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A NOTE ON MILNOR-WITT K -THEORY AND A THEOREM OF SUSLIN

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ABSTRACT. We give a simple presentation of the additive Milnor-Witt K -theory groups $K_n^{\text{MW}}(F)$ of the field F , for $n \geq 2$, in terms of the natural small set of generators. When $n = 2$, this specialises to a theorem of Suslin which essentially says that $K_2^{\text{MW}}(F) \cong \text{H}_2(\text{Sp}(F), \mathbb{Z})$.

1. INTRODUCTION

In [7], Suslin proved that for an infinite field F , $\text{H}_2(\text{Sl}(2, F), \mathbb{Z})$ is isomorphic to the fibre product $K_2^{\text{M}}(F) \times_{I^2/I^3} I^2(F)$, where $K_n^{\text{M}}(F)$ is the n -th Milnor K -group of F and $I = I(F)$ is the ideal of even-dimensional forms in the Witt ring $W(F)$. The proof uses the Matsumoto-Moore presentation of the group $\text{H}_2(\text{Sp}(F), \mathbb{Z}) = \text{H}_2(\text{Sl}(2, F), \mathbb{Z})$ as well as the characterisation of the 2-torsion of $K_2^{\text{M}}(F)$ as the set of all elements of the form $\{-1, a\}$. (More recently, Mazzoleni, [3], has given an alternative proof of this theorem which by-passes the theorem of Matsumoto-Moore.)

More recently, F. Morel has introduced the Milnor-Witt K -theory, $K_*^{\text{MW}}(F)$ ([4], [5]). This is a graded algebra given by a simple presentation, due to Morel and M. Hopkins, from which the following properties are easily deduced: $K_n^{\text{MW}}(F) \cong W(F)$ for all $n < 0$; $K_0^{\text{MW}}(F) \cong \text{GW}(F)$, the Grothendieck-Witt ring of isometry classes of quadratic forms over F ; there is an element η , of degree -1 , such that $K_*^{\text{MW}}(F)/\langle \eta \rangle \cong K_*^{\text{M}}(F)$. The main result about Milnor-Witt K -theory is that it gives an exact description of certain operations in stable motivic homotopy theory ; namely there is a natural isomorphism of graded rings

$$K_*^{\text{MW}}(F) \cong [\mathbb{S}^0, (\mathbb{G}_m)^*]$$

where \mathbb{S}^0 is the ‘motivic’ sphere spectrum, and $[,]$ denotes the group of morphisms in the stable \mathbb{A}^1 -homotopy category ([4], section 6).

Morel has shown (see [5], for example) that, for all $n \geq 0$,

$$K_n^{\text{MW}}(F) \cong K_n^{\text{M}}(F) \times_{I^n/I^{n+1}} I^n(F).$$

In fact this result is essentially a reformulation of some of the main results of Arason and Elman, [1], on the powers of $I(F)$. Their work, in turn, relies heavily on the work of Voevodsky, Orlov and Vishik on the Milnor conjecture. In view of Morel’s result, Suslin’s theorem can be re-formulated as the statement that $\text{H}_2(\text{Sl}(2, F), \mathbb{Z}) \cong K_2^{\text{MW}}(F)$, at least when F is infinite. Elsewhere ([6]), Morel has sketched a direct proof of this fact, using the machinery of \mathbb{A}^1 -homotopy theory.

In this note, which is more elementary in nature than any of the references above, we prove that the Matsumoto-Moore relations give a simple presentation of the additive

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group $K_n^{\text{MW}}(F)$, for all $n \geq 2$, in terms of the natural set of generators. When $n = 2$, this statement specializes to Suslin's theorem, as re-formulated above.

As another application of our main theorem, we give an abstract additive presentation of the group $I^n(F)$ with n -fold Pfister forms as generators. (Corollary 2.16).

2. MILNOR-WITT K -THEORY

Definition 2.1 (Hopkins-Morel, [4]). The Milnor-Witt K -theory of the field F is the graded associative ring $K_*^{\text{MW}}(F)$ generated by the symbols $[u]$, $u \in F^\times$, of degree $+1$ and one symbol η of degree -1 subject to the following relations:

1. For each $a \in F^\times \setminus \{1\}$, $[a] \cdot [1 - a] = 0$.
2. For each $a, b \in F^\times$, $[ab] = [a] + [b] + \eta[a][b]$.
3. For each $u \in F^\times$, $[u]\eta = \eta[u]$.
4. $\eta^2[-1] + 2\eta = 0$.

The following result is easily deduced ([6], Lemma 2.4):

Lemma 2.2. *For all $n \in \mathbb{Z}$, $K_n^{\text{MW}}(F)$ has the following presentation as an additive group: It is generated by the elements $\eta^m[a_1] \cdots [a_r]$, $m \geq 0$, $r = n + m \geq 0$ subject to the following relations:*

- (1) $\eta^m[a_1] \cdots [a_r] = 0$ if $r \geq 2$ and $a_{i-1} + a_i = 1$ for some $i \geq 2$.
- (2) $\eta^m[a_1] \cdots [a_{i-1}][a_i a'_i] \cdots [a_r] = \eta^m[a_1] \cdots [a_{i-1}][a'_i] \cdots [a_r] + \eta^m[a_1] \cdots [a_{i-1}][a_i] \cdots [a_r] + \eta^{m+1}[a_1] \cdots [a_{i-1}][a_i][a'_i] \cdots [a_r]$
- (3) $\eta^{m+2}[a_1] \cdots [a_{i-1}][-1][a_{i+1}] \cdots [a_{r+2}] + 2\eta^{m+1}[a_1] \cdots [a_{i-1}][a_{i+1}] \cdots [a_{r+2}] = 0$

However, in view of the relation $\eta[a_1][a_2] = [a_1 a_2] - [a_1] - [a_2]$, it is clear that $K_n^{\text{MW}}(F)$ can be generated by the elements $[a_1] \cdots [a_n]$ whenever $n \geq 1$. Our main theorem is a presentation of $K_n^{\text{MW}}(F)$ in terms of these generators when $n \geq 2$.

The theorem of Matsumoto and Moore ([2]), for the case of the symplectic group $\text{Sp}(F)$, gives a presentation of the group $\text{H}_2(\text{Sp}(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_1 \rangle$, $a_i \in F^\times$, subject to the relations:

- (i) $\langle a_1, a_2 \rangle = 0$ if $a_i = 1$ for some i
- (ii) $\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle$
- (iii) $\langle a_1, a_2 a'_2 \rangle + \langle a_2, a'_2 \rangle = \langle a_1 a_2, a'_2 \rangle + \langle a_1, a_2 \rangle$
- (iv) $\langle a_1, a_2 \rangle = \langle a_1, -a_1 a_2 \rangle$
- (v) $\langle a_1, a_2 \rangle = \langle a_1, (1 - a_1) a_2 \rangle$

This motivates the following (provisional) definition:

Definition 2.3. Let $n \geq 2$. For a field F , $K_n^{\text{MM}}(F)$ (MM is for 'Matsumoto-Moore') will denote the additive group which has the following presentation: the generators are $\langle a_1, \dots, a_n \rangle$, $a_i \in F^\times$ subject to the following relations:

- (i) $\langle a_1, \dots, a_n \rangle = 0$ if $a_i = 1$ for some i
- (ii) $\langle a_1, \dots, a_{i-1}, a_i, \dots, a_n \rangle = \langle a_1, \dots, a_i^{-1}, a_{i-1}, \dots, a_n \rangle$
- (iii) $\langle a_1, \dots, a_{n-1}, a_n a'_n \rangle + \langle a_1, \dots, a_n, a'_n \rangle = \langle a_1, \dots, a_{n-1} a_n, a'_n \rangle + \langle a_1, \dots, a_{n-1}, a_n \rangle$
- (iv) $\langle a_1, \dots, a_{n-1}, a_n \rangle = \langle a_1, \dots, a_{n-1}, -a_{n-1} a_n \rangle$
- (v) $\langle a_1, \dots, a_{n-1}, a_n \rangle = \langle a_1, \dots, a_{n-1}, (1 - a_{n-1}) a_n \rangle$

Remark 2.4. In particular, $K_2^{\text{MM}}(F) \cong \text{H}_2(\text{Sp}(F), \mathbb{Z}) (= \text{H}_2(\text{Sl}(2, F), \mathbb{Z}))$ if F is infinite or at least sufficiently large) by the theorem of Matsumoto-Moore.

Observe that, using relation (ii) together with (iii), (iv) and (v), we easily deduce the following relations in $K_n^{\text{MM}}(F)$:

$$\begin{aligned} (iii)' \quad & \langle a_1, \dots, a_{i-1}, a_i a'_i, \dots, a_n \rangle + \langle a_1, \dots, a_i, a'_i, \dots, a_n \rangle = \langle a_1, \dots, a_{i-1} a_i, a'_i, \dots, a_n \rangle + \\ & \langle a_1, \dots, a_{i-1}, a_i, \dots, a_n \rangle \\ (iv)' \quad & \langle a_1, \dots, a_{i-1}, a_i, \dots, a_n \rangle = \langle a_1, \dots, a_{i-1}, -a_{i-1} a_i, \dots, a_n \rangle \\ (v)' \quad & \langle a_1, \dots, a_{i-1}, a_i, \dots, a_n \rangle = \langle a_1, \dots, a_{i-1}, (1 - a_{i-1}) a_i, \dots, a_n \rangle \end{aligned}$$

Theorem 2.5. $K_n^{\text{MW}}(F) \cong K_n^{\text{MM}}(F)$ for all $n \geq 2$ via an isomorphism sending $[a_1] \cdots [a_n]$ to $\langle a_1, \dots, a_n \rangle$.

Proof. The theorem follows from Lemmas 2.8 and 2.15 below. \square

Corollary 2.6. For all infinite fields F , $K_2^{\text{MW}}(F) \cong H_2(\text{Sl}(2, F), \mathbb{Z})$.

Lemma 2.7. The relations $[a][-a] = 0$ and $[a][b] = [b^{-1}][a]$ hold in $K_*^{\text{MW}}(F)$ for all $a, b \in F^\times$.

Proof. Using the identity

$$-a(1 - a^{-1}) = 1 - a$$

together with **2.** gives

$$[-a] = [1 - a] - [1 - a^{-1}] - \eta[-a][1 - a^{-1}].$$

Hence

$$\begin{aligned} [a][-a] &= -[a](1 + \eta[-a])[1 - a^{-1}] && \text{(using 1.)} \\ &= -(1 + \eta[-a])[a][1 - a^{-1}] && \text{(since } \eta[u][v] = \eta[v][u] \text{ by 2.)} \\ &= (1 + \eta[-a])(1 + \eta[a])[a^{-1}][1 - a^{-1}] && \text{(since } [a] = (1 + \eta[a])[a^{-1}] \text{ by 2.)} \\ &= 0 && \text{(by 1.)} \end{aligned}$$

We will need the identity

$$\eta([a] + [-a])[-1] = -2[-1]$$

obtained by letting $x = a$ and $x = -a$ in the identity $[x] = [-1] + [-x] + \eta[-x][-1]$ and adding.

Observe also that $[a]^2 = [a][-1]$ for all $a \in F^\times$ (let $x = a$ above and use $[a][-a] = 0$).

Now, for any $a, b \in F^\times$ we have

$$\begin{aligned} 0 &= [ab][-ab] \\ &= ([a] + [b] + \eta[a][b])([-a] + [b] + \eta[-a][b]) \\ &= [a][b] + [b][-a] + \eta([a] + [-a])[b]^2 + [b]^2 \\ &= [a][b] + [b][-a] + \eta([a] + [-a])[-1][b] + [b][-1] \\ &= [a][b] + [b][-a] - [b][-1] \\ &= [a][b] + [b]([-a] - [-1]) \\ &= [a][b] + [b][a](1 + \eta[-1]) \end{aligned}$$

and hence

$$[a][b] = -[b][a](1 + \eta[-1]) = -(1 + \eta[-1])[b][a].$$

Thus

$$\begin{aligned}
[a][b] - [b^{-1}][a] &= -((1 + \eta[-1])[b] + [b^{-1}])[a] \\
&= -([b] + [b^{-1}] + \eta[-1][b])[a] \\
&= -([b] + [b^{-1}] + \eta[b^{-1}][b])[a] = -[1][a] = 0
\end{aligned}$$

(where we have used $[-1][b] = [b^{-1}][b]$ which follows from $[-1] = [-b] + [b^{-1}] + \eta[-b][b^{-1}]$ and $[b][-b] = 0$).

□

Lemma 2.8. *Let $n \geq 2$. The map ϕ which sends the element $\langle a_1, \dots, a_n \rangle$ of $K_n^{\text{MM}}(F)$ to $[a_1] \cdots [a_n]$ in $K_n^{\text{MW}}(F)$ extends uniquely to a well-defined epimorphism of groups.*

Proof. Well-definedness is the issue; we must prove that relations (i)–(v) are preserved by ϕ .

Relation (i): This follows from the identity $[1] = 0$ in $K_*^{\text{MW}}(F)$ (since $[1] = 2[-1] + \eta[-1]^2$ by **2.** and hence $\eta[1] = 0$ by **4.** and thus $[1] = 2[1]$ by **2.** again).

Relation (ii): This follows immediately from the relation $[a_{i-1}][a_i] = [a_i^{-1}][a_{i-1}]$ (Lemma 2.7).

Relation (iii): We have

$$\begin{aligned}
[a_{n-1}][a_n a'_n] + [a_n][a'_n] &= [a_{n-1}]([a_n] + [a'_n] + \eta[a_n][a'_n]) + [a_n][a'_n] \\
&= ([a_{n-1}] + [a_n] + \eta[a_{n-1}][a_n])[a'_n] + [a_{n-1}][a_n] \\
&= [a_{n-1}a_n][a'_n] + [a_{n-1}][a_n]
\end{aligned}$$

Relation (iv): We have $[a_{n-1}][-a_{n-1}a_n] = [a_{n-1}]([-a_{n-1}] + [a_n] + \eta[-a_{n-1}][a_n]) = [a_{n-1}][a_n]$ by Lemma 2.7.

Relation (v): Similarly, $[a_{n-1}][(1 - a_{n-1})a_n] = [a_{n-1}][a_n]$ using **1.** and **2.**

□

Lemma 2.9. *Let $n \geq 2$. For $a_1, \dots, a_n, x \in F^\times$ let*

$$\rho_x(\langle a_1, \dots, a_n \rangle) := \langle a_1, \dots, a_n x \rangle - \langle a_1, \dots, a_n \rangle - \langle a_1, \dots, x \rangle.$$

Then ρ_x extends uniquely to an endomorphism of $K_n^{\text{MM}}(F)$.

Proof. We must prove that ρ_x preserves defining relations (i)–(v).

Relation (i) is clear.

Relation (ii): When $i < n$ in (ii), the result is clear. For the case $i = n$, we find:

$$\begin{aligned}
\rho_x(\langle a_1, \dots, a_{n-1}, a_n \rangle) &= \langle a_1, \dots, a_{n-1}, x a_n \rangle - \langle a_1, \dots, a_{n-1}, a_n \rangle - \langle a_1, \dots, a_{n-1}, x \rangle \\
&= (\langle a_1, \dots, a_{n-1} x, a_n \rangle + \langle a_1, \dots, a_{n-1}, x \rangle - \langle a_1, \dots, x, a_n \rangle) \\
&\quad - \langle a_1, \dots, a_{n-1}, a_n \rangle - \langle a_1, \dots, a_{n-1}, x \rangle \text{ (using (iii))} \\
&= \langle a_1, \dots, a_{n-1} x, a_n \rangle - \langle a_1, \dots, x, a_n \rangle - \langle a_1, \dots, a_{n-1}, a_n \rangle \\
&= \langle a_1, \dots, a_n^{-1}, a_{n-1} x \rangle - \langle a_1, \dots, a_n^{-1}, a_{n-1} \rangle - \langle a_1, \dots, a_n^{-1}, x \rangle \\
&= \rho_x(\langle a_1, \dots, a_n^{-1}, a_{n-1} \rangle).
\end{aligned}$$

Relation (iii):

$$\begin{aligned}
& \rho_x(\langle a_1, \dots, a_{n-1}, a_n a'_n \rangle) + \rho_x(\langle a_1, \dots, a_n, a'_n \rangle) \\
= & \langle a_1, \dots, a_{n-1}, a_n(a'_n x) \rangle - \langle a_1, \dots, a_{n-1}, x \rangle - \langle a_1, \dots, a_{n-1}, a_n a'_n \rangle \\
& + \langle a_1, \dots, a_n, a'_n x \rangle - \langle a_1, \dots, a_n, x \rangle - \langle a_1, \dots, a_n, a'_n \rangle \\
= & (\langle a_1, \dots, a_{n-1}, a_n(a'_n x) \rangle + \langle a_1, \dots, a_n, a'_n x \rangle) - (\langle a_1, \dots, a_{n-1}, a_n a'_n \rangle + \langle a_1, \dots, a_n, a'_n \rangle) \\
& - \langle a_1, \dots, a_{n-1}, x \rangle - \langle a_1, \dots, a_n, x \rangle \\
= & (\langle a_1, \dots, a_{n-1} a_n, a'_n x \rangle + \langle a_1, \dots, a_{n-1}, a_n \rangle) - (\langle a_1, \dots, a_{n-1} a_n, a'_n \rangle + \langle a_1, \dots, a_{n-1}, a_n \rangle) \\
& - \langle a_1, \dots, a_{n-1}, x \rangle - \langle a_1, \dots, a_n, x \rangle \quad (\text{using (iii) again}) \\
= & \langle a_1, \dots, a_{n-1} a_n, a'_n x \rangle - \langle a_1, \dots, a_{n-1} a_n, a'_n \rangle - \langle a_1, \dots, a_{n-1}, x \rangle - \langle a_1, \dots, a_n, x \rangle \\
= & \langle a_1, \dots, a_{n-1} a_n, a'_n x \rangle - \langle a_1, \dots, a_{n-1} a_n, a'_n \rangle - \langle a_1, \dots, a_{n-1}, x \rangle \\
& - (\langle a_1, \dots, a_{n-1} a_n, x \rangle + \langle a_1, \dots, a_{n-1}, a_n \rangle - \langle a_1, \dots, a_{n-1}, a_n x \rangle) \\
= & (\langle a_1, \dots, a_{n-1} a_n, a'_n x \rangle - \langle a_1, \dots, a_{n-1} a_n, a'_n \rangle - \langle a_1, \dots, a_{n-1} a_n, x \rangle) \\
& + (\langle a_1, \dots, a_{n-1}, a_n x \rangle - \langle a_1, \dots, a_{n-1}, x \rangle - \langle a_1, \dots, a_{n-1}, a_n \rangle) \\
= & \rho_x(\langle a_1, \dots, a_{n-1} a_n, a'_n \rangle) + \rho_x(\langle a_1, \dots, a_{n-1}, a_n \rangle)
\end{aligned}$$

Relations (iv) and (v) are immediate. \square

For $n \geq 2$, we will denote the element $\rho_x(\langle a_1, \dots, a_n \rangle)$ in $K_n^{\text{MM}}(F)$ by $[a_1, \dots, a_n, x]$.

Lemma 2.10. *Let $n \geq 2$. For any permutation, σ , of $\{1, \dots, n+1\}$, $[a_1, \dots, a_{n+1}] = [a_{\sigma(1)}, \dots, a_{\sigma(n+1)}]$.*

Proof. It is immediate from the definition that $[a_1, \dots, a_n, a_{n+1}] = [a_1, \dots, a_{n+1}, a_n]$; i.e the result is true when σ is the transposition $(n \ n+1)$. From this it follows that

$$\begin{aligned}
[a_1, \dots, a_{n-1}, a_n, x] &= \rho_x(\langle a_1, \dots, a_n \rangle) \\
&= \rho_{a_n}(\langle a_1, \dots, a_{n-1}, x \rangle) \\
&= \rho_{a_n}(\langle a_1, \dots, x^{-1}, a_{n-1} \rangle) \\
&= [a_1, \dots, x^{-1}, a_{n-1}, a_n] = [a_1, \dots, x^{-1}, a_n, a_{n-1}] \\
&= \rho_x(\langle a_1, \dots, a_n, a_{n-1} \rangle) \\
&= [a_1, \dots, a_n, a_{n-1}, x]
\end{aligned}$$

This fact, together with relation (ii), now implies that

$$\begin{aligned}
[a_1, \dots, a_{i-1}, a_i, \dots, a_n, x] &= \rho_x(\langle a_1, \dots, a_{i-1}, a_i, \dots, a_n \rangle) \\
&= \rho_x(\langle a_1, \dots, a_i, a_{i-1}, \dots, a_n \rangle) = [a_1, \dots, a_{i-1}, a_i, \dots, a_n, x]
\end{aligned}$$

proving the lemma. \square

Lemma 2.11. *For all $x, y \in F^\times$, $\rho_x \rho_y = \rho_y \rho_x$.*

Proof. Let $a_1, \dots, a_n \in F^\times$. Then

$$\rho_y(\langle a_1, \dots, a_n \rangle) = [a_1, \dots, a_n, y] = [y, a_1, \dots, a_n]$$

and hence

$$\begin{aligned}
& \rho_x(\rho_y(\langle a_1, \dots, a_n \rangle)) = \rho_x([y, a_1, \dots, a_n]) \\
&= \rho_x(\langle y, a_1, \dots, a_{n-1}a_n \rangle) - \rho_x(\langle y, a_1, \dots, a_{n-1} \rangle) - \rho_x(\langle y, a_1, \dots, a_{n-2}, a_n \rangle) \\
&= [y, \dots, a_{n-1}a_n, x] - [y, a_1, \dots, a_{n-1}, x] - [y, a_1, \dots, a_n, x] \\
&= [x, \dots, a_{n-1}a_n, y] - [x, a_1, \dots, a_{n-1}, y] - [x, a_1, \dots, a_n, y] \\
&= \rho_y(\rho_x(\langle a_1, \dots, a_n \rangle)).
\end{aligned}$$

□

More generally, we define elements $[a_1, \dots, a_r] \in K_n^{\text{MM}}(F)$ ($r > n$) recursively by the formula

$$[a_1, \dots, a_{r+1}] := \rho_{a_{r+1}}([a_1, \dots, a_r]).$$

We will also use the notation $[a_1, \dots, a_n] := \langle a_1, \dots, a_n \rangle$ (i.e., when $r = n$).

Corollary 2.12. *Fix $n \geq 2$. For all $r > n$ and for all permutations, σ , of $\{1, \dots, r\}$*

$$[a_1, \dots, a_r] = [a_{\sigma(1)}, \dots, a_{\sigma(r)}].$$

Proof. We use induction on r . The case $r = n + 1$ has already been proved.

For permutations of $\{1, \dots, r + 1\}$ which fix $r + 1$, the result holds by induction since

$$[a_1, \dots, a_r, a_{r+1}] = \rho_{a_{r+1}}([a_1, \dots, a_r]).$$

On the other hand, when $r > n$, the result holds for the permutation $\sigma = (r \ r + 1)$ since

$$[a_1, \dots, a_r, a_{r+1}] = \rho_{a_{r+1}}(\rho_{a_r}([a_1, \dots, a_{r-1}])) = \rho_{a_r}(\rho_{a_{r+1}}([a_1, \dots, a_{r-1}])) = [a_1, \dots, a_{r+1}, a_r].$$

□

Remark 2.13. Observe that it follows that the relations (i)-(v) extend to the symbols $[a_1, \dots, a_r]$ ($r \geq n$) since we can always permute the key entries to before the n -th position and then use the fact that $[a_1, \dots, a_r] = \phi(\langle a_1, \dots, a_n \rangle)$ for an appropriate endomorphism ϕ . Furthermore, property (ii) and symmetry (Corollary 2.12) imply that $[a_1, \dots, a_i, \dots, a_r] = [a_1, \dots, a_i^{-1}, \dots, a_r]$ for any i .

Corollary 2.14. *Let $n \geq 2$. If $r > n$ and if $a_1, \dots, a_r \in F^\times$ then $[a_1, \dots, a_r] = 0$ if a_i is a square in F^\times for some $i \leq r$.*

Proof. By symmetry we can suppose that $i > 1$. Suppose that $a_i = b_i^2$. We thus have

$$\begin{aligned}
[a_1, \dots, a_{i-1}, b_i^2, \dots] &= [a_1, \dots, a_{i-1}b_i, b_i, \dots] + [a_1, \dots, a_{i-1}, b_i, \dots] - [a_1, \dots, b_i, b_i, \dots] \\
&= [a_1, \dots, a_{i-1}b_i, b_i^{-1}, \dots] + [a_1, \dots, a_{i-1}, b_i, \dots] - [a_1, \dots, b_i, b_i^{-1}, \dots] \\
&= [a_1, \dots, a_{i-1}, b_i \cdot b_i^{-1}, \dots] = [a_1, \dots, a_{i-1}, 1, \dots] = 0.
\end{aligned}$$

□

Lemma 2.15. *Let $n \geq 2$. There is a unique epimorphism $\lambda : K_n^{\text{MW}}(F) \rightarrow K_n^{\text{MM}}(F)$ satisfying*

$$\lambda(\eta^m[a_1] \cdots [a_r]) = [a_1, \dots, a_r] \quad (r = n + m)$$

Proof. We must show that λ preserves relations (1)-(3) of Lemma 2.2.

Relation (1) follows from (i) and (v) (see Remark 2.13).

Relation (2): We must prove that, for $r \geq n$ and $i \leq r$,

$$[a_1, \dots, a_i a'_i, \dots, a_r] = [a_1, \dots, a_i, \dots, a_r] + [a_1, \dots, a'_i, \dots, a_r] + [a_1, \dots, a_i, a'_i, \dots, a_r].$$

By symmetry we can assume $i \leq n$ and thus we reduce to the key case $r = n$; i.e. we must prove

$$\langle a_1, \dots, a_i a'_i, \dots, a_n \rangle = \langle a_1, \dots, a_i, \dots, a_n \rangle + \langle a_1, \dots, a'_i, \dots, a_n \rangle + [a_1, \dots, a_i, a'_i, \dots, a_n].$$

By property (ii), and by symmetry, we can assume $i = n$. The identity is now just the definition of $[a_1, \dots, a_n, a'_n]$.

Relation (3): We must prove that for $r \geq n$

$$[a_1, \dots, a_{i-1}, -1, a_{i+1}, \dots, a_{r+2}] = -2[a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{r+2}].$$

By symmetry, we can suppose that $r = n$ and $i = n + 1$. So we must show that $[a_1, \dots, a_n, -1, a_{n+2}] = -2[a_1, \dots, a_n, a_{n+2}]$. Now

$$\begin{aligned} [a_1, \dots, a_n, -1] &= [a_1, \dots, a_n, a_n] \quad (\text{by (iv)}) \\ &= \langle a_1, \dots, a_n^2 \rangle - 2\langle a_1, \dots, a_n \rangle \end{aligned}$$

and thus

$$[a_1, \dots, a_n, -1, a_{n+2}] = [a_1, \dots, a_n^2, a_{n+2}] - 2[a_1, \dots, a_n, a_{n+2}] = -2[a_1, \dots, a_n, a_{n+2}]$$

by Corollary 2.14. \square

As an application, we derive a simple additive presentation of the ideals $I^n(F)$, $n \geq 2$, in the Witt Ring of a field F :

Corollary 2.16. *For any field F , let $I(F)$ be the ideal of even-dimensional forms in the Witt Ring, $W(F)$, of the field F . As an additive group, $I^n(F) = I(F)^n$ has the following abstract presentation:*

It is generated by the classes of Pfister forms $\langle\langle a_1, \dots, a_n \rangle\rangle$, $a_i \in F^\times$ subject to the following relations:

- (i) $\langle\langle a_1, \dots, a_n \rangle\rangle = 0$ if a_i is a square for some i
- (ii) $\langle\langle a_1, \dots, a_{i-1}, a_i, \dots, a_n \rangle\rangle = \langle\langle a_1, \dots, a_i, a_{i-1}, \dots, a_n \rangle\rangle$
- (iii) $\langle\langle a_1, \dots, a_{n-1}, a_n a'_n \rangle\rangle + \langle\langle a_1, \dots, a_n, a'_n \rangle\rangle = \langle\langle a_1, \dots, a_{n-1} a_n, a'_n \rangle\rangle + \langle\langle a_1, \dots, a_{n-1}, a_n \rangle\rangle$
- (iv) $\langle\langle a_1, \dots, a_{n-1}, a_n \rangle\rangle = \langle\langle a_1, \dots, a_{n-1}, (1 - a_{n-1})a_n \rangle\rangle$

Proof. Morel's theorem ([5], Théorème 5.3), shows that there is an exact sequence

$$0 \rightarrow K_n^M(F)^2 \rightarrow K_n^{MW}(F) \rightarrow I^n(F) \rightarrow 0$$

where the first (nontrivial) homomorphism maps $\{a_1, \dots, a_n\}^2 = \{a_1, \dots, a_i^2, \dots, a_n\}$ to $[a_1] \cdots [a_i^2] \cdots [a_n]$ (for any i) and the next homomorphism sends $[a_1] \cdots [a_n]$ to the class of the Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$. Combining this with Theorem 2.5 give the result, since the identity $-a = (1 - a)/(1 - a^{-1})$ shows that (i) and (iv) imply the identity $\langle\langle a_1, \dots, a_{n-1}, a_n \rangle\rangle = \langle\langle a_1, \dots, a_{n-1}, -a_{n-1} a_n \rangle\rangle$. \square

Remark 2.17. Compare this with the presentation of $I^n(F)$ given by Arason and Elman ([1], Theorem 3.1). Of course, Corollary 2.16 – like the result of Arason and Elman – requires the proof of the Milnor conjecture (since it is needed for Morel's theorem), and conversely easily implies the Milnor conjecture.

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