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Boundary behaviour of Dirichlet series with applications to universal series

Stephen J. Gardiner and Myrto Manolaki

Abstract

This paper establishes connections between the boundary behaviour of functions representable as absolutely convergent Dirichlet series in a half-plane and the convergence properties of partial sums of the Dirichlet series on the boundary. This yields insights into the boundary behaviour of Dirichlet series and Taylor series which have universal approximation properties.

1 General Dirichlet series

We consider a general Dirichlet series of the form $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where (λ_n) is an unbounded strictly increasing sequence of nonnegative real numbers. If this series converges when $s = s_0$, then, as is well known, it converges uniformly throughout any angular region of the form $\{s \in \mathbb{C} : |\arg(s - s_0)| < \pi/2 - \delta\}$, where $\delta \in (0, \pi/2)$. In particular, f is defined on the half-plane $\{\operatorname{Re} s > \operatorname{Re} s_0\}$ and has a nontangential limit at s_0 . No such conclusion may be drawn if we know merely that some subsequence of the partial sums of the series converges at $s = s_0$ (see, for example, Bayart [3]). Nevertheless, we show below that there are strong connections between the nontangential boundary behaviour of f and the limiting behaviour of such subsequences. This has implications for the boundary behaviour of universal Dirichlet series and universal Taylor series.

A typical point of the complex plane will be written as $s = \sigma + it$. We denote by \mathbb{C}_+ the right-hand half-plane $\{\sigma > 0\}$, and by $D(\mathbb{C}_+)$ the space of all holomorphic functions on \mathbb{C}_+ which can be represented there as an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$. Such a representation is unique (see Theorem 6 of [15]). We define the partial sum

$$S_m(s) = \sum_{n=1}^m a_n e^{-\lambda_n s} \quad (s \in \mathbb{C}),$$

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and denote by $\text{nt} \lim_{s \rightarrow \zeta} f(s)$ the nontangential limit, when it exists (finitely), of f at a point ζ of the imaginary axis $i\mathbb{R}$.

Theorem 1 *Let $f \in D(\mathbb{C}_+)$, let (S_{m_k}) be a subsequence of (S_m) , and define*

$$E = \{\zeta \in i\mathbb{R} : S(\zeta) := \lim_{k \rightarrow \infty} S_{m_k}(\zeta) \text{ exists}\}$$

and

$$F = \{\zeta \in i\mathbb{R} : f(\zeta) := \text{nt} \lim_{s \rightarrow \zeta} f(s) \text{ exists}\}.$$

Then $f = S$ almost everywhere (Lebesgue) on $E \cap F$.

Theorem 1 may be applied to the Riemann zeta function ζ , since the function $s \mapsto \zeta(s+1)$ belongs to $D(\mathbb{C}_+)$. It tells us that, if a subsequence $(\sum_1^{m_k} n^{-s})$ of the Dirichlet polynomials is pointwise convergent on a set $E \subset \{\sigma = 1\}$, then the limit function must equal $\zeta(s)$ almost everywhere on E .

Definition By a *fat approach region* (relative to \mathbb{C}_+) to a point it_0 of the imaginary axis, we mean a set of the form

$$\Omega(it_0) = \{\sigma + it : |t - t_0| < a \text{ and } \phi(t - t_0) < \sigma < b\},$$

where $a, b > 0$ and $\phi : [-a, a] \rightarrow [0, \infty)$ is a Lipschitz function satisfying

$$\int_{-a}^a y^{-2} \phi(y) dy < \infty.$$

It is easy to see that such a set contains Stolz regions at it_0 of arbitrarily large aperture, that is, $\text{nt} \lim_{s \rightarrow it_0} \chi_{\Omega(it_0)}(s) = 1$, where χ_A denotes the characteristic function of a set A . Examples of fat approach regions to 0 are the half-discs $\{s : |s - a| < a, \sigma < a\}$, where $a > 0$, and sets of the form $\{\sigma + it : |t|^\alpha < \sigma < 1\}$, where $\alpha > 1$.

Theorem 2 *Let $f \in D(\mathbb{C}_+)$, let h be a positive harmonic function on \mathbb{C}_+ , and suppose that $e^{-h} f$ is bounded on two fat approach regions, $\Omega(it_1)$ and $\Omega(it_2)$, where $t_1 < t_2$. If a subsequence (S_{m_k}) is uniformly bounded on an open interval of $i\mathbb{R}$ containing it_1 and it_2 , then f is bounded on any rectangle of the form $\{\sigma + it : 0 < \sigma < 1, t_1 + \varepsilon < t < t_2 - \varepsilon\}$, where $\varepsilon > 0$.*

Remark 3 (a) *In Theorem 2 the function f must then have a nontangential limit $f(\zeta)$ at almost every point ζ of the interval (it_1, it_2) , by the local version of the Fatou theorem. Thus, by Theorem 1, $f(\zeta) = \lim_{k \rightarrow \infty} S_{m_k}(\zeta)$ at almost every point of (it_1, it_2) where the latter limit exists.*

(b) *In the special case where the function f itself is bounded on two fat approach regions, $\Omega(it_1)$ and $\Omega(it_2)$, the proof of Theorem 2 shows that (S_{m_k}) is uniformly bounded on the rectangle $\{\sigma + it : 0 < \sigma < 1, t_1 < t < t_2\}$, and so f is also bounded there.*

The High Indices Theorem of Hardy and Littlewood (see [14] or Theorem 23.1 of [16]) says that, if $\inf_n \lambda_{n+1}/\lambda_n > 1$ and $f(it_0) := \lim_{\sigma \rightarrow 0^+} f(\sigma + it_0)$ exists, then $S_n(it_0) \rightarrow f(it_0)$. We will now see a complementary result concerning Dirichlet series which possess a different type of gap structure.

If f is a holomorphic function on \mathbb{C}_+ which has bounded derivative on a fat approach region to $\zeta \in i\mathbb{R}$, it is easy to see that f has a finite nontangential limit there, which we again denote by $f(\zeta)$ in the next result.

Theorem 4 *Let $f \in D(\mathbb{C}_+)$ and suppose that f' is bounded on a fat approach region to $\zeta \in i\mathbb{R}$. If a subsequence (S_{m_k}) is uniformly bounded on an open interval of $i\mathbb{R}$ containing ζ , and $\lambda_{m_k+1}/\lambda_{m_k} \rightarrow \infty$, then $S_{m_k}(\zeta) \rightarrow f(\zeta)$.*

Theorem 4 fails if we drop the requirement that $\lambda_{m_k+1}/\lambda_{m_k} \rightarrow \infty$. To see this, let $f(s) = (e^s - 1)^{-1}$. Then $f \in D(\mathbb{C}_+)$, since $f(s) = \sum_1^\infty e^{-ns}$ when $s \in \mathbb{C}_+$, and f' is bounded on $\{s \in \mathbb{C}_+ : |s - i\pi| < \pi/2\}$. However, if we take $\zeta = i\pi$ and $\lambda_{m_k} = m_k = 2k$, then

$$|S_{m_k}(it)| = \left| \frac{1 - e^{-m_k it}}{e^{it} - 1} \right| \leq \frac{2}{\sqrt{2}} = \sqrt{2} \quad (\pi/2 < t < 3\pi/2)$$

and $S_{m_k}(\zeta) = 0$ for all $k \in \mathbb{N}$, yet $f(\zeta) = -1/2$. (Analogous reasoning for the function $f(s) = (2^s - 1)^{-1}$ yields a corresponding counterexample for ordinary Dirichlet series.)

Theorems 1 and 2 provide analogues for Dirichlet series of results recently established for Taylor series in [12] and [10], respectively. Theorem 4 implies a corresponding result for Taylor series $\sum a_n z^n$ in the unit disc \mathbb{D} with (pure) Ostrowski gaps, which can readily be deduced using the substitution $z = e^{-s}$ (see Corollary 14 in Section 6). In the next section we present applications of the above theorems to universal Dirichlet series and universal Taylor series.

2 Applications to universal series

We now focus on ordinary Dirichlet series, where $\lambda_n = \log n$. We say that a function f in $D(\mathbb{C}_+)$ belongs to the collection UD_0 , if it is of the form $f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$ and, for any continuous function $g : i\mathbb{R} \rightarrow \mathbb{C}$ and any $b > 0$, there is a subsequence (S_{m_k}) of the partial sums that converges to g uniformly on $[-bi, bi]$. Since UD_0 contains all Dirichlet series with universal approximation properties in $\mathbb{C} \setminus \mathbb{C}_+$ as described by Bayart [3], it follows from Theorem 6 of that paper that membership of UD_0 is topologically generic in $D(\mathbb{C}_+)$. (The topology is defined using the family of seminorms $\|\sum a_n n^{-s}\|_\sigma = \sum |a_n| n^{-\sigma}$, $\sigma > 0$.) Further results on universal Dirichlet series may be found in [4], [8], [17], and [20]. Theorems 1 and 2 yield the

following strong information about the boundary behaviour of functions in UD_0 , and so about universal Dirichlet series in particular.

Corollary 5 *Let $f \in UD_0$. Then, for almost every $\zeta \in i\mathbb{R}$, the set $f(\Gamma)$ is dense in \mathbb{C} for every open isosceles triangle $\Gamma \subset \mathbb{C}_+$ with vertex at ζ and horizontal axis of symmetry.*

Corollary 6 *Let $f \in UD_0$. Then, for any disc D centred on $i\mathbb{R}$, the set $\mathbb{C} \setminus f(D \cap \mathbb{C}_+)$ has zero logarithmic capacity.*

We recall that a subset of \mathbb{R} is called *residual* if its complement is of first Baire category.

Corollary 7 *Let $f \in UD_0$. Then there is a residual set $Z \subset \mathbb{R}$ such that $\{f(\sigma + it) : 0 < \sigma < 1\}$ is dense in \mathbb{C} for every $t \in Z$.*

Corollary 8 *Let $f \in D(\mathbb{C}_+)$ and h be a positive harmonic function on \mathbb{C}_+ , and suppose that $e^{-h}f$ is bounded on two fat approach regions $\Omega(it_1)$ and $\Omega(it_2)$, where $t_1 \neq t_2$. Then $f \notin UD_0$.*

The above corollaries are inspired by results recently established for universal Taylor series in [9] and [10]. In the version of Corollary 6 for universal Taylor series an even stronger conclusion was subsequently established in [11], but its proof does not readily transfer to Dirichlet series. The hypotheses of Corollary 8 clearly hold if, for example, f is bounded on two discs in \mathbb{C}_+ that are tangent to $i\mathbb{R}$ at distinct points.

Next we use Theorem 4 to give a new result for Taylor series. We recall that a holomorphic function f on \mathbb{D} is said to possess a universal Taylor series if, for any compact set $K \subset \mathbb{D}^c$ with connected complement and any continuous function $g : K \rightarrow \mathbb{C}$ that is holomorphic on K° , there is a subsequence (T_{m_k}) of the partial sums of the Taylor series of f about 0 which converges to g uniformly on K . The collection of all such functions is denoted by $U(\mathbb{D}, 0)$. Nestoridis [19] has shown that this universal approximation property is topologically generic for holomorphic functions on \mathbb{D} (with respect to the topology of local uniform convergence). Let $\omega \subset \mathbb{D}$ be open and $w \in \partial\mathbb{D}$. We call ω a fat approach region to w (relative to \mathbb{D}) if the set $\{1 - w^{-1}z : z \in \omega\}$ is a fat approach region to 0 (relative to \mathbb{C}_+).

Corollary 9 *Let f be a holomorphic function on \mathbb{D} . If f' is bounded on a fat approach region to some point of $\partial\mathbb{D}$, then $f \notin U(\mathbb{D}, 0)$.*

Corollary 9 complements a result of Costakis and Katsoprinakis (Theorem 2 of [7]) concerning the behaviour of antiderivatives of universal Taylor series on internally tangential discs.

3 Proof of Theorem 1

Our proof of Theorem 1 relies on the following convergence result for harmonic measures, taken from [12]. Although it is valid more generally, we will only need the case where Ω is a plane domain possessing a Green function $G_\Omega(\cdot, \cdot)$. For any (nonempty) open set ω , any Borel set A and any point z in ω , we denote by $\mu_z^\omega(A)$ the harmonic measure of $A \cap \partial\omega$ for ω evaluated at z . We refer to the book [2] for potential theoretic concepts.

Theorem 10 *Let $\xi_0 \in \Omega$ and ω be an open subset of Ω . Suppose that (v_k) is a decreasing sequence of subharmonic functions on ω such that $v_1/G_\Omega(\xi_0, \cdot)$ is bounded above and $\lim_{k \rightarrow \infty} v_k < 0$ on ω . If $\mu_{z_1}^\omega(\partial\Omega) > 0$ for some z_1 , then $\mu_{z_1}^{\{v_k < 0\}}(\partial\Omega) > 0$ for all sufficiently large k .*

We will also repeatedly use the following lemmas. As usual, the positive and negative parts of a real number t will be denoted by t^+ and t^- , respectively.

Lemma 11 *Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where $f \in D(\mathbb{C}_+)$, and let $u_m = \lambda_{m+1}^{-1} \log |S_m - f|$. Then, for any $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that*

$$u_m(s) \leq -\sigma + \varepsilon \quad (s = \sigma + it, \sigma > \varepsilon, m \geq m_0).$$

Proof. Given $\varepsilon > 0$, we have

$$\begin{aligned} u_m(s) &= \frac{\log \left| \sum_{n=m+1}^{\infty} a_n e^{-\lambda_n s} \right|}{\lambda_{m+1}} \\ &\leq \frac{\log \left(e^{\lambda_{m+1}(\varepsilon - \sigma)} \sum_{n=m+1}^{\infty} |a_n| e^{-\lambda_n \varepsilon} \right)}{\lambda_{m+1}} \leq \varepsilon - \sigma \quad (\sigma > \varepsilon), \end{aligned}$$

for all sufficiently large m , since $f \in D(\mathbb{C}_+)$. ■

Lemma 12 *Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where $f \in D(\mathbb{C}_+)$, let $\sigma_0 < 0$ and let $K \subset \{\sigma_0 < \sigma \leq 0\}$ be a nonpolar compact set. Then there exists $c > 0$ such that*

$$|S_m| \leq \max \left\{ \sup_K |S_m|, 1 \right\} e^{c \lambda_m G_{\mathbb{C} \setminus K}(\sigma_0 - 1, \cdot)} \quad \text{on } \{\sigma > \sigma_0\} \setminus K \quad \text{for all } m.$$

Proof. We dismiss the trivial case where $m = 1$ and $\lambda_1 = 0$. Since $f \in D(\mathbb{C}_+)$, we know (cf. Theorem 8.1 of [1]) that

$$\limsup_{m \rightarrow \infty} \frac{\log \left(\sum_{n=1}^m |a_n| \right)}{\lambda_m} \leq 0.$$

Also,

$$|S_m(s)| \leq \sum_{n=1}^m |a_n| e^{-\lambda_n \sigma} \leq e^{\lambda_m \sigma^-} \sum_{n=1}^m |a_n|.$$

Thus the functions

$$s \mapsto \left(\frac{\log |S_m(s)|}{\lambda_m} - \sigma^- \right)^+ \quad (m \in \mathbb{N})$$

are uniformly bounded on \mathbb{C} and converge uniformly to 0 as $m \rightarrow \infty$.

Let $b_m = \max\{\sup_K |S_m|, 1\}$. By the previous paragraph there exists a positive constant c_1 such that

$$\frac{\log(|S_m|/b_m)}{\lambda_m} \leq \frac{\log(|S_m|)}{\lambda_m} \leq c_1 \quad \text{on } \{\sigma + it : \sigma \geq \sigma_0\} \text{ for all } m.$$

Also, $G_{\mathbb{C} \setminus K}(\sigma_0 - 1, \cdot)$ has a positive limit at ∞ . Thus, applying the maximum principle in $\{\sigma > \sigma_0\} \setminus K$, we see that there is a positive constant c such that

$$\log \left(\frac{|S_m|}{b_m} \right) \leq c \lambda_m G_{\mathbb{C} \setminus K}(\sigma_0 - 1, \cdot) \quad \text{on } \{\sigma > \sigma_0\} \setminus K,$$

as required. ■

Proof of Theorem 1. We adapt an argument from [12]. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where $f \in D(\mathbb{C}_+)$, and let (S_{m_k}) , E and F be as in the statement of Theorem 1. For each $\zeta \in i\mathbb{R}$, let $\Gamma(\zeta)$ denote the open triangular region with vertices $\{\zeta, \zeta + 1 \pm i\}$. We suppose, for the sake of contradiction, that the conclusion of the theorem fails. Then, first multiplying f by a suitable unimodular constant, we may fix a number $a \geq 1$ large enough so that the set

$$\{\zeta \in E \cap F : \operatorname{Re}(S(\zeta) - f(\zeta)) \geq a^{-1}, |f| \leq a \text{ on } \Gamma(\zeta), |S_{m_k}(\zeta)| \leq a \text{ for all } k\}$$

has positive Lebesgue measure. Let K be a compact subset of the above set, also of positive measure, let $\Omega = \mathbb{C} \setminus K$ and $\omega = \cup_{\zeta \in K} \Gamma(\zeta)$. (We can choose K so that ω is connected.)

We define the subharmonic functions

$$u_k = \frac{1}{\lambda_{m_k+1}} \log \left(\frac{|S_{m_k} - f|}{2a} \right) \quad \text{on } \mathbb{C}_+ \quad (k \in \mathbb{N}).$$

By Lemma 11 we may choose a sequence $\sigma_k \downarrow 0$ such that

$$u_j(s) \leq -\sigma/2 \quad (\sigma_k \leq \sigma \leq 1, j \geq k). \quad (1)$$

Also,

$$u_k \leq \frac{1}{\lambda_{m_k}} \log^+ \left(\frac{\max\{|S_{m_k}|, |f|\}}{a} \right) \leq cG_\Omega(-2, \cdot) \text{ on } \omega \cup K \quad (k \geq m_0), \quad (2)$$

by Lemma 12 (where $\sigma_0 = -1$, say). We now define v_k to be the solution, $H_{\psi_k}^\omega$, to the Dirichlet problem in ω with boundary data ψ_k , where

$$\psi_k(s) = \begin{cases} -\sigma/2 & \text{if } \sigma \geq \sigma_k \\ cG_\Omega(-2, \cdot) & \text{if } 0 < \sigma < \sigma_k \\ 0 & \text{if } \sigma = 0 \end{cases} .$$

Then $u_k \leq v_k$ on ω , by (1) and (2), and (v_k) is a decreasing sequence of harmonic functions on ω with limit $-\sigma/2$.

Since K has positive linear measure, we know that $\mu_z^\omega(K) > 0$ when $z \in \omega$. (This follows from the F. and M. Riesz theorem and the fact that $\partial\omega$ is rectifiable; see Section VI.1 of [13].) Thus, by Theorem 10, there exists $k' \in \mathbb{N}$ such that the open set $\omega_1 := \omega \cap \{v_{k'} < 0\}$ is nonempty and

$$\mu_w^{\omega_1}(\partial\Omega) > 0 \text{ for some } w \in \omega_1. \quad (3)$$

When $k \geq k'$ we have $u_k < 0$ on ω_1 , so $|S_{m_k} - f| < 2a$, and hence $|S_{m_k}| < 3a$, on ω_1 . Now $S_{m_k} = H_{S_{m_k}}^{\omega_1}$ on ω_1 , so by dominated convergence

$$f = H_\phi^{\omega_1} \text{ on } \omega_1, \text{ where } \phi = \begin{cases} f & \text{on } \partial\omega_1 \cap \mathbb{C}_+ \\ S & \text{on } \partial\omega_1 \cap i\mathbb{R} \subset K \end{cases} .$$

However, we also know that

$$f = H_f^\omega = H_{H_f^\omega}^{\omega_1} = H_f^{\omega_1} \text{ on } \omega_1$$

(see Theorem 6.3.6 of [2]). Thus $H_{(S-f)\chi_K}^{\omega_1} \equiv 0$. This yields a contradiction, since our choice of K and ω_1 ensures that

$$\operatorname{Re} H_{(S-f)\chi_K}^{\omega_1}(w) \geq a^{-1} \mu_w^{\omega_1}(K) > 0.$$

Theorem 1 is now established. ■

4 Proof of Theorem 2

We will need the following variant of Proposition 4 in [12].

Proposition 13 *Let ω be a fat approach region to $it_0 \in i\mathbb{R}$ relative to \mathbb{C}_+ . Further, let (u_k) be a sequence of subharmonic functions on ω such that $u_k(s) \leq b\sigma$ for all k , where $b \geq 0$, and $\limsup_{k \rightarrow \infty} u_k < 0$. Then there exists $k_0 \in \mathbb{N}$ such that*

$$\operatorname{nt} \limsup_{s \rightarrow it_0} \frac{\sup_{k \geq k_0} u_k(s)}{\sigma} < 0.$$

Proof of Proposition. We choose another fat approach region ω_1 to it_0 such that $\bar{\omega}_1 \setminus \{it_0\} \subset \omega$, and observe (by Corollary 5.7.2 of [2]) that there are strictly decreasing sequences $\sigma_k \rightarrow 0$ and $\tau_k \rightarrow 0$ such that

$$u_j(s) \leq -\tau_k \quad (s \in \partial\omega_1, \sigma \geq \sigma_k, j \geq k).$$

Let $v_k = H_{\psi_k}^{\omega_1}$, where

$$\psi_k(s) = \begin{cases} -\tau_1 & (\sigma \geq \sigma_1) \\ -\tau_j & (\sigma_j \leq \sigma < \sigma_{j-1}; j = 2, \dots, k) \\ b\sigma & (0 \leq \sigma < \sigma_k) \end{cases}.$$

Then $u_j \leq v_k$ on ω_1 whenever $j \geq k$, and (v_k) is a decreasing sequence of harmonic functions on ω_1 with $\lim v_k < 0$ there.

We now recall (see Chapter 9 of [2]) that a set $E \subset \mathbb{C}_+$ is said to be minimally thin at it_0 (with respect to \mathbb{C}_+) if there is a positive superharmonic function h on \mathbb{C}_+ such that

$$\inf_E \frac{h}{P_{it_0}} > \inf_{\mathbb{C}_+} \frac{h}{P_{it_0}},$$

where P_{it_0} denotes the Poisson kernel for \mathbb{C}_+ with pole at it_0 ; that is,

$$P_{it_0}(s) = \frac{\sigma}{\sigma^2 + (t - t_0)^2} \quad (s = \sigma + it \in \mathbb{C}_+).$$

Theorem 9.7.1 of [2] tells us that, because ω_1 is a fat approach region at it_0 , the set $\mathbb{C}_+ \setminus \omega_1$ is minimally thin at it_0 . We can now apply Proposition 4 of [12] to see that there exist $k_0 \in \mathbb{N}$ and a set $E \subset \mathbb{C}_+$, minimally thin at it_0 , such that

$$\lim_{s \rightarrow it_0, s \in \mathbb{C}_+ \setminus E} \frac{v_{k_0}(s)}{\sigma} < 0.$$

Next, we appeal to Theorem 6 of [5] to see that

$$\text{nt} \lim_{s \rightarrow it_0} \frac{v_{k_0}(s)}{\sigma} < 0.$$

The result now follows, since $u_k \leq v_{k_0}$ whenever $k \geq k_0$. ■

Proof of Theorem 2. Now let f , h , $\Omega(it_j)$ and (S_{m_k}) be as in the statement of Theorem 2, and let $0 < \varepsilon < (t_1 - t_2)/2$. Then h has a Poisson integral representation of the form

$$h(s) = \frac{\sigma}{\pi} \int_{\mathbb{R}} |s - i\zeta|^{-2} d\mu(\zeta) \quad (s \in \mathbb{C}_+),$$

where μ is a positive measure on \mathbb{R} . If we define $I = [t_1 + \frac{\varepsilon}{2}, t_2 - \frac{\varepsilon}{2}]$ and

$$h_1(s) = \frac{\sigma}{\pi} \left(\int_{\mathbb{R} \setminus I} \frac{1}{|s - i\zeta|^2} d\mu(\zeta) + \frac{\mu(I)}{|s - i(t_1 + \frac{\varepsilon}{2})|^2} + \frac{\mu(I)}{|s - i(t_1 - \frac{\varepsilon}{2})|^2} \right),$$

then $e^{-h_1} |f|$ is also bounded above, by $d \geq 1$ say, on the sets $\Omega(it_j)$.

There exists $b \geq 1$ such that $|S_{m_k}| \leq b$ on the set $K = [i(t_1 - \varepsilon), i(t_2 + \varepsilon)]$, provided ε has been chosen small enough. By Lemma 12 there is a constant $c > 0$ such that

$$\log(|S_{m_k}|/b) \leq c\lambda_{m_k} G_{\mathbb{C} \setminus K}(-2, \cdot) \quad \text{on } \{\sigma > -1\} \setminus K \quad (k \in \mathbb{N}).$$

Hence there is a constant $c_1 > 0$ such that

$$\log(|S_{m_k}(s)|/b) \leq c_1 \lambda_{m_k} \sigma \quad \text{on } \Omega(it_j) \quad (j = 1, 2; k \in \mathbb{N})$$

(cf. Lemma 8.5.1 of [2]). If we define the subharmonic functions

$$u_k = \frac{1}{\lambda_{m_k+1}} \log \left(\frac{|S_{m_k} - f|}{2bde^{h_1}} \right) \quad \text{on } \mathbb{C}_+ \quad (k \in \mathbb{N}),$$

then

$$\begin{aligned} u_k(s) &\leq \frac{1}{\lambda_{m_k+1}} \log^+ \left(\frac{\max\{|S_{m_k}(s)|, |f(s)|\}}{bde^{h_1(s)}} \right) \\ &\leq \frac{1}{\lambda_{m_k+1}} \log^+ \max \left\{ \frac{|S_{m_k}(s)|}{b}, \frac{e^{-h_1(s)} |f(s)|}{d} \right\} \\ &\leq c_1 \sigma \quad (s \in \Omega(it_1) \cup \Omega(it_2), k \in \mathbb{N}). \end{aligned}$$

Since we know from Lemma 11 that $\limsup_{k \rightarrow \infty} u_k(s) \leq -\sigma < 0$ on \mathbb{C}_+ , it follows from Proposition 13 that there exists $k_0 \in \mathbb{N}$ such that $u_k(s) < 0$ on the line segments $(it_j, it_j + 1)$ ($j = 1, 2$) when $k \geq k_0$. Hence $|S_{m_k} - f| \leq 2bde^{h_1}$, and so $\log |S_{m_k}| \leq \log(2bd + d) + h_1$ on these line segments whenever $k \geq k_0$. Since (S_{m_k}) is uniformly bounded on $[it_1, it_2] \cup [it_1 + 1, it_2 + 1]$, it follows from the maximum principle that there exists $c_2 > 0$ such that $\log |f| \leq c_2 + h_1$ on the rectangle with vertices $\{it_j, it_j + 1 : j = 1, 2\}$, and the result follows from the boundedness of h_1 near points of the line segment $[i(t_1 + \varepsilon), i(t_2 - \varepsilon)]$. ■

5 Proof of Theorem 4

Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, where $f \in D(\mathbb{C}_+)$, and let $\zeta = it_0$. We assume that $|f'| \leq b$ on some fat approach region $\Omega(it_0)$ and that $|S_{m_k}| \leq b$ on the line segment $K = [i(t_0 - \varepsilon), i(t_0 + \varepsilon)]$ for some $b \geq 1$ and $\varepsilon \in (0, 1)$. We know from Lemma 12 that there is a constant $c > 0$ such that

$$\log(|S_{m_k}|/b) \leq c\lambda_{m_k} G_{\mathbb{C} \setminus K}(-2, \cdot) \quad \text{on } \{\sigma > -1\} \setminus K \quad (k \in \mathbb{N}).$$

We choose k_0 large enough so that $\lambda_{m_k}^{-1} < \varepsilon/3$ when $k \geq k_0$. Since there is a constant $c_1 > 0$ such that

$$G_{\mathbb{C} \setminus K}(-2, s) \leq c_1 |\sigma| \quad (|\sigma| < \varepsilon/3, |t - t_0| < 2\varepsilon/3),$$

we see that

$$\log \left(\frac{|S_{m_k}(s)|}{b} \right) \leq cc_1 \quad (|\sigma| \leq \frac{1}{\lambda_{m_k}}, |t - t_0| \leq \frac{\varepsilon}{3} + \frac{1}{\lambda_{m_k}}, k \geq k_0).$$

It follows from Cauchy's integral formula for derivatives, applied to circles of radius $\lambda_{m_k}^{-1}$ centred on L , that

$$|S'_{m_k}| \leq c_2 \lambda_{m_k} \quad \text{on } L = [i(t_0 - \varepsilon/3), i(t_0 + \varepsilon/3)] \quad \text{whenever } k \geq k_0,$$

where $c_2 = be^{cc_1}$.

Since $f' \in D(\mathbb{C}_+)$ (see Theorem 4 of [15]), we can appeal to Lemma 12 once again to see that

$$\log \left(\frac{|S'_{m_k}|}{c_2 \lambda_{m_k}} \right) \leq c_3 \lambda_{m_k} G_{\mathbb{C} \setminus L}(-2, \cdot) \quad \text{on } \mathbb{C}_+ \quad (k \geq k_0),$$

for some constant $c_3 > 0$. We define the subharmonic functions

$$u_k = \frac{1}{\lambda_{m_k+1}} \log \left(\frac{|S'_{m_k} - f'|}{2c_2 \lambda_{m_k}} \right) \quad \text{on } \mathbb{C}_+ \quad (k \in \mathbb{N}),$$

and note from Lemma 11 that $\limsup_{k \rightarrow \infty} u_k(s) \leq -\sigma < 0$ on \mathbb{C}_+ . Further,

$$u_k \leq \frac{1}{\lambda_{m_k+1}} \log \left(\frac{\max\{|S'_{m_k}|, |f'|\}}{c_2 \lambda_{m_k}} \right) \leq c_3 G_{\mathbb{C} \setminus L}(-2, \cdot) \quad \text{on } \Omega(it_0) \quad (k \geq k_0).$$

Since $G_{\mathbb{C} \setminus L}(-2, s)$ is comparable to $|\sigma|$ near it_0 , we see from Proposition 13 that there exist $k_1 \in \mathbb{N}$ and $c_4 > 0$ such that

$$|S'_{m_k} - f'| (s) \leq 2c_2 \lambda_{m_k} \exp(-c_4 \lambda_{m_k+1} \sigma) \quad (s \in (it_0, \xi_0], k \geq k_1),$$

where $\xi_0 = it_0 + 1$. Hence

$$\begin{aligned} ||S_{m_k}(it_0) - f(it_0)| - |S_{m_k}(\xi_0) - f(\xi_0)|| &\leq \int_{[it_0, \xi_0]} |S'_{m_k}(s) - f'(s)| |ds| \\ &\leq 2c_2 \lambda_{m_k} \int_0^1 \exp(-c_4 \lambda_{m_k+1} \sigma) d\sigma \\ &\leq 2 \frac{c_2}{c_4} \frac{\lambda_{m_k}}{\lambda_{m_k+1}} \rightarrow 0 \quad (k \rightarrow \infty), \end{aligned}$$

by hypothesis. Since $S_{m_k}(\xi_0) \rightarrow f(\xi_0)$ we also have $S_{m_k}(it_0) \rightarrow f(it_0)$, as required.

6 Proofs of Applications

Proof of Corollary 5. Let f be a holomorphic function on \mathbb{C}_+ . Plessner's theorem tells us that, at almost every point ζ of $i\mathbb{R}$, either f has a finite nontangential limit or $f(\Gamma)$ is dense in \mathbb{C} for every open isosceles triangle $\Gamma \subset \mathbb{C}_+$ with vertex at ζ and horizontal axis of symmetry. Theorem 1 tells us that f cannot both belong to UD_0 and also have a finite nontangential limit on a subset of $i\mathbb{R}$ of positive measure. Thus Corollary 5 is proved. ■

Proof of Corollary 6. Let $f \in UD_0$. To see why Corollary 6 holds, suppose (for the sake of contradiction) that there is a disc D , centred at a point of $i\mathbb{R}$, for which the set $\mathbb{C} \setminus f(D \cap \mathbb{C}_+)$ has positive logarithmic capacity. Myrberg's theorem (Theorem 5.3.8 of [2]) then shows that there is a positive harmonic function h on the open set $f(D \cap \mathbb{C}_+)$ satisfying $h(z) \geq \log |z|$ there. Thus $\log |f| \leq h \circ f$ on $D \cap \mathbb{C}_+$. Fatou's theorem now tells us that $h \circ f$, and hence f , is nontangentially bounded at almost every point of $D \cap i\mathbb{R}$, contradicting Corollary 5. ■

Proof of Corollary 7. Let $f \in UD_0$. By Corollary 6, the unrestricted cluster set of f at each boundary point of \mathbb{C}_+ is all of \mathbb{C} . The Collingwood Maximality Theorem (Theorem 4.8 of [6]), when adapted to the half-plane, tells us that there is a residual set $Z \subset \mathbb{R}$ such that, for each $t \in Z$, the cluster set of f along $\{\sigma + it : 0 < \sigma < 1\}$ as $\sigma \rightarrow 0$ is also all of \mathbb{C} . The result now follows. ■

Proof of Corollary 8. Let $f \in D(\mathbb{C}_+)$ and h be a positive harmonic function on \mathbb{C}_+ , and suppose that $|f| \leq e^h$ on two fat approach regions $\Omega(it_j)$, where $j = 1, 2$ and $t_1 < t_2$. If $f \in UD_0$, then we can find a subsequence (S_{m_k}) of partial sums converging uniformly to 0 on $[i(t_1 - 1), i(t_2 + 1)]$. Theorem 2 and Remark 3 then show that f is bounded near all points of the interval (it_1, it_2) and has nontangential limit 0 almost everywhere on this line segment. This leads to the contradictory conclusion that $f \equiv 0$. ■

In what follows the partial sums of a power series $\sum_0^\infty a_n z^n$ will be denoted by (T_m) . The series is said to have pure Ostrowski gaps (p_k, q_k) if

$$1 \leq p_1 < q_1 \leq p_2 < q_2 \leq \dots \quad \text{and} \quad q_k/p_k \rightarrow \infty,$$

and if

$$a_n = 0 \quad \text{whenever} \quad p_k + 1 \leq n \leq q_k \quad \text{for some} \quad k \in \mathbb{N}.$$

The following consequence of Theorem 4 is an intermediate step towards proving Corollary 9.

Corollary 14 *Let f be a holomorphic function on \mathbb{D} such that f' is bounded on a fat approach region to some point w of $\partial\mathbb{D}$ (whence $f(w)$ exists as a*

nontangential limit). If the Taylor series of f about 0 has pure Ostrowski gaps (p_k, q_k) , and (T_{p_k}) is uniformly bounded on an open arc of $\partial\mathbb{D}$ containing w , then $T_{p_k}(w) \rightarrow f(w)$.

Proof of Corollary 14. Without loss of generality we may assume that $w = 1$. Then the function $g : s \mapsto f(e^{-s})$ belongs to $D(\mathbb{C}_+)$ and has bounded derivative on a fat approach region to $0 \in i\mathbb{R}$. Since g has the form

$$g(s) = \sum_{k=1}^{\infty} \sum_{n=q_{k-1}+1}^{p_k} a_n e^{-ns} \quad (s \in \mathbb{C}_+),$$

where q_0 is interpreted as -1 , and $(q_k + 1)/p_k \rightarrow \infty$, the result follows from Theorem 4. ■

Proof of Corollary 9. Let $f(z) = \sum_0^\infty a_n z^n$ be a holomorphic function on \mathbb{D} such that f' is bounded on a fat approach region to some point $e^{i\phi}$ of $\partial\mathbb{D}$. Now suppose, for the sake of contradiction, that $f \in U(\mathbb{D}, 0)$. It follows from Theorem 2 of [18] that we can find sequences (p_k) and (q_k) such that

- (i) $f = g + h$ on \mathbb{D} , where h is entire and g possesses pure Ostrowski gaps (p_k, q_k) ; and
- (ii) $T_{p_k} \rightarrow f(e^{i\phi}) + 1$ uniformly on $\{e^{i\theta} : |\theta - \phi| \leq \pi/2\}$.

We can now apply Corollary 14 to the function $f - h$ to see that $T_{p_k}(e^{i\phi}) \rightarrow f(e^{i\phi})$, which contradicts (ii). ■

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