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# Averaging Techniques for the Analysis of Event Driven Models of All Digital PLLs

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**Abstract**—In this paper, we introduce a statistical approach for studying a special class of nonlinear dynamical systems such as ADPLLs and ADPLL networks, where the process driving the adjustment of the DCO frequency can be seen as  $\Sigma\Delta$  modulation. We showed that, by applying the Frobenius-Perron operator to the governing equation, it is possible to find the invariant probability density which is valid for dynamically changing input of  $\Sigma\Delta$  modulator. By using this, we show that the average behaviour of the corresponding complex system can be dramatically simplified and studied analytically.

## I. INTRODUCTION

All Digital Phase Locked Loops (ADPLLs) are compulsory components in many microelectronic systems since they generate clocking signals. In recent applications, ADPLLs are interconnected into trees or networks to generate a synchronised distributed signal [1]–[9]. The research on ADPLLs and their networks is driven by the following major issues: how to ensure the synchronisation and stability of a single ADPLL or a network [10]–[15] and how to minimise their jitter [16]–[25]. To address these issues, one should have a greater theoretical insight into the behaviour of the system.

This, however, is a challenging task since all signals in digital circuits are quantized and non-smooth. As a result, the full model that describes an ADPLL takes the form of a set of nonlinear piece-wise discrete-time equations [4], [5] in the most general case. It is difficult to perform their analysis in terms of naive continuous approach. Moreover, equations which describe quantised signals are entirely nonlinear and do not allow decomposition that includes principal linear term. Such an example could be the first-order  $\Sigma\Delta$  modulator with  $\text{sgn}(x)$  nonlinearity. Therefore, to date there is only one reliable method for studying these systems which is straightforward simulations which rely on numerical solutions of nonlinear finite-difference equations. The common disadvantages of this method are slow computation rates (especially for Monte Carlo simulations) and lack of understanding of the asymptotic behaviour of the system.

In spite of mathematical difficulties in the analysis of nonlinear iterative maps relevant for electronics and phase locked loops, there is a certain class of maps whose properties are studied quite well —  $\Sigma\Delta$  modulators. This type of nonlinear map originated from Analogue-to-Digital converters [16], [26], [27] and find themselves in Bang-Bang All-Digital Phase-Locked Loops (BBADPLL) [12]–[14], [28]. For example, statistical properties of oversampled  $\Sigma\Delta$  modulators were studied by Gray in [26]. It was shown that for a constant input the basic properties of a quantizer such as the average output and quantization noise can be obtained through ergodic properties of the corresponding iterative map. In study [27],

Koski showed the feasibility of the ergodic approach for studying the same system in the presence of circuit noise. In particular, it was shown that Gaussian circuit noise does not eliminate spikes in the power spectrum of a binary quantizer for constant input. For a periodic input, Teplinsky *et al.* showed that  $\Sigma\Delta$  modulators and BBADPLLs have common limit behaviours in terms of driven interval shift dynamics, and under certain conditions the trajectories remain within invariant belts [16]. Using sign dependent random walking in study [14], Tertinek *et al.* performed the statistical analysis of a first-order BBADPLL in the presence of accumulative phase noise of a reference clock. It was shown that the static timing jitter of an PLL can be represented explicitly via basic components: reference clock jitter and quantizer jitter. In Reference [28], the behavior of the same system was studied subject to jitter of the digital controlled oscillator (DCO).

In recent years ADPLLs have found implementation as a part of even more complex digital circuits — All-Digital PLL networks [1]–[3], [11]. This idea was inherited from works by Pratt and Nguyen [15], Gutnik and Chandrakasan [29] and Fridman [30]. These networks utilize ADPLLs distributed in a System-on-a-Chip (SoC), where DCOs interact with each other by means of average errors coming from adjacent error detectors. These networks can provide a coherent clocking signal distributed across the surface of a microchip. This approach could be an alternative to conventional single ADPLL clocking systems, where due to increasing topological complexity, it becomes impossible to supply a subsystem with a signal from a single clocking source.

It is not a surprise that the theoretical analysis of these types of dynamical systems faces tremendous difficulties especially when it comes to determining the set of control parameters where the system is stable in steady-state. However, as we show later in this paper, if it is possible to represent the principal dynamics of the system in the form of a first-order  $\Sigma\Delta$  modulator, the stability analysis of the system dramatically simplifies and can be performed in terms of linear finite-difference equations. The trade-off for such simplicity is the loss of information about jitter. However, taking into account the complexity of ADPLL networks, this trade-off is fair.

The paper is organised as follows. In Section II, we show how the  $\Sigma\Delta$  process appears as a principal driving term in the equations for ADPLLs. In Section III, we show the feasibility of an ergodic approach in studying the average dynamics of the first-order  $\Sigma\Delta$  modulator with respect to an *arbitrary input signal*. In Section IV, we apply the method established in Section III to study the stability of the ADPLL developed in [1]. Finally, the conclusions are made in Section V.

## II. STATEMENT OF THE PROBLEM

In this Section, we introduce a  $\Sigma\Delta$  modulator and nonlinear finite-difference equations for an ADPLL related to it. The mathematical model of the  $\Sigma\Delta$  modulator can be represented via the nonlinear map:

$$x_{n+1} = x_n - \text{sgn}(x_n) + \gamma_n \quad (1)$$

where the integer  $n$  is a sampling instance,  $x_n$  is the current state of the system,  $\gamma_n$  is the current input and  $\text{sgn}(x)$  is the signum function:  $\text{sgn}(x) = -1$  if  $x < 0$  and  $\text{sgn}(x) = 1$  if  $x \geq 0$ .

In Reference [12], the equations were introduced to describe the dynamics of a special kind of a second order BBADPLL (see Fig. 1):

$$\begin{cases} \Delta t_{n+1} = \Delta t_n + T^R - NT_0 - N\beta\Delta T\psi_{n-D} - N\alpha\Delta T\text{sgn}(\Delta t_{n-D}) \\ \psi_{n+1} = \psi_n + \text{sgn}(\Delta t_{n+1}) \end{cases}$$

where  $\Delta t_n$  is the time difference between the rising edges of local and reference oscillators,  $T^R$  is the reference clock period,  $T_0$  is the initial period of the local clock,  $\Delta T$  is the tuning step of the local clock,  $\text{sgn}(\varepsilon_n)$  is the proportional error,  $\psi_n$  is the accumulated error,  $N$  is the divider ratio,  $D$  is the digital delay and  $\alpha$  and  $\beta$  are proportional and integral gains in proportional-integral filter (PI). By making the variable transform:

$$\begin{aligned} \Delta t &= x_n \alpha N \Delta T \\ \gamma_n &= -(NT_0 + N\beta\Delta T\psi_{n-D} - T^R)/(\alpha N \Delta T) \end{aligned}$$

we obtain:

$$\begin{cases} x_{n+1} = x_n - \text{sgn}(x_{n-D}) + \gamma_n \\ \gamma_{n+D+1} = \gamma_{n+D} - R \text{sgn}(x_{n+1}) \end{cases} \quad (2)$$

where  $R = \beta/\alpha$ . Now the new variable  $x_n$  describes the normalised time difference between the reference and local signals. When the delay  $D = 0$ , the first equation corresponds to (1).

In our previous studies [4], [5], we introduced a model of the ADPLL implemented and verified in [1] (for the block diagram of the system, please refer again to Fig. 1). By making a variable substitution, one can show that the dynamics of such a system can be simplified to:

$$\begin{cases} x_{n+1} = x_n - \text{sgn}(x_n) + \gamma_n \\ \gamma_{n+1} = \gamma_n - \alpha(\varepsilon_{n+1} - \varepsilon_n) - 0.5\beta\varepsilon_n(1 - \sigma_n) \\ \varepsilon_{n+3} = 0.5\varepsilon_{n+2}(1 - \sigma_n\sigma_{n+1}) + 0.5(\sigma_n + \sigma_{n+1}) \end{cases} \quad (3)$$

where  $\sigma_n = \text{sgn}(x_n)$  and

$$\begin{aligned} \alpha &= 0.5K_p H_{\max} \Delta f_{\text{DCO}} / f^R \\ \beta &= 0.5K_i H_{\max} \Delta f_{\text{DCO}} / f^R \end{aligned} \quad (4)$$

In this system,  $K_p$  and  $K_i$  stand for the proportional and integral gain coefficients,  $\Delta f_{\text{DCO}}$  is the tuning step for the DCO,  $f^R$  is the frequency of the reference clock and  $H_{\max}$  is the maximum quantisation value for the time-to-digital converter in the digital time detector (DTD). The variable  $\varepsilon_n$  represents a normalized quantised error (difference between the reference and local signals). The variable  $\gamma_n$  is related to

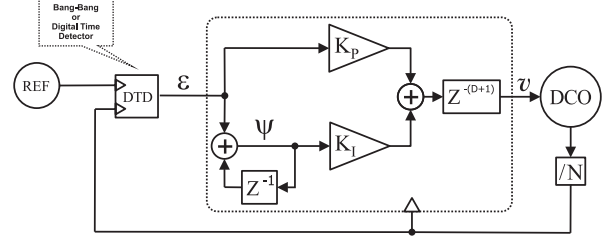


Fig. 1. Block diagram of an All-Digital PLL. The system contains a digital time detector (DTD) that determines the difference ( $\varepsilon$ ) between the reference and local signals, proportional-integral (PI) filter and a digitally controlled oscillator (DCO). The PI controller provides a control signal ( $v$ ) that changes the frequency of the DCO. Reference [12] utilises a Bang-Bang DTD whereas study [2] utilises a particular finite state machine DTD.

the reference  $f^R$  and local DCO  $f_n^D$  frequencies as follows:  $\gamma_n = (f^R - f_n^D)/(f^R + f_n^D)$ . Here we assume that normalised control parameters  $\alpha$  and  $\beta$  are small, therefore, the frequency dynamics are slow:  $f_{n+1}^D \approx f_n^D$  hence  $\gamma_{n+1} \approx \gamma_n$ .

In the model discussed above, the frequency of the local clock changes linearly with respect to a control code  $v_n = K_i\varepsilon_n + K_p\psi_n$ . However, due to memory restrictions in the integrator which is used for cumulative error computations, the frequency range of the DCO is limited between minimum and maximum values:  $f_n^D \in [f_{\min}; f_{\max}]$ . Clearly, in order for an ADPLL to operate stably, the reference frequency has to lie within that range as well. This assumption gives us the upper bound for  $|\gamma_n| < (f_{\max} - f_{\min})/(2f_{\min})$ . For the typical range (e.g., in [2]), we may conclude that  $|\gamma_n| < |\gamma_{\max}| \approx 0.2$ . Even more, in the vicinity of a stable region (if it exists),  $f_n^D \approx f^R$ , thus for this case  $|\gamma| \ll |\gamma_{\max}|$  and can be treated as a small value.

## III. AVERAGING TECHNIQUES

The equations given in the previous Section describe the behaviour of a single ADPLL. In particular, equation (2) represents the dynamics of a Bang-Bang ADPLL [12], whereas (3) represents the simplified dynamics of an ADPLL from [2]. In this section, we introduce an averaging technique which significantly simplifies the stability analysis of the corresponding systems. This can be expanded to the case of multiple ADPLLs and be used for the analysis of their networks.

The first equations in (2) (for  $D = 0$ ) and (3) have the same form as the equation of the first order  $\Sigma\Delta$  modulator provided that  $\gamma_n$  is approximately constant. It is known that for  $|\gamma_n| < 1$ ,  $|x_n|$  is bounded and  $1 - \gamma_n < |x_{n+1} - x_n| < 1 + \gamma_n$ . In (3) it means that the rate of change of  $x_n$  is much faster than for  $\gamma_n$  (where we again remind that  $\alpha$  and  $\beta$  are small). Therefore, inspired by the averaging idea developed in the theory of continuous nonlinear systems, we will describe the averaged dynamics for  $\varepsilon_n$  and  $\gamma_n$  by assuming that the initial conditions for  $x_0$  are continuously distributed according to the probability density function  $\omega_0(x)$  of the variable  $x_n$ .

From now on, we focus our attention on the system (3) due to its practical implementation described in [1]–[3]. In this paper, we will use the following parameters of its ASIC implementation:  $V_{\text{DD}} = 1.0$  V,  $\Delta f_{\text{DCO}} = 150$  kHz,  $f_{\min} = 135$  MHz,  $f_{\max} = 175$  MHz, resolution time of a time-to-digital converter in a digital time detector  $\tau_{\text{TDC}} = 20$  ps, number of levels in DTD  $N_{\text{DTD}} = 7$ .

By applying averaging by  $x_n$  to (3) we get:

$$\begin{cases} \gamma_{n+1} = \gamma_n - \alpha(\varepsilon_{n+1} - \varepsilon_n) - 0.5\beta\varepsilon_n(1 - \langle \sigma_n \rangle) \\ \varepsilon_{n+3} = 0.5\varepsilon_{n+2}(1 - \langle \sigma_n \sigma_{n+1} \rangle) + 0.5(\langle \sigma_n \rangle + \langle \sigma_{n+1} \rangle) \end{cases} \quad (5)$$

where  $\gamma_n$  and  $\varepsilon_n$  are slow variables, and  $\langle \cdot \rangle$  denotes averaging over  $x_n$ . It is clear that during such an averaging procedure, information about high frequency dynamics and digital jitter wipes out. However, the average dynamics and its stability analysis simplify dramatically and can be applied for even more complex systems.

In order to calculate  $\langle \sigma_n \rangle$ ,  $\langle \sigma_{n+1} \rangle$  and  $\langle \sigma_n \sigma_{n+1} \rangle$ , we use the ergodic principle that can be applied to iterative map (1). For an ergodic process, all average characteristics can be calculated using the invariant probability density distribution function (PDF)  $\omega(x)$ . Using the Frobenius-Perron operator, one can show that for a constant (and irrational)  $\gamma$ , where  $|\gamma| < 1$ , the PDF for (1) satisfies the equation:

$$\omega(x) = \int_{-\infty}^{\infty} \omega(x') P(x|x') dx' \quad (6)$$

$$P(x|x') = \delta[x - (x' - \text{sgn}(x') + \gamma)] \quad (7)$$

where  $\delta(x)$  is the Dirac delta-function. The solution of this is a uniform distribution over a range  $x \in (\gamma - 1; \gamma + 1)$ :

$$\omega(x, \gamma) = u(x, \gamma - 1, \gamma + 1) \quad (8)$$

$$u(x, a, b) = \frac{1}{2} \begin{cases} 1 & a < x < b \\ 0 & x \notin (a; b) \end{cases} \quad (9)$$

Iterative map (1) and its invariant distribution (8) possess a remarkable property. Namely, by applying transform (7) with a different parameter  $\gamma'$  to a set  $x$  given by the PDF  $\omega(x, \gamma)$ , we get a new set  $x$  with a similar PDF, that characterises by parameter  $\gamma'$  instead of  $\gamma$ , i.e.,  $\omega(x, \gamma')$ . Therefore, for an arbitrary sequence  $\{\gamma_n\}$ , the PDF for  $x$  evolves as:

$$\omega_n(x) = u(x, \gamma_{n-1} - 1, \gamma_{n-1} + 1) \quad (10)$$

if the initial PDF is from the class given by eq. (8).

We can prove statement (10) as follows. Let  $n \in \mathbb{Z}$  and  $x, x', \gamma_n$  and  $\gamma_{n+1} \in \mathbb{R}$ . We also assume that  $|\gamma_n| < 1$ ,  $|\gamma_{n+1}| < 1$  and  $\omega_n(x) = u(x, \gamma_{n-1} - 1, \gamma_{n-1} + 1)$ . Similarly to (6) and (7), we obtain:

$$\omega_{n+1}(x) = \int_{-\infty}^{\infty} \omega_n(x') \delta[x - (x' - \text{sgn}(x') + \gamma_n)] dx' \quad (11)$$

By splitting the integral into two parts, where  $x' \leq 0$  and  $x' > 0$ , and taking into account  $\delta(-x) = \delta(x)$ , we get:

$$\begin{aligned} \omega_{n+1}(x) &= \int_{-\infty}^0 \omega_n(x') \delta[x' - (x - (\gamma_n + 1))] dx' + \\ &\quad \int_0^{\infty} \omega_n(x') \delta[x' - (x - (\gamma_n - 1))] dx' = \\ &= \omega_n(x - (\gamma_n + 1)) \theta^-(x - (\gamma_n + 1)) + \omega_n(x - (\gamma_n - 1)) \theta^+(x - (\gamma_n - 1)) \end{aligned}$$

where  $\theta^+(x)$  is the Heaviside step function and  $\theta^-(x) = \theta^+(-x)$ . According to the definition of  $\omega(x)$  and  $\theta^-(x)$ ,  $\omega_n(x - (\gamma_n + 1)) \theta^-(x - (\gamma_n + 1)) = 0.5$  if

$$\gamma_{n-1} + \gamma_n < x < \gamma_n + 1$$

Therefore, using notation (9) we get:

$$\omega_n(x - (\gamma_n + 1)) \theta^-(x - (\gamma_n + 1)) = u(x, \gamma_{n-1} + \gamma_n, \gamma_n + 1)$$

By analogy, performing similar calculations we get:

$$\omega_n(x - (\gamma_n - 1)) \theta^+(x - (\gamma_n - 1)) = u(x, \gamma_n - 1, \gamma_{n-1} + \gamma_n)$$

By adding them together we obtain the following result:

$$\begin{aligned} u(x, \gamma_n - 1, \gamma_{n-1} + \gamma_n) + u(x, \gamma_{n-1} + \gamma_n, \gamma_n + 1) = \\ u(x, \gamma_n - 1, \gamma_n + 1) = \omega_{n+1}(x) \end{aligned}$$

which proves statement (10).

Using PDF (10), we calculate  $\langle \sigma_n \rangle$  and  $\langle \sigma_n \sigma_{n+1} \rangle$  for arbitrary  $n$ :

$$\langle \sigma_n \rangle = \frac{1}{2} \int_{\gamma_{n-1}-1}^{\gamma_{n-1}+1} \text{sgn}(x) dx = \gamma_{n-1}$$

In order to calculate  $\langle \sigma_n \sigma_{n+1} \rangle$ , we use map (1):

$$\sigma_n \sigma_{n+1} = \text{sgn}(x_n) \cdot \text{sgn}(x_{n+1}) = \text{sgn}(x_n) \cdot \text{sgn}(x_n - \text{sgn}(x_n) + \gamma_n)$$

and so

$$\begin{aligned} \langle \sigma_n \sigma_{n+1} \rangle &= \frac{1}{2} \int_{\gamma_{n-1}-1}^{\gamma_{n-1}+1} \text{sgn}(x) \text{sgn}(x - \text{sgn}(x) + \gamma_n) dx = \\ &= -\frac{1}{2} \int_{\gamma_{n-1}-1}^0 \text{sgn}(x + 1 + \gamma_n) dx + \frac{1}{2} \int_0^{\gamma_{n-1}+1} \text{sgn}(x - 1 + \gamma_n) dx = \\ &= \frac{1}{2} \int_{1+\gamma_n}^{\gamma_n+\gamma_{n-1}} \text{sgn}(y) dy + \frac{1}{2} \int_{\gamma_{n-1}}^{\gamma_n+\gamma_{n-1}} \text{sgn}(y) dy = |\gamma_n + \gamma_{n-1}| - 1 \end{aligned}$$

By using that, we can rewrite (5) as follows:

$$\begin{cases} \gamma_{n+1} = \gamma_n - \alpha(\varepsilon_{n+1} - \varepsilon_n) - 0.5\beta\varepsilon_n(1 - \gamma_{n-1}) \\ \varepsilon_{n+3} = 0.5\varepsilon_{n+2}(2 - |\gamma_n + \gamma_{n-1}|) + 0.5(\gamma_{n-1} + \gamma_n) \end{cases} \quad (12)$$

We can see that the averaged equations are much simpler than the equations for the original system (3). In the next Section, we shall analyse their stability by using a linear perturbation technique.

#### IV. STABILITY ANALYSIS

From the second equation in (12), we can see that for all  $\gamma_n + \gamma_{n-1}$  of the same sign there is a steady state error  $\varepsilon_n = \text{const}$ , and  $\varepsilon_n = \text{sgn}(\gamma_{n-1} + \gamma_n)$ . However, because of negative feedback implemented in the second equation of (12), such deviations lead to decreasing  $\gamma_n$  towards zero, so does  $\varepsilon_n$ . Therefore, a synchronous regime with  $\gamma_n = 0$  and  $\varepsilon_n = 0$  can exist. We will check its stability by linearising (12) in the vicinity of  $\gamma_n = 0$  and  $\varepsilon_n = 0$ :

$$\begin{cases} \gamma_{n+1} = \gamma_n - \alpha(\varepsilon_{n+1} - \varepsilon_n) - 0.5\beta\varepsilon_n \\ \varepsilon_{n+3} = \varepsilon_{n+2} + 0.5(\gamma_{n-1} + \gamma_n) \end{cases} \quad (13)$$

In order to perform the stability analysis of this map we represent it in a conventional form by variable transformations. Let  $\mathbf{x}_n$  be a vector with components:

$$\mathbf{x}_n = [\gamma_n, \gamma_{n-1}, \varepsilon_{n+2}, \varepsilon_{n+1}, \varepsilon_n]^T$$

then (13) corresponds to the linear map:

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$$

where the matrix  $\mathbf{A}$  is:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -\alpha & \alpha - 0.5\beta \\ 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution  $\mathbf{x}_n = 0$  is stable when all eigenvalues  $\lambda_k$  of the matrix  $\mathbf{A}$  lie inside a unit circle  $|\lambda_k| < 1$ . The corresponding characteristic equation for eigenvalues is:

$$\lambda^5 - 2\lambda^4 + \lambda^3 + 0.5\lambda^2\alpha + 0.25\lambda\beta - 0.5\alpha + 0.25\beta = 0$$

In order to analyse the conditions for  $|\lambda_k| < 1$  to be true, we will use the conformal transform  $\lambda \mapsto \mu$ :

$$\lambda = \frac{1 + \mu}{1 - \mu}$$

This map transforms the domain  $|\lambda_k| < 1$  into the domain  $\text{Re}(\mu) < 0$ , where  $\mu_k$  determine by the equation:

$$\begin{aligned} \mu^5 + \mu^4 \left( \frac{\beta}{8} - \frac{\alpha}{2} + 3 \right) + \mu^3 \left( -\frac{\beta}{2} + \frac{3\alpha}{2} + 3 \right) + \\ \mu^2 \left( \frac{3\beta}{4} - \frac{3\alpha}{2} + 1 \right) + \mu \left( -\frac{\beta}{2} + \frac{\alpha}{2} \right) + \frac{\beta}{8} = 0 \end{aligned}$$

By applying the Routh-Hurwitz stability criterion to this equation, we find an analytic expression for the stable operating region of the system (3) and (13). The result shows that the stable operating region spanned by parameters  $\alpha > 0, \beta > 0$  and

$$32\alpha^4 - 8\alpha^3(24 + 7\beta) + 4\alpha^2(-96 + 76\beta + 9\beta^2) + \quad (14)$$

$$\beta(-448 - 224\beta + 28\beta^2 + \beta^3) - 2\alpha(-128 - 240\beta + 76\beta^2 + 5\beta^3) > 0$$

For small  $\alpha$  and  $\beta$ , the domain can be simplified to:

$$(\alpha > 0) \cap \left( 0 < \beta < \frac{4}{7}\alpha \right) \quad (15)$$

Figure 2 summarises the main results of the work, including the domain of stability in the  $(\alpha, \beta)$  plane and the digital jitter displayed by the system. The two cases shown in this figure correspond to two different quantisation levels of the digital time detector ( $N_{\text{DTD}} = 1$  and  $N_{\text{DTD}} = 7$  respectively). The line reflecting the necessary condition of stability is  $\beta = (4/7)\alpha$  and separates the area where synchronisation is not possible under any conditions (above the line), from the area where synchronisation can be achieved with different levels of digital jitter (below the line). We particularly note that the stability line is the same for  $N_{\text{DTD}}$  in this example. This stability condition provides a simple and clear guideline for the practical design of ADPLLs with any  $N_{\text{DTD}}$  and configurations of the DCO controller. This result is general, not limited to a particular configuration of an ADPLL and can be extended to other PLL topologies.

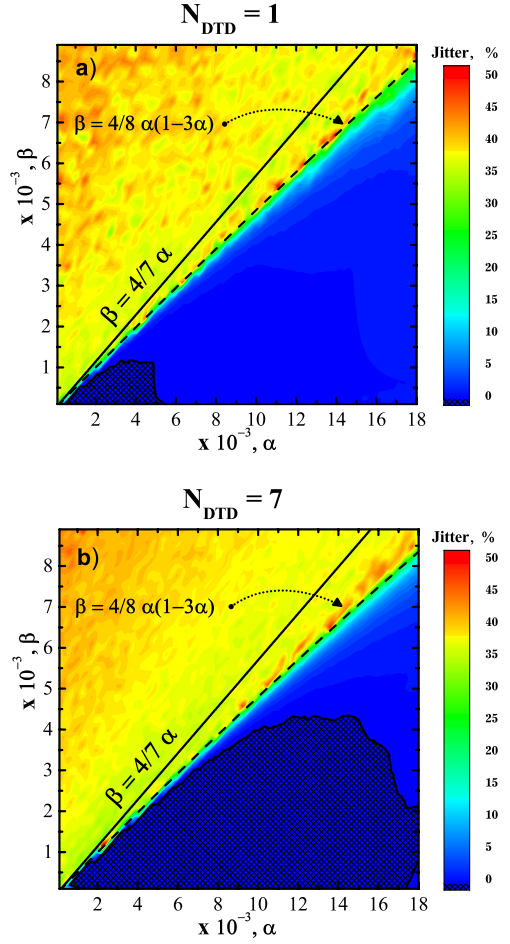


Fig. 2. The dependence of the relative timing jitter on PI control parameters for two different quantisation levels of the digital time detector in a single ADPLL. The solid line represents the theoretical boundary that distinguishes stable and unstable worst case dynamics of (3). The dashed line along with the colour plot represents the numerical result from [4]. The shaded region corresponds to jitter  $< 0.5\%$

## V. CONCLUSIONS

In this study, we introduced an averaging technique for the analysis of All-Digital PLLs. The aim of the analysis was to obtain an insight into the stability of ADPLLs and their digital jitter, without limiting ourselves to a particular configuration of the system. We made only one assumption about the operation of an ADPLL (“adiabatic process”) meaning that the dynamics experienced by the DCO frequency are relatively slow, which is the case for a reasonably wide range of control parameters. We began with equations of an ADPLL we derived and verified experimentally in our previous work, and we showed that these can be related (but not reduced or simplified) to equations known in the literature. Having taken  $\Sigma\Delta$  modulation as the driving process, we proposed an averaging technique based on the ergodic properties of the signals generated in an ADPLL. For these purposes, we used the Frobenius-Perron operator for obtaining the evolution of the probability density function to describe the system response to an arbitrary input signal. This technique allowed us to obtain much simpler equations in terms of the average properties of the signals generated in the system and to analyse these equations analytically. As a result, we obtained a necessary condition for synchronisation which can be a broader class of ADPLLs.

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