3] We estimate by replacing $\tilde{\kappa}_0$ by $\hat{\kappa}_0$.

$$\times \sup_{\tau\in[0,D+t_0]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}}$$
$$+ (D_1\tilde{C}_1\sqrt{1+\epsilon}+D_2)C_4\sqrt{\frac{C_6}{2\kappa_0}(1+\epsilon)}e^{\delta_\epsilon D}$$
$$\times \sup_{\tau\in[D+t_0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}}$$
$$+ (D_1\tilde{C}_2\sqrt{1+\epsilon}+D_3)C_4\sqrt{\frac{C_6}{2\kappa_0}(1+\epsilon)}e^{\delta_\epsilon D}$$
$$\times \sup_{\tau\in[0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|v(\tau)\|.$$

As $\delta_\epsilon \underset{\epsilon\rightarrow 0^+}{\longrightarrow} 0^+$ and because of the small gain assumption (28), there exists $\epsilon > 0$ such that

$$(D_1\tilde{C}_1\sqrt{1+\epsilon}+D_2)C_4\sqrt{\frac{C_6}{2\kappa_0}(1+\epsilon)}e^{\delta_\epsilon D} < 1.$$

We fix such $\epsilon > 0$, which is independent of the initial condition (x_0, X_0) and the disturbance v. Therefore, we obtain that, for all $t \geq 0$,

$$\begin{aligned} & e^{\delta_\epsilon (t+(D+t_0))}\|X(t+(D+t_0))\|_{\mathcal{H}} \\ & \leq \sup_{\tau\in[D+t_0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}} \\ & \leq E_1\sqrt{V(D+t_0)} + E_2\|x_0\| + E_3 \sup_{\tau\in[0,D+t_0]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}} \\ & \quad + E_4 \sup_{\tau\in[0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|v(\tau)\| \end{aligned} \tag{51}$$

for some constants $E_1, E_2, E_3, E_4 > 0$. From (49), we have, for all $t \geq 0$,

$$\begin{aligned} & e^{\delta_\epsilon (t+(D+t_0))}\|x(t+(D+t_0))\| \\ & \leq \tilde{C}_0\|x_0\| + \tilde{C}_1\sqrt{1+\epsilon} \sup_{\tau\in[0,D+t_0]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}} \\ & \quad + \tilde{C}_1\sqrt{1+\epsilon} \sup_{\tau\in[D+t_0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}} \\ & \quad + \tilde{C}_2\sqrt{1+\epsilon} \sup_{\tau\in[0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|v(\tau)\| \\ & \leq F_1\sqrt{V(D+t_0)} + F_2\|x_0\| + F_3 \sup_{\tau\in[0,D+t_0]} e^{\delta_\epsilon \tau}\|X(\tau)\|_{\mathcal{H}} \\ & \quad + F_4 \sup_{\tau\in[0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|v(\tau)\|, \end{aligned} \tag{52}$$

where $F_1 = \tilde{C}_1 E_1\sqrt{1+\epsilon}$, $F_2 = \tilde{C}_0 + \tilde{C}_1 E_2\sqrt{1+\epsilon}$, $F_3 = \tilde{C}_1(1+E_3)\sqrt{1+\epsilon}$, and $F_4 = (\tilde{C}_2+\tilde{C}_1 E_4)\sqrt{1+\epsilon}$. Combining (51-52) and noting that (obtained from (23-24))

$$\begin{aligned} & V(D+t_0) \\ & \leq C_9\|X_0\|_{\mathcal{H}}^2 + D_1 C_{10} \sup_{\tau\in[0,D+t_0]} \|x(\tau)\|^2 \\ & \quad + D_2 C_{10} \sup_{\tau\in[0,D+t_0]} \|X(\tau)\|_{\mathcal{H}}^2 + D_3 C_{10} \sup_{\tau\in[0,D+t_0]} \|v(\tau)\|^2 \end{aligned} \tag{53}$$

we obtain the existence of constants $G_i \geq 0$, independent of the initial condition (x_0, X_0) and the disturbance v, such that (29) holds for all $t \geq D + t_0$. Finally, based on (47) and (40), we estimate the control input as follows. For all $t \geq 0$,

$$\begin{aligned} & e^{\delta_\epsilon t}\|u(t+(D+t_0))\| \leq \frac{\|K\|}{\sqrt{C_2(\gamma_1)}}\sqrt{V(t+(D+t_0))}e^{\delta_\epsilon t} \\ & \leq \frac{\|K\|\tilde{C}_0}{\sqrt{C_2(\gamma_1)}}\sqrt{V(D+t_0)} \\ & \quad + \|K\|\sqrt{\frac{C_6}{2\kappa_0 C_2(\gamma_1)}(1+\epsilon)}e^{-\delta_\epsilon t_0} \sup_{\tau\in[t_0,t+(D+t_0)]} e^{\delta_\epsilon \tau}\|d(\tau)\|_{\mathcal{H}}. \end{aligned}$$

Therefore, we infer from (50) and (53) the existence of constants $H_i \geq 0$, independent of the initial condition (x_0, X_0) and the disturbance v, such that (30) holds. This concludes the proof of Theorem 5.4.

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