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The Wolff hull of a compact holomorphic self map on an infinite dimensional ball

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Abstract

For large classes of (finite and) infinite dimensional complex Banach spaces Z, B its open unit ball and $f : B \to B$ a compact holomorphic fixed-point free map, we introduce and define the *Wolff hull*, $W(f)$, of f in ∂B and prove that $W(f)$ is proximal to the images of all subsequential limits of the sequences of iterates $(f^n)_n$ of f. The Wolff hull generalises the concept of a Wolff point, where such a point can no longer be uniquely determined, and coincides with the Wolff point if Z is a Hilbert space.

Recall that $(f^n)_n$ does not generally converge even in finite dimensions, compactness of f (i.e. $f(B)$ is relatively compact) is necessary for convergence in the infinite dimensional Hilbert ball and all accumulation points $\Gamma(f)$ of $(f^n)_n$ map B into ∂B (for any topology finer than the topology of pointwise convergence on B). The target set of f, is

$$T(f) = \bigcup_{g \in \Gamma(f)} g(B).$$

To locate $T(f)$ we use a concept of closed convex holomorphic hull, $\mathrm{Ch}(x) \subset \partial B$ for each $x \in \partial B$ and define a distinguished Wolff hull $W(f)$. We show that the Wolff hull intersects all hulls from $T(f)$, namely,

$$W(f) \cap \mathrm{Ch}(x) \neq \emptyset \quad \text{for all} \quad x \in T(f).$$

If B is the Hilbert ball, $W(f)$ is the Wolff point and this is the usual Denjoy-Wolff result. Our results are for all reflexive Banach spaces having a homogeneous ball (or equivalently, for all finite rank JB^* triples). These include many well-known operator spaces, for example, $L(H,K)$, where either H or K is finite dimensional.

Introduction

The behaviour of the sequence of iterates $(f^n)_n$ of a holomorphic self-map f depends largely on its domain of definition. We are interested here in the large class of domains that can be realised as homogeneous balls, namely the open unit ball of Banach spaces

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known as the JB^*-triples, including the balls of all Hilbert spaces, C^*-algebras, JB^*-algebras and more.

Classical results from the 1920s [9, 31, 32] prove that for holomorphic $f : \Delta \to \Delta$ without fixed point in Δ, the iterates (f^n) converge to a constant ξ, the *Wolff point*, in $\partial\Delta$. While four decades later this was extended to the finite dimensional Hilbert ball [15], it does not extend to the infinite dimensional Hilbert ball [30]. On the other hand, Chu and Mellon [8] prove the result for compact f (i.e. $f(B)$ is relatively compact) on an infinite dimensional Hilbert ball, by intrinsic use of its strong convexity. Since then many authors [1, 2, 3, 5, 4, 17, 18, 16, 6, 12, 29] have studied iterates of maps on infinite dimensional balls possessing extra convexity properties. That approach fails however if the ball is not strongly convex, even, for example, in the bidisc [8]. More importantly, since the ball B of a JB*-triple is strongly convex if, and only if, the triple is already a Hilbert space, none of those approaches usefully generalise here.

JB*-triples do, however, possess a triple algebraic structure which encodes the holomorphic structure of the ball. As (f^n) does not generally converge, our goal is to locate cluster points h of (f^n) and their images $h(B) \subseteq \partial B$. It is recently proved in [23] that the topology of pointwise convergence on B coincides on $\{f^n : n \in \mathbb{N}\}$ with the topology of local uniform convergence (and more). Therefore we may not explicitly mention the topology hereafter but assume it is any topology on $H(B, Z)$, the holomorphic maps from B to Z, that is finer than the topology of pointwise convergence on B [23, Corollary 3.9].

Henceforth Z denotes a complex Banach space known as a JB*-triple, B its open unit ball and $f : B \mapsto B$ a compact holomorphic map with no fixed point in B. For example, B might be the open unit ball of $\mathcal{L}(H, \mathbb{C}^n)$, where H is any Hilbert space.

Since holomorphic functions $h : B \to B$ contract the Kobayashi distance κ, cf. [10], we have

$$\kappa(h(z), h(w)) \leq \kappa(z, w), \text{for all } z, w, \in B.$$

In particular, if h has a fixed point $z = h(z)$, then

$$\kappa(z, w) < r \Longrightarrow \kappa(z, h(w)) < r$$

implies that every Kobayashi ball centred at z is h-invariant. Kobayashi balls on a triple were shown in [25, Prop 2.3] to have a simple description as operator balls, $c + T(B)$, where T is a linear (Bergman) operator and $c \in B$.

Although we are interested here in a function f which has no fixed point in B, for a sequence $(\alpha_k), 0 < \alpha_k < 1, \lim_k \alpha_k = 1$, each scaled map $f_k = \alpha_k f$ has a fixed point z_k in B by the Earle-Hamilton theorem [11] (see also [13, p. 96]). From the above therefore, every Kobayashi ball centred at z_k, described as $D_k := c_k + T_k(B)$, is f_k-invariant. Compactness of f implies that, passing to a subsequence if necessary, z_k converges to a point ξ which must lie in ∂B as f has no fixed point in B. One can then examine the limits, c of c_k in Z and T of T_k in $L(Z)$, to produce a limiting domain $E := c + T(B)$ in B which is invariant for f. We produce limiting domains E_λ of all "sizes" λ.

While the arguments and tools involved in calculating the domains E_λ are quite subtle and require extensive use of Jordan theory, the overall process is nonetheless similar in form to the original arguments for the disc Δ. Indeed, the limiting domains E_λ are exact infinite dimensional analogues of the invariant horocycles in Δ (see [7] for a summary of the classical case). For details we refer to [25], [22] and [28]. We recall that B is homogeneous in a reflexive Banach space, if and only if, Z is a finite rank JB*-triple.

In [22], the parametrised family of f-invariant domains, E_λ, $\lambda > 0$, is used to locate the constant subsequential limits of the iterates of f. There, the focus on limits which are constant functions is in the spirit of the original Denjoy-Wolff theorem and its generalisation to the strictly convex setting of a Hilbert ball in finite [15] or infinite dimensions [8]. Here however, our aim is to use the f-invariant domains E_λ to locate the set of *all* accumulation points $\Gamma(f)$ of $(f^n)_n$ (and their images in $\overline{B}$). It is already known (see [17, Theorem 2.5]) that if f is fixed point free and $g \in \Gamma(f)$ then $g(B)$ must lie on ∂B. The concept of holomorphic boundary component [20] has a concrete algebraic realisation for JB*-triple balls and for each $g \in \Gamma(f)$ there is (a unique tripotent) $d \in \partial B$ such that $g(B)$ lies in the single boundary component K_d.

If we let

$$\mathrm{Ch}(d) := \overline{K_d},$$

this set then acts as a closed convex holomorphic hull of d in the boundary of B. In finite dimensions, $\mathrm{Ch}(d)$ is shown in [26, Proposition 3.2] to coincide with a convex hull used in [3] defined in terms of complex supporting hyperplanes.

The behaviour of the f-invariant domains E_λ as $\lambda \to 0$, or equivalently, as E_λ approaches the boundary, is our best clue to determining where on the boundary we should focus our search for images of accumulation points of $(f^n)_n$. This is best stated in terms of the target set, $T(f)$, of f, given by,

$$T(f) = \bigcup_{g\in\Gamma(f)} g(B) \subseteq \partial B.$$

We begin by extending a finite dimensional result [26, Prop. 2.6, Cor. 2.10] to show (Corollary 2.7 below) that there exists (a tripotent) $\widetilde{e}$ in ∂B such that

$$\bigcap_\lambda E_\lambda = \emptyset \quad \text{and} \quad \bigcap_\lambda \overline{E_\lambda} = \mathrm{Ch}(\widetilde{e}). \tag{1}$$

In the Hilbert ball, $\mathrm{Ch}(\widetilde{e})$ coincides with the Wolff point ξ. In the general setting, $\bigcap_\lambda \overline{E_\lambda} = \mathrm{Ch}(\widetilde{e})$ shows $\mathrm{Ch}(\widetilde{e})$ to be the smallest possible replacement for the (generally no longer unique) Wolff point. As such, we will refer to $\mathrm{Ch}(\widetilde{e})$ as the (closed convex) *Wolff hull* of f, denoted by $W(f)$.

Our main result then shows proximity of $T(f)$ to $W(f)$.

0.1 Theorem. *For all $x \in T(f)$,*

$$W(f) \cap \mathrm{Ch}(x) \neq \emptyset.$$

We remark that this reproduces the classical Denjoy-Wolff theorem in the case of the Hilbert ball since, there, $W(f) = \{\xi\} \in \mathrm{Ch}(x) = \{x\}$ for all $x \in T(f)$ giving $T(f) = \{\xi\}$. In particular, all subsequential limits of $(f^n)_n$ coincide with ξ and hence $(f^n)_n$ itself converges to ξ.

1 Notation and background

Throughout, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For X and Y complex Banach spaces, $\mathcal{L}(X, Y)$ denotes the space of continuous linear maps from X to Y, $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\mathsf{GL}(X)$ is all invertible elements in $\mathcal{L}(X)$. For domains $D \subset X$ and $\widetilde{D} \subset Y$ we denote the set of all holomorphic maps from D to $\widetilde{D}$ by $H(D, \widetilde{D})$, with $H(D) = H(D, D)$. For $f \in H(D)$, the iterates of f are $f^n := f \circ f^{n-1}, n \in \mathbb{N}, n > 1$ and $f^1 = f$.

We will study compact ($f(B)$ is relatively compact) holomorphic maps on the open unit ball, B, of a complex Banach space in infinite dimensions and use $\Gamma(f)$ for the set of accumulation points of $(f^n)_n$ with respect to the topology of local uniform convergence on B. Elements of $\Gamma(f)$ are holomorphic maps from B to $\overline{B}$ and compactness of f implies $\Gamma(f) \neq \emptyset$ [8, Lemma 1]. The following result shows however that, in regard to $(f^n)_n$, we can replace the topology of local uniform convergence by any topology finer than the topology of pointwise convergence on B.

1.1 Theorem. *[22, Corollary 3.10, Remark 3.7] Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Let ν be any topology on $H(B, Z)$ finer than the topology of pointwise convergence on B and let $(f^{n_\alpha})_\alpha$ be any subnet of the sequence of iterates. Then the following are equivalent.*

(i) $(f^{n_\alpha})_\alpha$ is pointwise convergent;

(ii) $(f^{n_\alpha})_\alpha$ is ν convergent;

(iii) $(f^{n_\alpha})_\alpha$ is locally uniformly convergent.

Morover, the set $\Gamma(f)$ of all accumulation points of $(f^n)_n$ coincides with the set of all subsequential limits of (f^n) (for all the above topologies).

1.1 JB*-triples

Every homogeneous open unit ball is biholomorphically equivalent to a bounded symmetric domain and bounded symmetric domains are classified [19] as the open unit balls of JB^*-triples. JB^*-triples include all C^*-, JB^*- and J^*-algebras.

1.2 Definition. A JB*-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x, y, z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.

(ii) The map $z \mapsto \{x, x, z\}$, denoted $x \,\Box\, x$, is Hermitian, $\sigma(x \,\Box\, x) \geq 0$ and $\|x \,\Box\, x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum.

(iii) The product satisfies the following "triple identity"

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

The triple product satisfies $\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|$ so is continuous and gives rise to the linear maps: $x \,\Box\, y \in \mathcal{L}(Z) : z \mapsto \{x, y, z\}$, $Q_x \in \mathcal{L}_{\mathbb{R}}(Z) : z \mapsto \{x, z, x\}$, and the geometrically significant Bergman operators $B(x, y) = I - 2x \,\Box\, y + Q_x Q_y \in \mathcal{L}(Z)$.

1.2 Tripotents and Ordering

Tripotents here replace idempotents for an algebra, namely, $e \in Z$ is a tripotent if $\{e, e, e\} = e$. Every tripotent e induces a splitting of Z, as $Z = Z_0(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_1(e)$, where $Z_k(e)$ is the k eigenspace of $e \Box e$ and the linear maps $P_0(e) = B(e, e)$, $P_{\frac{1}{2}}(e) = 2(e \,\Box\, e - Q_e Q_e)$, and $P_1(e) = Q_e Q_e$ are mutually orthogonal projections of Z onto $Z_0(e)$, $Z_{\frac{1}{2}}(e)$, and $Z_1(e)$ respectively. $Z_0(e)$ and $Z_1(e)$ are themselves triples whose open unit balls,

$B_0(e)$ and $B_1(e)$, are therefore bounded symmetric domains.
Elements $x, y \in Z$ are orthogonal, $x\perp y$, if $x\Box y = 0$ (or equivalently [21] if $y\Box x = 0$). In particular, if c and e are orthogonal tripotents then $c + e$ is also a tripotent, giving a natural partial order on the set, M, of all tripotents in Z as follows.

1.3 Definition. For tripotents c and e we say $c < e$ if $0 \neq e - c \in M$ and $(e - c)\perp c$.

Then e is maximal if $Z_0(e) = 0$ and e is minimal if $Z_1(e) = \mathbb{C}e$. Z is said to have finite rank r if every element $z \in Z$ is contained in a subtriple of (complex) dimension $\leq r$, and r is minimal with this property. The rank 1 triples are the Hilbert spaces. If Z has finite rank r, a frame is a set $\{e_1, \ldots, e_r\}$ of non-zero pairwise orthogonal minimal tripotents and $z \in Z$ has a unique spectral decomposition, called its Peirce decomposition, as $z = \lambda_1 e_1 + \cdots + \lambda_r e_r$, for some frame $\{e_1, \ldots, e_r\}$ and scalars $0 \leq \lambda_1 \leq \ldots \leq \lambda_r = \|z\|$. Real and complex extreme points of $\overline{B}$ coincide with the set of maximal tripotents. For details see [21].

1.3 Boundary structure of bounded symmetric domains

Let E be a complex Banach space with open unit ball B_E.

1.4 Definition. $A \subset \overline{B}_E$, $A \neq \emptyset$ is a holomorphic boundary component of B_E if A is minimal with respect to the fact that either $f(\Delta) \subset A$ or $f(\Delta) \subset \overline{B}_E \setminus A$, for all $f \in \mathcal{F} = \{f : \Delta \to Z$ holomorphic with $f(\Delta) \subset \overline{B}_E\}$.

These holomorphic boundary components form a partition of $\overline{B_E}$ and the holomorphic boundary component of B_E containing $a \in \overline{B_E}$ is written K_a. By replacing $\mathcal{F}$ in the above definition with the set of all complex (real) affine maps : $\Delta \to \overline{B}$ we get the definition of complex (real) affine boundary component. For Z a finite rank JB^*-triple holomorphic and affine boundary components coincide (we refer simply to boundary components) and can be described in terms of tripotents as follows.

1.5 Theorem. *([21, Theorem 6.3], [20, Proposition 4.3])*
Let Z be a JB^*-triple with open unit ball B. The following assertions hold.

(i) *Holomorphic and affine boundary components coincide and are precisely the sets*

$$K_e = e + B_0(e)$$

where e is a tripotent and $B_0(e) = B \cap Z_0(e)$ is the bounded symmetric domain associated with the triple $Z_0(e)$. Moreover, the map $e \to K_e$ is a bijection between the set, M, of tripotents and the set of boundary components of B.

(ii) *An element x in Z belongs to K_e if, and only if, $e = \lim_{n\to\infty} x^{2n+1}$, where $x^{2n+1} := \{x, x^{2n-1}, x\}$, $n \geq 1$.*

(iii) *The boundary components of K_e are K_d for $d \geq e$. In particular,*

$$\overline{K}_e = \bigcup_{d\geq e} K_d$$

for tripotents $e, d \in Z$.

1.6 Remarks. It follows that $B = K_0$ is the only open boundary component, while the only closed boundary components are those singletons corresponding to extreme points. In particular, if x is not an extreme point of $\overline{B}$ then $\overline{K}_x$ is strictly larger than K_x.

We recall a concept of closed convex (holomorphic) hull, coinciding with a hull used in [3] defined in terms of complex supporting hyperplanes.

1.7 Proposition. *[26, Definition 3.1, Proposition 3,2]*
For $x \in \overline{B}$, let $\mathrm{Ch}(x) := \overline{K}_x$. Then

(i) $\mathrm{Ch}(x)$ *is the smallest closed convex set containing the holomorphic boundary component of x.*

(ii) $\mathrm{Ch}(x)$ *is affinely (or holomorphically) connected $\Leftrightarrow$ x is extreme $\Leftrightarrow$ $\mathrm{Ch}(x) = \{x\}$.*

(iii)

$$\mathrm{Ch}(x) = \overline{K}_u = \bigcup_{d \geq u} K_d = \bigcup_{d \geq u} d + B_0(d), \text{ for tripotent } u := \lim_n x^{2n+1}.$$

In particular, for $x \in B$, $\mathrm{Ch}(x) = \overline{B}$.

1.4 A Wolff theorem for bounded symmetric domains

For a fixed-point free compact holomorphic self map, f, of B, Jordan theory has been used to produce f-invariant domains [25] which have a simple algebraic description in terms of Bergman operators and which generalise the f-invariant horocycles of Δ. See [27] for a comprehensive introduction to this Jordan approach.

1.8 Theorem. *[22, Theorem 2.1, Remarks 2.2] Let Z be a finite rank JB*-triple with open unit ball B and $f : B \to B$ be a compact fixed-point free holomorphic map. Then for all $\lambda > 0$, there exists $c_\lambda \in B$ and $T_\lambda \in GL(Z)$ such that the domain*

$$E_\lambda := c_\lambda + T_\lambda(B)$$

is f-invariant and is a non-empty convex affine subset of B.

For each $y \in B$ there exists $\lambda_y > 0$ such that $y \in \partial E_{\lambda_y}$ ([25, Theorem 3.10]) showing that we can produce domains E_λ of arbitrary size λ. Furthermore [22, Proposition 2.5], there is a unique tripotent $\widetilde{e} \in \partial B$ such that

$$\lim_{\lambda \to 0^+} c_\lambda = \widetilde{e} \text{ and } \lim_{\lambda \to 0^+} T_\lambda = B(\widetilde{e}, \widetilde{e}) = P_0(\widetilde{e})$$

where $P_0(\widetilde{e})$ is the projection of Z onto the subtriple $Z_0(\widetilde{e})$.

2 New Results on Finite Rank JB*-triples

Let Z be a finite rank JB^*-triple with open unit ball B, $f : B \to B$ be a compact holomorphic map with no fixed point in B and $\Gamma(f)$ be as above. Our quest for the images of elements in $\Gamma(f)$ motivates the following definition, cf. [3].

2.1 Definition. The *target set* of f is

$$T(f) := \bigcup_{g \in \Gamma(f)} g(B).$$

By [17, Theorem 2.5], $T(f) \subseteq \partial B$. We seek to locate $T(f)$ by studying the limit of the domains E_λ as $\lambda \to 0$. We begin with some necessary definitions.

The "Wolff point", ξ, of f is found in the usual way, noting that it is not necessarily unique. Choose $(\alpha_k)_k$, $0 < \alpha_k < 1$, $\alpha_k \uparrow 1$ and let $f_k := \alpha_k f$ for all k. Then f_k has a fixed point, z_k, in $\alpha_k B$ [11] and, by compactness of f, we may assume $z_k \to \xi \in \overline{B}$ and hence $\xi \in \partial B$, as otherwise it would be a fixed point of f.

Let the Peirce decomposition of ξ be $\xi = \mu_1 e_1 + \cdots + \mu_r e_r$ for a frame $\{e_1, \cdots, e_r\}$ and $0 \leq \mu_1 \leq \cdots \leq \mu_r = \|\xi\| = 1$. Let now

$$p \in \{0, \ldots, r-1\} \text{ be such that } p + 1 = \min_k \{1 \leq k \leq r : \mu_k = 1\}.$$

We let $v := \mu_1 e_1 + \cdots + \mu_p e_p$ if $p \neq 0$ and $v = 0$ otherwise and let e be the tripotent $e = e_{p+1} + \cdots + e_r \in \partial B$. Then $\xi = e + v$, $e = \lim_n \xi^{2n+1}$ and Theorem 1.5 gives $K_\xi = K_e$. Of course, $p = 0 \Leftrightarrow v = 0 \Leftrightarrow \xi = e$ is maximal $\Leftrightarrow \xi = e$ is extreme in $\overline{B} \Leftrightarrow K_\xi = K_e = \{\xi\}$.

As each E_λ is described as a limit [22, Theorem 2.1] of Kobayashi balls $D_{z_k,\lambda}$, in order to identify the limiting behaviour of E_λ, we return to the Peirce decompositions for each z_k. Namely, $z_k = \gamma_{k1} e_{k1} + \cdots + \gamma_{kr} e_{kr}$ for some frame $\{e_{k1}, \ldots, e_{kr}\}$ and scalars $0 \leq \gamma_{k1} \leq \cdots \leq \gamma_{kr} = \|z_k\|$.

Essentially, we must now determine a convergence of domains "$\lim_\lambda E_\lambda$" as a double limit "$\lim_\lambda \lim_k D_{z_k,\lambda}$" in the infinite dimensional case. Motivated by insights and techniques from [25, 28], we make several definitions, passing to a subsequence whenever necessary for convergence.

2.2 Definition. For $1 \leq i \leq r$, let

$$a_i := \lim_k \frac{1 - \gamma_{kr}^2}{1 - \gamma_{ki}^2} = \lim_k \frac{1 - \|z_k\|^2}{1 - \gamma_{ki}^2} \tag{2}$$

and, for $\lambda > 0$, let $s_i \geq 0$ be determined by $(1 - s_i^2)^2 = \frac{\lambda}{a_i + \lambda}$.

Clearly, $0 \leq a_1 \leq \cdots \leq a_r = 1$ and $s_i = 0 \Leftrightarrow a_i = 0$. It follows from [28, Theorem 2.1] that $\lim_k \gamma_{ki} < 1 \Leftrightarrow 1 \leq i \leq p$. In particular, $0 = a_1 = \cdots = a_p \leq a_{p+1} \leq \cdots \leq a_r = 1$. However, some of $\{a_{p+1}, \ldots, a_{r-1}\}$ may also be zero, and this prompts the following definition.

2.3 Definition. Let

$$\widetilde{e} = e_m + \cdots + e_r \tag{3}$$

where $m \in \{p+1, \ldots, r\}$ is chosen such that $a_1, \ldots, a_{m-1} = 0$ and $a_m \neq 0$.

Clearly $\widetilde{e} \leq e$ and $\widetilde{e} = e \iff m = p + 1 \iff a_{p+1} \neq 0$. Since $0 < e_r \leq \widetilde{e}$, we have $\widetilde{e}$ is non-zero. In general, unfortunately, e and $\widetilde{e}$ are different, as shown by the following example.

2.4 Example. Let $f : \Delta^2 \to \Delta^2$, $(w_1, w_2) \mapsto (\frac{1+w_1^2}{2}, \frac{1+w_2}{2})$ which is fixed point free. Take $\alpha_k = \frac{k-1}{k}$. Fixed points z_k of $\alpha_k f$ are given by

$$z_k = (\gamma_{k1}, \gamma_{k2}) = (\frac{k - \sqrt{2k-1}}{k-1}, \frac{k-1}{k+1}).$$

The norm is provided by the second coordinate and the spectral decomposition of z_k is $\gamma_{k1}e_1 + \gamma_{k2}e_2$ where $e_1 = (1,0)$ and $e_2 = (0,1)$. Clearly $z_k \to \xi = e = (1,1)$. As $a_i = \lim_k \frac{1-\|z_k\|^2}{1-\gamma_{ki}^2}$ we have

$$a_2 = 1, \quad a_1 = \lim_k \frac{4k}{(k+1)^2} \cdot \frac{(k-1)^2}{-4k+2+2k\sqrt{2k-1}} = 0.$$

From Definition 2.3 above, $\widetilde{e} = (0,1) < e = (1,1)$.

For an example on the bidisc where $\widetilde{e} = e$, but is not extreme, see [22, Example 2.14]. We now define our domains E_λ.

2.5 Definition.

$$E_\lambda := c_\lambda + T_\lambda(B)$$

where

$$c_\lambda := \sum_{i=m}^{r} \left(\frac{a_i}{a_i+\lambda}\right) e_i, \quad v_\lambda := \sum_{i=m}^{r} s_i e_i, \quad \text{and} \quad T_\lambda := B(v_\lambda, v_\lambda) \in L(Z). \tag{4}$$

2.6 Remark. We note that Definition 2.5 of E_λ is independent of v satisfying $\xi = e + v$ above, so E_λ is determined only by the boundary component $K_\xi = K_e$ of ξ rather than by the point ξ itself.

If B is a Hilbert ball then $\overline{E_\lambda} \cap \partial B = \{\xi\}$ but generally this set can be large and is not affinely connected ([25] Example 4.6).

The next result refines [22, Propositions 2.6 and 2.7]. It proves that not only is $\bigcap_{\lambda>0} \overline{E_\lambda}$ non-empty but it is a (closed) convex subset of ∂B and, indeed, is the smallest such subset containing the boundary component $K_{\widetilde{e}}$.

2.7 Corollary. *(i)* $\bigcap_{\lambda>0} E_\lambda = \emptyset$.

(ii) $\bigcap_{\lambda>0} \overline{E_\lambda} = \mathrm{Ch}(\widetilde{e})$.

Proof. For (i), see [22, Proposition 2.6]. For (ii), we note that the proof of [22, Proposition 2.7] applies equally well to $\widetilde{e}$, as to e, to give $\mathrm{Ch}(\widetilde{e}) = \overline{K_{\widetilde{e}}} \subseteq \bigcap_{\lambda>0} \overline{E_\lambda}$. Then [22, Proposition 2.6] implies $\bigcap_{\lambda>0} \overline{E_\lambda} \subseteq \mathrm{Ch}(\widetilde{e})$ giving (ii).

■

Definition 2.3 implies that

$$\mathrm{Ch}(\xi) = \mathrm{Ch}(e) \subseteq \mathrm{Ch}(\widetilde{e}) \text{ with equality if, and only if, } a_{p+1} \neq 0.$$

Let $g \in \Gamma(f)$. Our next result, and key to everything that follows, relates $g(B)$ to the closed convex hull of the tripotent $\widetilde{e}$. This allows us to confine our search for $T(f)$ to certain specific boundary components by placing restrictions on those components K_d which contain $g(B)$.

2.8 Proposition. *Let B be the open unit ball of a finite rank JB*-triple, $f : B \to B$ be compact, holomorphic and fixed-point free. Let $g \in \Gamma(f)$. Then for a tripotent $d \in \partial B$ satisfying $g(B) \subseteq K_d$, we have*

$$\mathrm{Ch}(d) \cap \mathrm{Ch}(\widetilde{e}) \neq \emptyset.$$

Proof. Fix $g \in \Gamma(f)$. [17, Theorem 2.5] and Theorem 1.5 above give $g(B) \subseteq K_d \subseteq \partial B$, for a unique tripotent $d \in \partial B$. Fix now $\lambda > 0$ and $z_\lambda \in E_\lambda$. Then $K_d = K_{g(z_\lambda)}$. As E_λ is f invariant, $g(z_\lambda) \in \overline{E_\lambda} = c_\lambda + T_\lambda(\overline{B})$ [22, Lemma 2.3]. Therefore $g(z_\lambda) = c_\lambda + T_\lambda(w_\lambda)$, for some $w_\lambda \in \overline{B}$. Since also $g(z_\lambda) \in K_d$, we have $g(z_\lambda) = c_\lambda + T_\lambda(w_\lambda) = d + x_\lambda$, for some $x_\lambda \in B_0(d) = B \cap Z_0(d)$. Theorem 1.8 above gives

$$\lim_{\lambda \to 0^+} c_\lambda = \widetilde{e} \text{ and } \lim_{\lambda \to 0^+} T_\lambda = P_0(\widetilde{e}). \tag{5}$$

Using weak compactness of B, the Hahn-Banach theorem and passing to a subnet where necessary (see also proof of [22, Proposition 2.6]), we have for $w = \lim_\lambda w_\lambda \in \overline{B}$ and $x = \lim_\lambda x_\lambda \in \overline{B_0(d)}$ that

$$\lim_{\lambda \to 0^+} c_\lambda + T_\lambda(w_\lambda) = \lim_{\lambda \to 0^+} d + x_\lambda.$$

From (5) this gives

$$\widetilde{e} + P_0(\widetilde{e})(w) = d + x.$$

Now $\widetilde{e} + P_0(\widetilde{e})(w) \in \widetilde{e} + P_0(\widetilde{e})(\overline{B}) \subseteq \widetilde{e} + \overline{P_0(\widetilde{e})(B)} = \widetilde{e} + \overline{B_0(\widetilde{e})} = \overline{K_{\widetilde{e}}} = \mathrm{Ch}(\widetilde{e})$. This gives

$$\widetilde{e} + P_0(\widetilde{e})(w) = d + x \in \overline{K}_{\widetilde{e}} \cap \overline{K}_d = \mathrm{Ch}(\widetilde{e}) \cap \mathrm{Ch}(d)$$

and we are done.

■

As the Wolff point ξ is not generally unique (see Remark 2.6), and Proposition 2.8 above shows that $\bigcap_\lambda \overline{E_\lambda} = \mathrm{Ch}(\widetilde{e})$ is the smallest possible replacement for the Wolff point in the general setting, we introduce the following definition.

2.9 Definition. The *Wolff hull* of f is

$$W(f) := \mathrm{Ch}(\widetilde{e}) = \bigcap_{\lambda > 0} \overline{E_\lambda}.$$

We note that $W(f)$ is a closed convex subset of ∂B. Clearly, for a Hilbert ball, strong convexity means that the Wolff hull reduces to the Wolff point, $W(f) = \xi$.

We now rephrase Proposition 2.8 to give our main result which relates the target set of f to its Wolff hull. This reduces to the usual Denjoy-Wolff result in the case of the Hilbert ball.

2.10 Theorem. *Let B be the open unit ball of a finite rank JB*-triple, $f : B \to B$ be compact, holomorphic and fixed-point free. For all $x \in T(f)$*

$$\mathrm{Ch}(x) \cap W(f) \neq \emptyset.$$

Proof. Let $x \in T(f)$. Then $x \in g(B)$ for some $g \in \Gamma(f)$ and hence there is a tripotent d (unique to g) with $g(B) \subset K_d$. Then $x \in K_d$ implies $K_x = K_d$ and hence $\mathrm{Ch}(x) = \mathrm{Ch}(d)$. Proposition 2.8 then gives the stated result.

■

3 Further results on the target set

This section extends results for $T(f)$ on some finite dimensional domains [3] to the infinite dimensional setting.

The following Proposition for (not necessarily finite dimensional) Banach spaces is a consequence of Definition 1.4 and [23, Proposition 3.3, Corollary 3.9] and will be used implicitly hereafter.

3.1 Proposition. *Let D be a domain in any complex Banach space and E be a complex Banach space with open unit ball B_E. For every holomorphic map $h : D \to E$ such that $h(D) \subset \overline{B}_E$ then*

$$h(D) \subseteq \bigcap_{z \in D} K_{h(z)}.$$

If, in addition, $h(D) \cap \partial B_E \neq \emptyset$ we have $h(D) \subseteq \bigcap_{z \in D} K_{h(z)} \subset \partial B_E$.

In particular, if f is a fixed-point free compact holomorphic self map of B_E and g is an accumulation point of (f^n), for any topology finer than the topology of pointwise convergence on B_E, then

$$g(B_E) \subseteq \bigcap_{z \in B_E} K_{g(z)} \subset \partial B_E.$$

We return now to the finite rank bounded symmetric domain B and define

$$K_W = \bigcup_{z \in W} K_z, \text{ for } W \subseteq \overline{B},$$

emphasising, however, that K_W is no longer any kind of boundary component. The following extends the finite dimensional result [3, Lemma 6] to bounded symmetric domains.

3.2 Proposition. *Let B be a finite rank bounded symmetric domain, $f : B \to B$ be compact holomorphic and fixed-point free. Then*

$$T(f) \subseteq \bigcap_{\lambda > 0} K_{\partial E_\lambda \cap \partial B}.$$

Proof. Fix $\lambda > 0$ and choose $z_\lambda \in E_\lambda$. Let $g \in \Gamma(f)$. As above, $g(z_\lambda) \in \overline{E_\lambda} \cap \partial B = \partial E_\lambda \cap \partial B$ and hence $K_{g(z_\lambda)} \subseteq K_{\partial E_\lambda \cap \partial B}$. Proposition 3.1 gives

$$g(B) \subseteq \bigcap_{\lambda > 0} K_{g(z_\lambda)} \subseteq \bigcap_{\lambda > 0} K_{\partial E_\lambda \cap \partial B}.$$

■

We recall the following results from [22, Corollaries 2.11 and 2.13]

3.3 Corollary. *(i) All constant maps in $\Gamma(f)$ lie in $W(f)$.*

(ii) If $\widetilde{e}$ is extreme then ξ is extreme and is the only possible constant map in $\Gamma(f)$. (This holds, in particular, if ξ is extreme and $a_1 \neq 0$.)

3.4 Corollary. *If $\widetilde{e}$ is extreme then for $g \in \Gamma(f)$ there is a tripotent $d \in \partial B$ satisfying $d \leq \widetilde{e}$ such that*

$$g(B) \subseteq K_d = d + B_0(d).$$

Proof. Fix $g \in \Gamma(f)$. As $\widetilde{e}$ is extreme, $\text{Ch}(\widetilde{e}) = \{\widetilde{e}\}$ and hence, from Theorem 2.8, there is $d \in \partial B$ with $g(B) \subseteq K_d$ and $\widetilde{e} \in \text{Ch}(d) = \overline{K_d}$. We only need to show that $d \le \widetilde{e}$. Theorem 1.5 (iii) gives

$$\overline{K}_d = \bigcup_{q \ge d} K_q$$

and hence $\widetilde{e} \in K_q$ for some tripotent $q \ge d$. On the other hand, Theorem 1.5(ii) implies that $q = \lim_n \widetilde{e}^{2n+1} = \widetilde{e}$ and so $\widetilde{e} \ge d$.

∎

3.5 Corollary. *If $\widetilde{e}$ is extreme then*

$$T(f) \subseteq \bigcup_{d \le \widetilde{e}} K_d = \bigcup_{d \le \widetilde{e}} (d + B_0(d)).$$

For example, in the bidisc if we have $\widetilde{e} = (\nu_1, \nu_2)$, with $|\nu_1| = |\nu_2| = 1$, then the above implies that

$$T(f) \subseteq (\{\nu_1\} \times \Delta) \cup (\Delta \times \{\nu_2\}) \cup \{(\nu_1, \nu_2)\},$$

(compare also [14, Theorem 4]).

Our final example looks again at the bidisc $B = \Delta \times \Delta$, as the simplest possible domain B which is not strongly convex.

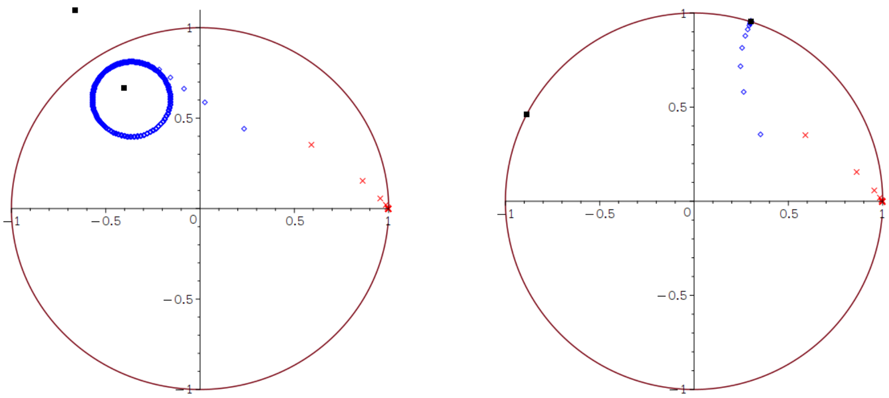
3.6 Example. Let $B = \Delta \times \Delta$ and $f : B \to B$ be given by

$$f(z, w) = (g_{\frac{1}{2}}(z), \mathrm{e}^{i\theta} g_{\frac{1}{2}}(zw)),$$

where $g_{\frac{1}{2}}(x) = \frac{1+2x}{2+x}$ is the Möbius map on the unit disc that maps the origin to $\frac{1}{2}$ and $\theta \in (-\pi, \pi]$. As f is fixed point free and has a continuous extension to the boundary, we can solve explicitly for any boundary fixed points of f (at least one must exist). The first coordinate of any fixed point is evidently 1. Solving for the second coordinate gives $w = e^{i\theta} g_{\frac{1}{2}}(w)$ which reduces to a quadratic whose discriminant vanishes only at $\theta = \pi/3$. Thus, any boundary fixed point of f is of the form $(1, \eta)$ for some $\eta \in \overline{\Delta}$ and $\theta = \pi/3$ is a critical value in deciding whether $\eta \in \Delta$ or in $\partial\Delta$.

When $|\theta| > \frac{\pi}{3}$, $\eta \in \Delta$ and hence $\xi = (1, \eta) \in \{1\} \times \Delta$ is the unique fixed point of f in $\overline{\Delta}^2$ and therefore must be the Wolff point. Then $e = \widetilde{e} = (1, 0)$ and $W(f) = 1 \times \overline{\Delta}$. Figure 1a below shows values of iterates of f from a starting point in $\Delta \times \Delta$ converging in the first coordinate but not in the second, with accumulation points lying on a circle containing η. This circle varies with the initial point showing that there are non-constant limits in $\Gamma(f)$. We have $T(f) \subset \{1\} \times \Delta \subsetneq W(f) = \{1\} \times \overline{\Delta}$.

For $|\theta| \le \frac{\pi}{3}$, the quadratic $w = e^{i\theta} g_{\frac{1}{2}}(w)$ provides two fixed points on $\partial\Delta$ only one of which, η, gives $(1, \eta)$ that is attracting for f and hence must be the Wolff point $\xi = (1, \eta) \in \{1\} \times \partial\Delta$. As ξ is an extreme point, we have $e = \xi$. Thus $\widetilde{e} = e$ or $\widetilde{e} = (1, 0)$. However, numerically solving (with use of Waterloo Maple [24]), $\alpha_k f(z_k) = z_k$ for scaling factors $\alpha_k = \frac{k-1}{k}$ and then calculating a_1 (as in (2) above), we find that $a_1 > 0$ and hence (Definition 2.3) $\widetilde{e} = e = (1, \eta)$. In this case therefore, $W(f)$ is the singleton $\{(1, \eta)\}$. The iterates of an initial point in $\Delta \times \Delta$ (see Figure 1b below) converge to $(1, \eta)$. The target set $T(f) = \{(1, \eta)\} = W(f)$.



(a) $\theta < \frac{\Pi}{3}$. (b) $\theta > \frac{\Pi}{3}$.

Figure 1: The first coordinate is shown by $\times$, the second coordinate by $\diamond$. Solved values for η depicted by ▪.

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