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The diagonalizable nonnegative inverse eigenvalue problem

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Abstract

In this article we provide some lists of real numbers which can be realized as the spectra of nonnegative diagonalizable matrices but which are not the spectra of nonnegative symmetric matrices. In particular, we examine the classical list $\sigma = (3 + t, 3 - t, -2, -2, -2)$ with $t \geq 0$, and show that σ is realizable by a nonnegative diagonalizable matrix only for $t \geq 1$. We also provide examples of lists which are realizable as the spectra of nonnegative matrices, but not as the spectra of nonnegative diagonalizable matrices by examining the Jordan Normal Form.

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1 Introduction

Classifying spectra of nonnegative matrices is known as the *nonnegative inverse eigenvalue problem* (or NIEP). The *real nonnegative inverse eigenvalue problem* (or RNIEP) is to determine necessary and sufficient conditions on the list σ of n real numbers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ so that σ is the spectrum of an entry-wise $n \times n$ nonnegative matrix A . If there exists such a nonnegative matrix A with spectrum σ , the list σ is said to be realizable or we say the matrix A *realizes* σ . If we further require that the realizing matrix be symmetric we call the problem the *symmetric nonnegative inverse eigenvalue problem* (or SNIEP). In a similar vein, the *diagonalizable (real) nonnegative inverse eigenvalue problem* (or D-(R)NIEP)

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requires that the realizing matrix be diagonalizable. If σ is symmetrically realizable, then, since every symmetric matrix is diagonalizable, it follows that σ is D-RNIEP realizable. Many examples in which realizable lists are not diagonalizably realizable are known [12]. Naturally, one may ask whether every D-RNIEP realizable list is symmetrically realizable. This is the main thrust of this work and an answer in the negative is given.

These problems are unsolved for $n \geq 5$ though the trace zero case for 5×5 matrices was solved (i) in the case of NIEP by Laffey and Meehan [16] and (ii) in the case of SNIEP by Spector [30]. The RNIEP and SNIEP are the same for $n \leq 4$ but are different for $n \geq 5$ as was first shown by Johnson, Laffey and Loewy in [9]. However, the NIEP in which we may augment the list σ by adding an arbitrary number N of zeros was solved theoretically by Boyle and Handelman [1] and a constructive version was found by Laffey [14]. For special cases that bound the size of N we refer the reader to [3], [18], [19]

The first significant result on NIEP was Suleimanova's result [32] on lists of real numbers with just one positive number, which says that the real list $\lambda_1 > 0 \geq \lambda_2 \geq \dots \geq \lambda_n$ is realizable if and only if $\lambda_1 + \lambda_2 + \dots + \lambda_n \geq 0$. The question of realizing real lists of five or more numbers containing just two positive numbers is still unsolved in general ([13], [2]).

1.1 Necessary conditions

The Perron-Frobenius theorem ([27],[6]) says (among other things) that the spectral radius of an irreducible nonnegative matrix must be contained in the spectrum of that matrix i.e.

$$\max\{|\lambda_j| : \lambda_j \in \sigma\} \in \sigma.$$

In this context the spectral radius ρ is known as the *Perron root*, and for irreducible nonnegative matrices $\rho > 0$, and it occurs just once as an eigenvalue. We define the Newton power sums s_k as follows:

$$s_k := \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k, \quad \text{for } k = 1, 2, \dots$$

Notice that if $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is the spectrum of a nonnegative matrix A then the power sum s_k is also the trace of the k^{th} power of a realizing matrix A for σ . Independently Loewy and London [23] and Johnson [9] derived an infinite set of inequalities which the

spectrum of a nonnegative matrix must satisfy, namely that

$$n^{m-1}s_{km} \geq s_k^m \text{ for } k, m = 1, 2, \dots$$

known as the JLL conditions. Necessary conditions for realizability in both the RNIEP and the SNIEP thus include:

$$\max\{|\lambda_j| : \lambda_j \in \sigma\} \in \sigma \tag{1}$$

$$s_k \geq 0 \text{ for } k = 1, 2, \dots \tag{2}$$

$$n^{m-1}s_{km} \geq s_k^m \text{ for } k, m, n = 1, 2, \dots \tag{3}$$

A new necessary condition for the SNIEP when $n = 5$ and when the trace is at least half the spectral radius is given by Loewy and Spector in [31]. A necessary condition for NIEP for general n involving only the first three Newton power sums s_k is given by Cronin and Laffey in [4].

2 A classic example $\sigma = \{3, 3, -2, -2, -2\}$

The list $\sigma = \{3, 3, -2, -2, -2\}$, in the guise $\tau = (1, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3})$ was first studied by Salzman in 1971 [28] and Friedland in 1977 [5]. As can be checked the list τ satisfies the necessary conditions (1), (2) and (3) for all positive integers k, m and n .

It is well known however that the list τ is not realizable. Paparella and Taylor [26] prove that a more general result generalizes lists like τ .

Laffey and Meehan in [17] showed that in order for a list of five numbers which sum to zero to be realizable, a refined JLL inequality must be satisfied, namely $4s_4 - s_2^2 \geq 0$. For σ (which is 3τ) we have $4s_4 - s_2^2 = 840 - 900 < 0$ and so again we see that a small perturbation of σ cannot be realizable.

We define

$$s_j := (3 + t)^j + 3^j + 3(-2)^j \text{ for } j = 1, 2, \dots$$

2.1 $\sigma_t = \{3 + t, 3, -2, -2, -2\}$

A result of Guo ([7], Theorem 2.1) implies that there is a minimum $t > 0$ for which $\sigma_t = \{3 + t, 3, -2, -2, -2\}$ is realizable. However determining the least positive t for which σ_t is realizable is not yet solved.

In her thesis, Meehan [25] showed that σ_t is realizable for $t \geq 0.519310982048\dots$ and a realizing matrix of the form

$$A = \begin{bmatrix} t & 1 & 0 & 0 & 0 \\ p & 0 & 1 & 0 & 0 \\ 0 & q & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & w & h & 0 \end{bmatrix}$$

is presented. She also shows that for a 5×5 extreme (or Perron extreme) matrix [15], $4s_4 - s_2^2 + s_1^2s_2 - \frac{s_1^4}{2} \geq 0$ which for σ_t , means $t \geq 0.39671\dots$.

Hence the range for the minimum value of t for which σ_t is realizable is

$$0.39671\dots \leq t \leq 0.51931\dots$$

but its precise value is not yet determined. This example highlights the difficulty in moving from the NIEP for $n = 5$ and trace zero to the positive trace case in the NIEP, even when the numbers of the list are all real.

2.1.1 $\sigma_t = \{3 + t, 3, -2, -2, -2\}$ in the symmetric case

We note that for $t = 1$, $\sigma_t = (3 + t, 3, -2, -2, -2)$, is *symmetrically* realizable by the matrix

$$A = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{6} \\ 0 & 0 & 0 & \sqrt{6} & 0 \end{bmatrix}$$

and this is the best possible t in the symmetric case. Thus the RNIEP and the SNIEP are different for $n = 5$, and this is the first case in which they differ [9]. The interested

reader should consult Loewy and London [23], for the proof that the RNIEP=SNIEP for $n \leq 4$.

2.2 $\hat{\sigma}_t = \{3 + t, 3 - t, -2, -2, -2\}$

A further related problem of finding the minimum value of $t > 0$ for which the list $\hat{\sigma}_t = (3 + t, 3 - t, -2, -2, -2)$ is realizable, is solved. Note that the sum of the elements in $\hat{\sigma}_t$ is zero and so any realizing matrix for $\hat{\sigma}_t$ must have trace zero. Also note, that any realizing matrix for $\hat{\sigma}_t$ must be irreducible.

The minimum value of $t > 0$ for which $\hat{\sigma}_t = (3 + t, 3 - t, -2, -2, -2)$ is realizable in the *symmetric* case must be at least one. For a list of five real numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$, McDonald and Neumann [24] showed that

$$\lambda_2 + \lambda_5 \leq \text{trace}(A)$$

where A is a symmetric realizing matrix. Hence for $\hat{\sigma}_t$ we must have that $(3-t) + (-2) \leq 0$ which implies $t \geq 1$. For $t = 1$, $\hat{\sigma}_t = (3 + t, 3 - t, -2, -2, -2)$ is symmetrically realizable by

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

and so $t = 1$ is the best possible result in the symmetric case for $\hat{\sigma}_t$. This was first proven by Loewy and Hartwig (unpublished).

2.2.1 $\hat{\sigma}_t = \{3 + t, 3 - t, -2, -2, -2\}$ in the general case

However in the general case as noted earlier, Laffey and Meehan show that $\hat{\sigma}_t$ must satisfy the necessary condition

$$4s_4 \geq s_2^2.$$

This requires that

$$t \geq t_0 := \sqrt{16\sqrt{6} - 39} = 0.43799 \dots$$

and they show that the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{15+t^2}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3t^4 + 58t^2 + 3 & \frac{t^4+78t^2-15}{4} & 10 + 6t^2 & \frac{15+t^2}{2} & 0 \end{bmatrix}$$

is nonnegative for $t \geq t_0$ and has spectrum $(3+t, 3-t, -2, -2, -2)$. Also note, that by the result of Guo [7], the list $(3 + \hat{t}, 3, -2, -2, -2)$ is realizable for all $\hat{t} \geq 2t_0 = 0.87598 \dots$, but that this is weaker than the bound cited in section 2.1 above.

3 D-RNIEP \neq SNIEP

Next we examine the subtle difference between the SNIEP and the Diagonalizable RNIEP or D-RNIEP, where the D-RNIEP is the problem of finding a nonnegative *diagonalizable* matrix realizing a given real spectrum σ .

Again we consider the list $\hat{\sigma}_t = (3 + t, 3 - t, -2, -2, -2)$, for $t > 0$. We note that for

$$t \geq t_0 = \frac{1}{10} \sqrt{120\sqrt{1066} - 3899} = 0.4354153419 \dots$$

the perturbed list $(3 + t, 3 - t, -1.9, -2, -2.1)$ is the spectrum of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1501+100t^2}{200} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{15000t^4+289950t^2+14649}{5000} & \frac{10000t^4+779800t^2-148199}{40000} & \frac{150t^2+249}{25} & \frac{1501+100t^2}{200} & 0 \end{bmatrix}.$$

The (5, 2) entry of the matrix A is nonnegative if $10000t^4 + 779800t^2 \geq 148199$ and this holds for $t \geq t_0$. Also note that for $t < 4.9$ this matrix is *diagonalizable* as it has five distinct eigenvalues.

However if the list $(3 + t, 3 - t, -1.9, -2, -2.1)$ is to be symmetrically realizable by a matrix A we can use the argument of McDonald and Neumann [24] which yields $t \geq 0.9$. Hence the two problems of D-RNIEP and SNIEP are different at least in this case (see

also the recent work on the diagonalizable real nonnegative inverse eigenvalue problem in [10], [11], [12] and [22]).

We also note that for sufficiently small $\epsilon > 0$, by continuity, the spectrum $(3 + t, 3 - t, -2 + \epsilon, -2, -2 - \epsilon)$ is diagonalizably realizable (five distinct eigenvalues) for t close to $\sqrt{16\sqrt{6} - 39} = 0.43799\dots$.

However this is not a continuous property in ϵ since we have the following:

Proposition 1. *If $\hat{\sigma}_t = (3 + t, 3 - t, -2, -2, -2)$ is realizable by a diagonalizable matrix A , then $t \geq 1$.*

To prove this result we will make use of the following result of Schur ([8], 0.8.5):

Lemma 2. *Let*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

be an $n \times n$ matrix.

If $\text{rank}(B) = \text{rank}(B_{11}) = k$ and B_{11}^{-1} exists, then $B_{22} = B_{21}B_{11}^{-1}B_{12}$.

Proof. Note that

$$\begin{aligned} \begin{bmatrix} I & 0 \\ -B_{21}B_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I & -B_{11}^{-1}B_{12} \\ 0 & I \end{bmatrix} &= \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{bmatrix} \begin{bmatrix} I & -B_{11}^{-1}B_{12} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{bmatrix}, \end{aligned}$$

where B_{22} is the Schur complement of B_{11} in B and where I and 0 are the identity and zero matrices of appropriate dimensions respectively.

Since $\text{rank}(B) = \text{rank}(B_{11}) = k$, it follows that $\text{rank}(B_{22} - B_{21}B_{11}^{-1}B_{12}) = 0$,

so $B_{22} = B_{21}B_{11}^{-1}B_{12}$, as claimed. □

Proof of Proposition 1

Proof. Suppose $0 < t < 1$, and that $\hat{\sigma}_t = (3 + t, 3 - t, -2, -2, -2)$ is realized by the nonnegative diagonalizable matrix A . Then A is irreducible.

Let $w^T > 0$ be the left eigenvector associated with the Perron eigenvalue $3 + t$, so that $w^T A = (3 + t)w^T$, where T denotes the transpose.

Let v be the right eigenvector associated with the eigenvalue $3 - t$, so $Av = (3 - t)v$.

Then $(3+t)w^T v = w^T Av = w^T(3-t)v$ which implies $w^T v = 0$ and so v must have some negative components. We now show that v cannot have just one positive (or negative) component. First note that v cannot have all its entries positive (or negative) as only the Perron eigenvector (and its scalar multiples) has this property. Thus v (or $-v$) has either one or two positive entries (and hence either four or three non-positive entries respectively). Without loss of generality we can permute the entries of v so that its positive entries, denoted with a $+$ symbol below, occur in the first position(s) of the vector v i.e.

$$v = \begin{bmatrix} + \\ - \\ - \\ - \\ - \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} + \\ + \\ - \\ - \\ - \end{bmatrix}$$

where the minus sign means the corresponding entry is less than or equal to zero.

Now if v has just one positive entry then $Av = (3-t)v$ implies

$$\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (3-t) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where A_{12} is (1×4) , A_{21} is (4×1) , and A_{22} is (4×4) , and $v = [v_1 \ v_2]^t$ where $v_1 > 0$ and $v_2 \leq 0$, for $v_1 \in \mathbb{R}$, and $v_2 \in \mathbb{R}^4$. But this implies $A_{12}v_2 = (3-t)v_1$ which is false as the left hand side of this equation is non-positive and the right hand side is positive.

Hence this sign pattern (one positive component followed by four non-positive) for v is not permitted. So v must have 2 positive entries and 3 non-positive entries.

Hence we can write $Av = (3-t)v$ as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = (3-t) \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

where A_{11} is (2×2) , A_{22} is (3×3) , and $v_1, v_2 \geq 0$ are column vectors in \mathbb{R}^2 and \mathbb{R}^3 respectively. Then, using a corollary of the Courant-Fischer argument (see Theorem 8.3.2 of [8]), the Perron eigenvalue of A_{11} is at least $3-t$ and hence A_{11} has spectrum $\pm(3-t+\epsilon)$ where $\epsilon \geq 0$, since $\text{tr}(A_{11}) = 0$. Since A is diagonalizable the minimum polynomial of A

is

$$m_A(x) = (x - (3 + t))(x - (3 - t))(x + 2)$$

so that

$$(A - (3 + t)I)(A - (3 - t)I)(A + 2I) = 0, \text{ which implies}$$

$$(A^2 - 6A + (9 - t^2)I)(A + 2I) = 0, \text{ and so}$$

$$A^3 - 4A^2 - (3 + t^2)A + (18 - 2t^2)I = 0.$$

Hence we have that

$$A^3 + (18 - 2t^2)I = 4A^2 + (3 + t^2)A. \quad (4)$$

Now A diagonalizable also implies there exists a nonsingular matrix T with

$$T^{-1}(A + 2I_5)T = (5 + t) \oplus (5 - t) \oplus (0_3).$$

Now $\text{rank}(A + 2I) = 2$ since $\text{rank}(A_{11} + 2I_2) = 2$.

Note that A_{11} has trace zero and so $A_{11} + 2I$ has the form

$$\begin{bmatrix} 2 & a_{12} \\ a_{21} & 2 \end{bmatrix}.$$

So $\det(A_{11} + 2I) = 4 - a_{12}a_{21}$

Since $\text{rank}(A + 2I) = 2$ we have that $\text{rank}(A_{22} + 2I_3) \leq 2$ since the rank of a leading principal submatrix cannot exceed the rank of the original matrix. Thus $A_{22} + 2I_3$ has zero as an eigenvalue (as it does not have full rank) and hence -2 is an eigenvalue of A_{22} .

Applying the Courant-Fischer argument once more we have that $A_{21}z_1 - A_{22}z_2 = -(3 - t)z_2$ so that $(A_{22} - (3 - t)I)z_2 = A_{21}z_1 \geq 0$ and so A_{22} has an eigenvalue $3 - t + \epsilon'$ where $\epsilon' \geq 0$.

Since $\text{tr}(A_{22}) = 0$, A_{22} has eigenvalues $3 - t + \epsilon'$, $-1 + t - \epsilon'$, -2 .

Now A^2 and A^3 have the form

$$A^2 = \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix},$$

$$\begin{aligned}
A^3 &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix} \\
&= \begin{pmatrix} A_{11}^3 + A_{11}A_{12}A_{21} + A_{12}(A_{21}A_{11} + A_{22}A_{21}) & * \\ ** & A_{21}(A_{11}A_{12} + A_{12}A_{22}) + A_{22}^3 + A_{22}A_{21}A_{12} \end{pmatrix}
\end{aligned}$$

where * and ** do not contribute to the following calculations.

Let

$$\alpha = \text{tr}(A_{12}A_{21}) = \text{tr}(A_{21}A_{12})$$

$$\beta = \text{tr}(A_{11}A_{12}A_{21}) \quad \text{and}$$

$$\gamma = \text{tr}(A_{12}A_{22}A_{21}).$$

Using (4) we see that the contribution to $\text{tr}(I)$, $\text{tr}(A)$, $\text{tr}(A^2)$ and $\text{tr}(A^3)$ from positions (1, 1) and (2, 2) yields that

$$4(2(3 - t + \epsilon)^2 + \alpha) = 2\beta + \gamma + 2(18 - 2t^2). \quad (5)$$

Applying Lemma 2 to

$$A + 2I = \begin{bmatrix} A_{11} + 2I_2 & A_{12} \\ A_{21} & A_{22} + 2I_3 \end{bmatrix}$$

we get that $A_{22} + 2I_3 = A_{21}(A_{11} + 2I_2)^{-1}A_{12}$.

Now $(A_{11} + 2I_2)^{-1}$

$$\begin{aligned}
&= \frac{1}{\det(A_{11} + 2I_2)} \begin{bmatrix} 2 & -a_{12} \\ -a_{21} & 2 \end{bmatrix} \\
&= \frac{-1}{(5 - t + \epsilon)(1 - t + \epsilon)} \begin{bmatrix} 2 & -a_{12} \\ -a_{21} & 2 \end{bmatrix} \\
&= \frac{1}{(5 - t + \epsilon)(1 - t + \epsilon)} \begin{bmatrix} -2 & a_{12} \\ a_{21} & -2 \end{bmatrix}
\end{aligned}$$

$$= \frac{(A_{11} - 2I_2)}{(5 - t + \epsilon)(1 - t + \epsilon)}.$$

So $A_{22} + 2I_3 = \frac{A_{21}(A_{11} - 2I_2)A_{12}}{(5 - t + \epsilon)(1 - t + \epsilon)}$. Note that $\text{tr}(A_{22}) = 0$ so upon comparing traces in this equation we get that

$$\begin{aligned} 6 &= \frac{\text{tr}(A_{21}A_{11}A_{12}) - 2\text{tr}(A_{21}A_{12})}{(5 - t + \epsilon)(1 - t + \epsilon)} \\ &= \frac{\beta - 2\alpha}{(5 - t + \epsilon)(1 - t + \epsilon)}. \end{aligned}$$

Now (5) gives

$$2\beta + \gamma + 36 - 4t^2 = 8(3 - t + \epsilon)^2 + 4\alpha.$$

This implies

$$12(5 - t + \epsilon)(1 - t + \epsilon) + \gamma + 36 - 4t^2 = 72 - 48t + 8t^2 + 16\epsilon(3 - t) + 8\epsilon^2$$

which yields

$$24 + 24\epsilon + 4\epsilon^2 + \gamma - (24t + 8\epsilon t) = 0.$$

But note that $24 - 24t > 0$ since $t < 1$, and $24\epsilon - 8\epsilon t = 16\epsilon + 8\epsilon - 8\epsilon t > 0$, implies that the left hand side of this equation is positive and the right hand side is zero which contradicts our hypothesis that $t < 1$. Hence $t \geq 1$. \square

Laffey and Smigoc [20] proved that $(3 + t, 3 - t, -2, -2, -2, 0)$ is symmetrically realizable by a 6×6 matrix for $t \geq \frac{1}{3}$ and we conjecture that $t = \frac{1}{3}$ is the best bound for any number of zeros added to the spectrum $\widehat{\sigma}_t$.

4 A note on the dependence of realizable spectra on the Jordan Normal Form structure

The matrix

$$A = \frac{1}{4} \begin{bmatrix} 0 & 8 & 1 & 0 & 0 \\ 8 & 0 & 1 & 0 & 0 \\ \frac{75}{2} & \frac{75}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 829 & 829 & 256 & 110 & 0 \end{bmatrix}$$

has spectrum $\sigma_{\frac{3}{4}} = (3 + \frac{3}{4}, 3 - \frac{3}{4}, -2, -2, -2)$ and minimal polynomial

$$\left(x - \left(3 + \frac{3}{4}\right)\right) \left(x - \left(3 - \frac{3}{4}\right)\right) (x + 2)^2$$

and so it distinguishes the diagonalizably realizable lists from those with Jordan Canonical Form structure

$$(3 + t) \oplus (3 - t) \oplus (-2) \oplus \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

with $t = \frac{3}{4}$. Hence the D-RNIEP is different to the general RNIEP. Note that this matrix was built up from a 4×4 nonnegative matrix with spectrum $(3 + \frac{3}{4}, 3 - \frac{3}{4}, -2, -2)$ having the entry 2 on the diagonal using the Šmigoc methods deployed in [29]. The question of whether every realizable spectrum can be realized by a nonderogatory matrix is open and will require further ideas related to those developed in this paper.

5 Conclusion

In this article we proved that the SNIEP \neq D-RNIEP and that the D-NIEP can be distinguished from the general NIEP by examining the Jordan Normal Form. To prove the main result (Proposition 1) we used a result of Schur (Lemma 2) and a necessary condition due to McDonald and Neumann ($\lambda_2 + \lambda_5 \leq \text{tr}A$) and a result of Courant-Fischer.

We also showed that the minimum $t > 0$ for which the classical list $(3+t, 3-t, -2, -2, -2)$ is realizable by a diagonalizable matrix is $t = 1$.

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