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# QUADRATIC FORMS AND FOUR PARTITION FUNCTIONS MODULO 3

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ABSTRACT. Recently, Andrews, Hirschhorn and Sellers have proven congruences modulo 3 for four types of partitions using elementary series manipulations. In this paper, we generalize their congruences using arithmetic properties of certain quadratic forms.

## 1. INTRODUCTION

A partition of a non-negative integer  $n$  is a non-increasing sequence whose sum is  $n$ . An overpartition of  $n$  is a partition of  $n$  where we may overline the first occurrence of a part. Let  $\overline{p}(n)$  denote the number of overpartitions of  $n$ ,  $\overline{p}_o(n)$  the number of overpartitions of  $n$  into odd parts,  $ped(n)$  the number of partitions of  $n$  without repeated even parts and  $pod(n)$  the number of partitions of  $n$  without repeated odd parts. The generating functions for these partitions are

$$\sum_{n \geq 0} \overline{p}(n)q^n = \frac{(-q; q)_\infty}{(q; q)_\infty}, \quad (1.1)$$

$$\sum_{n \geq 0} \overline{p}_o(n)q^n = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty}, \quad (1.2)$$

$$\sum_{n \geq 0} ped(n)q^n = \frac{(-q^2; q^2)_\infty}{(q; q^2)_\infty}, \quad (1.3)$$

$$\sum_{n \geq 0} pod(n)q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty}, \quad (1.4)$$

where as usual

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).$$

The infinite products in (1.1)–(1.4) are essentially the four different ways one can specialize the product  $(-aq; q)_\infty / (bq; q)_\infty$  to obtain a modular form whose level is relatively prime to 3.

A series of four recent papers examined congruence properties for these partition functions modulo 3 [1, 5, 6, 7]. Among the main theorems in these papers are the following congruences (see Theorem 1.3 in [6], Corollary 3.3 and Theorem 3.5 in [1], Theorem 1.1 in [5] and Theorem 3.2 in [7], respectively). For all  $n \geq 0$  and  $\alpha \geq 0$  we have

$$\overline{p}_o(3^{2\alpha}(An + B)) \equiv 0 \pmod{3}, \quad (1.5)$$

where  $An + B = 9n + 6$  or  $27n + 9$ ,

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$$\text{ped}\left(3^{2\alpha+3}n + \frac{17 \cdot 3^{2\alpha+2} - 1}{8}\right) \equiv \text{ped}\left(3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{3}, \quad (1.6)$$

$$\bar{p}(3^{2\alpha}(27n + 18)) \equiv 0 \pmod{3} \quad (1.7)$$

and

$$\text{pod}\left(3^{2\alpha+3} + \frac{23 \cdot 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod{3}. \quad (1.8)$$

We note that congruences modulo 3 for  $\bar{p}(n)$ ,  $\bar{p}_o(n)$  and  $\text{ped}(n)$  are typically valid modulo 6 or 12. The powers of 2 enter trivially (or nearly so), however, so we do not mention them here.

The congruences in (1.5)–(1.8) are proven in [1, 5, 6, 7] using elementary series manipulations. If we allow ourselves some elementary number theory, we find that much more is true.

With our first result we exhibit formulas for  $\bar{p}_o(3n)$  and  $\text{ped}(3n + 1)$  modulo 3 for all  $n \geq 0$ . These formulas depend on the factorization of  $n$ , which we write as

$$n = 2^a 3^b \prod_{i=1}^r p_i^{v_i} \prod_{j=1}^s q_j^{w_j}, \quad (1.9)$$

where  $p_i \equiv 1, 5, 7$  or  $11 \pmod{24}$  and  $q_j \equiv 13, 17, 19$  or  $23 \pmod{24}$ . Further, let  $t$  denote the number of prime factors of  $n$  (counting multiplicity) that are congruent to 5 or 11  $\pmod{24}$ . Let  $R(n, Q)$  denote the number of representations of  $n$  by the quadratic form  $Q$ .

**Theorem 1.1.** *For all  $n \geq 0$  we have*

$$\bar{p}_o(3n) \equiv f(n)R(n, x^2 + 6y^2) \pmod{3}$$

and

$$\text{ped}(3n + 1) \equiv (-1)^{n+1}R(8n + 3, 2x^2 + 3y^2) \pmod{3},$$

where  $f(n)$  is defined by

$$f(n) = \begin{cases} -1, & n \equiv 1, 6, 9, 10 \pmod{12}, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, we have

$$\bar{p}_o(3n) \equiv f(n)(1 + (-1)^{a+b+t}) \prod_{i=1}^r (1 + v_i) \prod_{j=1}^s \left(\frac{1 + (-1)^{w_j}}{2}\right) \pmod{3} \quad (1.10)$$

and

$$(-1)^n \text{ped}(3n + 1) \equiv \bar{p}_o(48n + 18) \pmod{3}. \quad (1.11)$$

There are many ways to deduce congruences from Theorem 1.1. For example, calculating the possible residues of  $x^2 + 6y^2$  modulo 9 we see that

$$R(3n + 2, x^2 + 6y^2) = R(9n + 3, x^2 + 6y^2) = 0,$$

and then (1.10) implies that  $\bar{p}_o(27n) \equiv \bar{p}_o(3n) \pmod{3}$ . This gives (1.5). The congruences in (1.6) follow from those in (1.5) after replacing  $48n + 18$  by  $3^{2\alpha}(48(3n + 2) + 18)$  and  $3^{2\alpha}(48(9n + 6) + 18)$  in (1.11). We record two more corollaries, which also follow readily from Theorem 1.1.

**Corollary 1.2.** *For all  $n \geq 0$  and  $\alpha \geq 0$  we have*

$$\overline{p}_o(2^{2\alpha}(An + B)) \equiv 0 \pmod{3},$$

where  $An + B = 24n + 9$  or  $24n + 15$ .

**Corollary 1.3.** *If  $\ell \equiv 1, 5, 7$  or  $11 \pmod{24}$  is prime, then for all  $n$  with  $\ell \nmid n$  we have*

$$\overline{p}_o(3\ell^2 n) \equiv 0 \pmod{3}. \tag{1.12}$$

For the functions  $\overline{p}(3n)$  and  $pod(3n + 2)$  we have relations not to binary quadratic forms but to  $r_5(n)$ , the number of representations of  $n$  as the sum of five squares. Our second result is the following.

**Theorem 1.4.** *For all  $n \geq 0$  we have*

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}$$

and

$$pod(3n + 2) \equiv (-1)^n r_5(8n + 5) \pmod{3}.$$

Moreover, for all odd primes  $\ell$  and  $n \geq 0$ , we have

$$\overline{p}(3\ell^2 n) \equiv \left( \ell - \ell \left( \frac{n}{\ell} \right) + 1 \right) \overline{p}(3n) - \ell \overline{p}(3n/\ell^2) \pmod{3} \tag{1.13}$$

and

$$(-1)^{n+1} pod(3n + 2) \equiv \overline{p}(24n + 15) \pmod{3}, \tag{1.14}$$

where  $\left( \frac{\bullet}{\ell} \right)$  denotes the Legendre symbol.

Here we have taken  $\overline{p}(3n/\ell^2)$  to be 0 unless  $\ell^2 \mid 3n$ . Again there are many ways to deduce congruences. For example, (1.7) follows readily upon combining (1.13) in the case  $\ell = 3$  with the fact that

$$r_5(9n + 6) \equiv 0 \pmod{3},$$

which is a consequence of the fact that  $R(9n + 6, x^2 + y^2 + 3z^2) = 0$ . One can check that (1.8) follows similarly. For another example, we may apply (1.13) with  $n$  replaced by  $n\ell$  for  $\ell \equiv 2 \pmod{3}$  to obtain

**Corollary 1.5.** *If  $\ell \equiv 2 \pmod{3}$  is prime and  $\ell \nmid n$ , then*

$$\overline{p}(3\ell^3 n) \equiv 0 \pmod{3}.$$

## 2. PROOFS OF THEOREMS 1.1 AND 1.4

*Proof of Theorem 1.1.* On page 364 of [6] we find the identity

$$\sum_{n \geq 0} \overline{p}_o(3n) q^n = \frac{D(q^3)D(q^6)}{D(q)^2},$$

where

$$D(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}.$$

Reducing modulo 3, this implies that

$$\begin{aligned} \sum_{n \geq 0} \bar{p}_o(3n)q^n &\equiv \sum_{x, y \in \mathbb{Z}} (-1)^{x+y} q^{x^2+6y^2} \pmod{3} \\ &\equiv \sum_{n \geq 0} f(n)R(n, x^2 + 6y^2)q^n \pmod{3}. \end{aligned}$$

Now it is known (see Corollary 4.2 of [3], for example) that if  $n$  has the factorization in (1.9), then

$$R(n, x^2 + 6y^2) = (1 + (-1)^{a+b+t}) \prod_{i=1}^r (1 + v_i) \prod_{j=1}^s \left( \frac{1 + (-1)^{w_j}}{2} \right). \quad (2.1)$$

This gives (1.10). Next, from [1] we find the identity

$$\sum_{n \geq 0} ped(3n + 1)q^n = \frac{D(q^3)\psi(-q^3)}{D(q)^2},$$

where

$$\psi(q) := \sum_{n \geq 0} q^{n(n+1)/2}.$$

Reducing modulo 3, replacing  $q$  by  $-q^8$  and multiplying by  $q^3$  gives

$$\sum_{n \geq 0} (-1)^{n+1} ped(3n + 1)q^{8n+3} \equiv \sum_{n \geq 0} R(8n + 3, 2x^2 + 3y^2)q^{8n+3} \pmod{3}.$$

It is known (see Corollary 4.3 of [3], for example) that if  $n$  has the factorization given in (1.9), then

$$R(n, 2x^2 + 3y^2) = (1 - (-1)^{a+b+t}) \prod_{i=1}^r (1 + v_i) \prod_{j=1}^s \left( \frac{1 + (-1)^{w_j}}{2} \right).$$

Comparing with (2.1) finishes the proof of (1.11). □

*Proof of Theorem 1.4.* On page 3 of [5] we find the identity

$$\sum_{n \geq 0} \bar{p}(3n)q^n \equiv \frac{D(q^3)^2}{D(q)} \pmod{3}.$$

Reducing modulo 3 and replacing  $q$  by  $-q$  yields

$$\sum_{n \geq 0} (-1)^n \bar{p}(3n)q^n \equiv \sum_{n \geq 0} r_5(n)q^n \pmod{3}.$$

It is known (see Lemma 1 in [4], for example) that for any odd prime  $\ell$  we have

$$r_5(\ell^2 n) = \left( \ell^3 - \ell \binom{n}{\ell} + 1 \right) r_5(n) - \ell^3 r_5(n/\ell^2).$$

Here  $r_5(n/\ell^2) = 0$  unless  $\ell^2 \mid n$ . Replacing  $r_5(n)$  by  $(-1)^n \bar{p}(3n)$  throughout gives (1.13). Now equation (1) of [7] reads

$$\sum_{n \geq 0} (-1)^n pod(3n+2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}.$$

Reducing modulo 3 we have

$$\begin{aligned} \sum_{n \geq 0} (-1)^n pod(3n+2)q^n &\equiv \psi(q)^5 \pmod{3} \\ &\equiv \sum_{n \geq 0} r_5(8n+5)q^n \pmod{3} \\ &\equiv -\sum_{n \geq 0} \bar{p}(24n+15)q^n \pmod{3}, \end{aligned}$$

where the second congruence follows from Theorem 1.1 in [2]. This implies (1.14) and thus the proof of Theorem 1.4 is complete. □

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