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A SCHWARZ LEMMA AND COMPOSITION OPERATORS

M. MACKEY AND P. MELLON

ABSTRACT. We give an alternative description of the Carathéodory pseudo-distance on a domain D in an arbitrary complex Banach space. This gives a Schwarz lemma for holomorphic maps of the domain. We specialise to the case of a bounded symmetric domain and obtain some applications. In particular, we give the connected components of the space of composition operators with symbol in a bounded symmetric domain. This generalises results for the space of composition operators on $H^\infty(\Delta)$ in [12] and for $H^\infty(B)$, B the unit ball of a Hilbert space or commutative C^*-algebra in [2].

INTRODUCTION

Let D be a domain in a complex Banach space E and let Δ be the open unit disc in $\mathbb{C}$. We define the following pseudo-distance on D,

$$d_D(z,w) := \sup\{|f(z) - f(w)| : f : D \to \Delta \text{ holomorphic}\} \quad \text{for } z, w \in D.$$

We prove that

$$\log \frac{2 + d_D}{2 - d_D}$$

is in fact the Carathéodory pseudo-distance C_D on D. This results in a Schwarz Lemma for holomorphic maps from D to Δ. When we specialise this to B_E, the open unit ball of a Banach space E, we prove firstly that d_{B_E} can be expressed in terms of holomorphic self-maps of B_E, namely,

$$d_{B_E}(z,w) = \sup\{\|f(z) - f(w)\| : f : B_E \to B_E \text{ holomorphic}\}.$$

Since the Carathéodory distance on B_E satisfies

$$C_{B_E}(z,0) = \tanh^{-1} \|z\|$$

we obtain, among others, the following Schwarz Lemma for all $f : B_E \to B_E$ holomorphic:

$$\|f(z) - f(0)\| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \text{ for all } z \in B_E.$$

If B is a bounded symmetric domain and $f : B \to B$ is holomorphic, we get

$$\|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \text{ for all } z, w \in B.$$

The description of d_{B_E} in terms of holomorphic self-maps of B_E makes it suited to the study of composition operators on the space $H^\infty(B_E)$ and, indeed, this is the motivation behind the introduction of d_Δ in the one variable case in [12]. The set-up is as follows:

to every $\phi : B_E \to B_E$ holomorphic we associate a linear map C_ϕ, called a composition operator, on the space $H^\infty(B_E)$ of all bounded holomorphic functions on B_E by

$$C_\phi(f) = f \circ \phi$$

for $f \in H^\infty(B_E)$. The idea is to associate the function theoretic properties of ϕ with the properties of C_ϕ as a linear mapping.

For $B = \Delta$, a survey of the classical theory of composition operators on the Hardy and Bergman spaces is given in [4] and [16]. To extend the classical results where ϕ is taken as a holomorphic function on Δ to the case where ϕ is a function of several or even infinitely many variables, one can head in a variety of directions. For example, if B_n is the open unit ball of $\mathbb{C}^n$, MacCluer, Shapiro and Luecking, among others have looked at the action of C_ϕ on the Hardy spaces $H^p(B_n)$, $0 < p < \infty$ and the Bergman spaces $A^p(B_n)$, $0 < p < \infty$. Jafari, Li, Russo and others have studied C_ϕ on the Hardy and Bergman spaces of finite dimensional bounded symmetric domains and strongly pseudo-convex domains. We refer to the survey of Russo [15] for references and more information. In the infinite dimensional case, we refer to [1, 2, 7] which study composition operators on the space $H^\infty(B_E)$, for E a complex Banach space.

Our aim is to extend to a bounded symmetric domain B results of MacCluer, Ohno and Zhao for the one variable case in [12] that determine the connected components of the topological space of composition operators on $H^\infty(\Delta)$ with the natural uniform norm topology. These results were extended in [2] when B is the open unit ball of a Hilbert space or commutative C^*-algebra, and in [17] when[1] B is the open unit ball of $\mathbb{C}^n$.

We recall that every bounded symmetric domain B can be realised as the open unit ball of a Banach space Z, known as a JB^*-triple [8]. The algebraic properties of Z, in particular the properties of the Bergman operator $B(z, w)$ and the quasi-inverse map $z \to z^a$ are then used, together with the distance d_B, to determine the connected components of the space of composition operators on $H^\infty(B)$. For a general survey and background details on the Poincaré distance, Carathéodory pseudo-distance and JB^*-triples we refer to [5].

1. Notation and Background

We let E and F denote complex Banach spaces and D and $\tilde{D}$ domains in E and F respectively. The set of all holomorphic maps from D to $\tilde{D}$ is denoted by $H(D, \tilde{D})$. We write $H^\infty(D)$ for the space of all bounded $\mathbb{C}$-valued holomorphic functions on D and $\|f\|_\infty := \sup_{z\in D} |f(z)|$ for all $f \in H^\infty(D)$.

Definition 1.1. The Poincaré distance ρ on Δ is

$$\rho(z, w) := \tanh^{-1} \left| \frac{z - w}{1 - \overline{z}w} \right| \quad \text{for } z, w \in \Delta.$$

[1]We thank the refereee for drawing our attention to this reference.

The Carathéodory pseudo-distance can be defined on any complex manifold [5], although we restrict our attention here to the case of a domain D.

Definition 1.2. The Carathéodory pseudo-distance on a domain D is given by

$$C_D(z,w) := \sup\{\rho(f(z),f(w)) : f \in H(D,\Delta)\} \quad \text{for } z,w \in D.$$

The Carathéodory pseudo-distances form a Schwarz-Pick system (cf. [5]) for which holomorphic functions act as contractions, namely,

$$C_{D_2}(f(z),f(w)) \le C_{D_1}(z,w) \;\text{ for all } f \in H(D_1,D_2),\; z,w \in D_1.$$

In fact, this is the smallest of all Schwarz-Pick systems. For bounded domains, cf. [5, chapters 4 and 5], it turns out that C_D is continuous and generates the original topology thus ensuring that it is actually a distance on D.

We now introduce the class of Banach spaces known as the JB^*-triples. We use H and K to denote arbitrary complex Hilbert spaces and $\mathcal{L}(X,Y)$ to denote the space of continuous linear operators from a Banach space X to a Banach space Y. We let $\mathcal{L}(X) = \mathcal{L}(X,X)$ and $\mathsf{GL}(X)$ be all invertible elements in $\mathcal{L}(X)$.

Definition 1.3. A JB^*-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x,y,z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.

(ii) The map $z \to \{x,x,z\}$, denoted $x \,\square\, x$, is Hermitian, $\sigma(x \,\square\, x) \ge 0$ and $\| x \,\square\, x \| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum.

(iii) The product satisfies the following "triple identity"

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

Let Z be a JB^*-triple. Several types of linear operators on Z arise naturally from the triple product:

$$x \,\square\, y \in \mathcal{L}(Z) : z \to \{x,y,z\},$$

$$Q_x \in \mathcal{L}_{\mathbb{R}}(Z) : z \to \{x,z,x\},$$

and the important Bergman operators

$$B(x,y) = I - 2\,x \,\square\, y + Q_x Q_y \in \mathcal{L}(Z).$$

Example 1.4. (i) $\mathcal{L}(H,K)$ is a JB^*-triple for the product $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$ where y^* denotes the usual adjoint of y and $B(x,y)z = (1-xy^*)z(1-y^*x)$.

(ii) $\mathcal{C}_0(X)$, the continuous $\mathbb{C}$-valued functions vanishing at infinity on a locally compact Hausdorff space X, is a JB^*-triple for the product $\{x,y,z\} = x\overline{y}z$ and $B(x,y)z = (1-x\overline{y})^2 z$.

As Banach spaces the JB^*-triples are characterised by the fact that their open unit balls are homogeneous. In fact, if we let $\mathsf{Aut}(B)$ denote all biholomorphic maps from B to B then for all a in B, we have $g_a \in \mathsf{Aut}(B)$ defined by

$$g_a(z) = a + B(a,a)^{\frac{1}{2}}(I + z \,\square\, a)^{-1}z$$

(cf. [9]) which satisfies $g_a(0) = a$, $g_a^{-1} = g_{-a}$ and $g_a'(0) = B(a,a)^{\frac{1}{2}}$ (defined in terms of a functional calculus). We note the fundamental formula [10]

$$\|B(a,a)^{-\frac{1}{2}}\| = \frac{1}{1 - \|a\|^2} \tag{1}$$

for $a \in B$. For $z, a \in B$, $z^a := (I - z \,\square\, a)^{-1}z$ is called the quasi-inverse of z with respect to a and satisfies

$$\|z^a\| \le \frac{\|z\|}{1 - \|z\|\|a\|}. \tag{2}$$

The quasi-inverse also satisfies $(z^a)^b = z^{a+b}$ whenever both sides of this equation are well defined. For further details see [11, Chapter 7] or [6].

It is known [8] that every bounded symmetric domain is biholomorphically equivalent to the open unit ball of a JB^*-triple and vice versa. From the homogeneity therefore one can easily see that on a bounded symmetric domain B the Carathéodory distance is given by $C_B(z,w) = \tanh^{-1}\|g_{-z}(w)\|$. For a recent survey of JB^*-triples and bounded symmetric domains we refer to [3].

2. The Carathéodory pseudo-distance

To study composition operators on $H^\infty(\Delta)$ MacCluer et al. [12] introduce the distance d_∞ on Δ

$$d_\infty(z,w) := \sup\{|f(z) - f(w)| : f \in H^\infty(\Delta), \|f\|_\infty \le 1\}, \quad z, w \in \Delta.$$

It is not too difficult to see [13] that

$$d_\infty(z,w) = \frac{2 - 2\sqrt{1 - \beta(z,w)^2}}{\beta(z,w)} \quad \text{for } \beta(z,w) := \left|\frac{z - w}{1 - \overline{z}w}\right|$$

or in terms of the Poincaré metric ρ on Δ

$$\rho(z,w) = \tanh^{-1}\beta(z,w) = \log\frac{2 + d_\infty(z,w)}{2 - d_\infty(z,w)}. \tag{3}$$

Motivated by this we introduce the following pseudo-distance on an arbitrary domain D

$$d_D(z,w) := \sup\{|f(z) - f(w)| : f \in H(D,\Delta)\}.$$

We note that $d_\Delta = d_\infty$ above. Clearly

$$\begin{aligned} d_D(z,w) &= \sup\{|g(f(z)) - g(f(w))| : g \in H(\Delta,\Delta), f \in H(D,\Delta)\} \\ &= \sup_{f \in H(D,\Delta)} d_\infty(f(z), f(w)) \qquad \text{for } z, w \in D. \end{aligned}$$

Since the map $t \to \log \frac{2+t}{2-t}$ is strictly increasing on $[0,2)$ it follows that

$$\begin{aligned}\log \frac{2+d_D(z,w)}{2-d_D(z,w)} &= \sup_{f\in H(D,\Delta)} \log \frac{2+d_\infty(f(z),f(w))}{2-d_\infty(f(z),f(w))} \\ &= \sup_{f\in H(D,\Delta)} \rho(f(z),f(w)) \quad \text{from (3)} \\ &= C_D(z,w).\end{aligned}$$

In other words, $\log \frac{2+d_D}{2-d_D}$ is the Carathéodory pseudo-distance on D, or equivalently for any domain D

$$d_D(z,w) = \frac{2-2\sqrt{1-(\tanh C_D(z,w))^2}}{\tanh C_D(z,w)} \qquad \text{for } z,w \in D. \tag{4}$$

Throughout, we use B_E to denote the open unit ball of an arbitrary complex Banach space E and reserve B to denote a bounded symmetric domain.

We now present a series of Schwarz Lemmas arising from (4).

Lemma 2.1. *(i) Let D be an arbitrary domain and $f : D \to \Delta$ be holomorphic. Then*

$$|f(z)-f(w)| \le \frac{2-2\sqrt{1-(\tanh C_D(z,w))^2}}{\tanh C_D(z,w)} \quad \textit{for } z,w \in D.$$

In particular, if $D = B_E$ is the open unit ball of a Banach space E then

$$|f(z)-f(0)| \le \frac{2-2\sqrt{1-\|z\|^2}}{\|z\|} \quad \textit{for } z \in B_E.$$

(ii) Let B be a bounded symmetric domain and $f : B \to \Delta$ be holomorphic. Then

$$\begin{aligned}|f(z)-f(w)| &\le \frac{2-2\sqrt{1-\|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \\ &= 2\frac{\sqrt{\|B_w^{-1}B(w,z)B_z^{-1}\|}-1}{\sqrt{\|B_w^{-1}B(w,z)B_z^{-1}\|-1}} \qquad \textit{for } z,w \in B\end{aligned}$$

where g_z is an automorphism of B taking 0 to z and $B_z := B(z,z)^{\frac{1}{2}}$.

Proof. (i) is immediate from (4). The first part of (ii) follows from (i) since on a bounded symmetric domain B

$$C_B(z,w) = \tanh^{-1}\|g_{-z}(w)\| \ \text{ for } z,w \in B \tag{5}$$

where g_z is an automorphism of B taking 0 to z.

For the second part of (ii) we rewrite

$$\frac{2-2\sqrt{1-\|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|}$$

in terms of Bergman operators using the fact [14, Proposition 3.1] that

$$\frac{1}{1-\|g_{-z}(w)\|^2} = \|B_w^{-1}B(w,z)B_z^{-1}\| \quad \text{for } z,w \in B. \tag{6}$$

□

For the purpose of studying composition operators on $H^\infty(B_E)$ the distance we really need on B_E is written in terms of self-maps of B_E, namely,

$$\tilde{d}_{B_E}(z,w) := \sup\{\|f(z)-f(w)\| : f \in H(B_E)\}.$$

Proposition 2.2. *The distance $\tilde{d}_{B_E}$ coincides with d_{B_E}.*

Proof. Fix z,w in B_E and $f \in H(B_E)$. By the Hahn-Banach theorem there exists $\lambda = \lambda(z,w,f) \in Z^*$, $\|\lambda\| \le 1$ with

$$\|f(z)-f(w)\| = \lambda(f(z)-f(w))$$

and hence $\tilde{d}_{B_E}(z,w) \le d_{B_E}(z,w)$. On the other hand, if $g \in H(B_E,\Delta)$ then for any fixed $u \in \partial B_E$ the map $z \to g(z)u$ is in $H(B_E)$ and this implies $d_{B_E}(z,w) \le \tilde{d}_{B_E}(z,w)$. □

Proposition 2.2 together with (4), (5) and (6) now easily gives the following.

Corollary 2.3. *(i) Let E be a Banach space and $f : B_E \to B_E$ be holomorphic. Then*

$$\|f(z)-f(w)\| \le \frac{2-2\sqrt{1-(\tanh C_{B_E}(z,w))^2}}{\tanh C_{B_E}(z,w)} \quad \text{for } z,w \in B_E.$$

In particular,

$$\|f(z)-f(0)\| \le \frac{2-2\sqrt{1-\|z\|^2}}{\|z\|} \quad \text{for } z \in B_E.$$

(ii) Let B be a bounded symmetric domain and $f : B \to B$ be holomorphic. Then

$$\begin{aligned}\|f(z)-f(w)\| &\le \frac{2-2\sqrt{1-\|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \\ &= 2\frac{\sqrt{\|B_w^{-1}B(w,z)B_z^{-1}\|}-1}{\sqrt{\|B_w^{-1}B(w,z)B_z^{-1}\|-1}} \quad \text{for } z,w \in B,\end{aligned}$$

where g_z is an automorphism of B taking 0 to z and $B_z := B(z,z)^{\frac{1}{2}}$.

As the Bergman operators play a fundamental role in the holomorphy of B and $B(a,a)^{\frac{1}{2}} = g_a'(0)$, $a \in B$ the inequality [10]

$$\|B(a,a)^{\frac{1}{2}}\| \le 1$$

is crucial to the geometry of B. We are able to obtain a simple direct proof of this result.

Corollary 2.4. *For $a \in B$, $\|B(a,a)^{\frac{1}{2}}\| \le 1$.*

Proof. Fix $a \in B$. For all $z \in B$

$$\frac{2-2\sqrt{1-\|z\|^2}}{\|z\|} = d_B(z,0) \geq \|g_a(z) - g_a(0)\| = \|B(a,a)^{\frac{1}{2}} z^{-a}\|.$$

Since $z^a \in B$ if $\|z\| < \frac{1}{1+\|a\|}$ and $(z^a)^{-a} = z$ this implies that

$$\|B(a,a)^{\frac{1}{2}} z\| \leq \frac{2-2\sqrt{1-\|z^a\|^2}}{\|z^a\|}$$

when $\|z\| < \frac{1}{1+\|a\|}$. Fix $0 < t < \frac{1}{1+\|a\|}$. For $\|z\| \leq t$, we have from (2) that

$$\|z^a\| \leq \frac{\|z\|}{1-\|z\|\|a\|} \leq \frac{t}{1-t\|a\|}$$

and since $h(t) = (2-2\sqrt{1-t^2})/t$ is strictly increasing on $[0,1)$ this gives

$$\|B(a,a)^{\frac{1}{2}} z\| \leq h(\|z^a\|) \leq h(\frac{t}{1-t\|a\|}).$$

Then

$$\begin{aligned} \|B(a,a)^{\frac{1}{2}}\| = \sup_{\|z\|=1} \|B(a,a)^{\frac{1}{2}} z\| &= \frac{1}{t} \sup_{\|z\|=t} \|B(a,a)^{\frac{1}{2}} z\| \\ &\leq \frac{1}{t}\, h(\frac{t}{1-t\|a\|}) \\ &= 2\left(1-t\|a\| + \sqrt{(1-t\|a\|)^2 - t^2}\right)^{-1}. \end{aligned}$$

As $t \to 0$ this gives $\|B(a,a)^{\frac{1}{2}}\| \leq 1$ as required. □

3. Composition operators on $H^\infty(B)$

In this section we study the connected components of the space of composition operators on $H^\infty(B)$ with the uniform norm topology where B is a bounded symmetric domain. Our motivation was to extend the one variable results in [12] to the case of infinitely many variables. In the case where B is the open unit ball of a Hilbert space or of a commutative C^*-algebra we refer to [2]. The key to this study is the distance d_B which gives a formula for the hyperbolic distance C_B, namely,

$$C_B = \log \frac{2+d_B}{2-d_B}.$$

Just as the Möbius maps are crucial when studying Δ, so the analogous automorphisms $\{g_a : a \in B\}$ of B are essential here and we establish some simple identities.

Lemma 3.1. *For $a, b \in B$,*

$$g_{-a}(a+b) = (B(a,a)^{-\frac{1}{2}} b)^a \qquad \textit{when } a+b \in B, \tag{7}$$

and

$$g_{-a}(b) = (B(a,a)^{-\frac{1}{2}}(b-a))^a. \tag{8}$$

Proof. Clearly, the two expressions are equivalent. Recall that $g_a(z) = a + B(a,a)^{\frac{1}{2}} z^{-a}$. Since the inverse of $z \to z^a$ is $z \to z^{-a}$ and the inverse of g_a is g_{-a} it follows that $g_{-a}(b) = g_a^{-1}(b) = (B(a,a)^{-\frac{1}{2}}(b-a))^a$. □

For $z, w \in B$, we define $\beta(z,w) := \|g_{-z}(w)\|$.

Definition 3.2. For $\phi, \psi \in H(B)$ we let

$$d_\beta(\phi,\psi) := \sup_{z\in B} \beta(\phi(z),\psi(z)).$$

We note that d_β is a metric on $H(B)$ and, by virtue of the following result, it is the topological structure of $(H(B), d_\beta)$ that interests us.

Proposition 3.3. *Let $\phi, \psi \in H(B)$. Then*

$$\|C_\phi - C_\psi\| = \frac{2 - 2\sqrt{1 - d_\beta(\phi,\psi)^2}}{d_\beta(\phi,\psi)}.$$

In particular, the space of composition operators on $H^\infty(B)$ with the uniform norm topology is homeomorphic as a topological space to $(H(B), d_\beta)$.

Proof. Proposition 2.2 together with (4) and (5) gives

$$d_B(z,w) = \sup\{\|f(z) - f(w)\| : f \in H(B)\} = \frac{2 - 2\sqrt{1 - \beta(z,w)^2}}{\beta(z,w)}.$$

Since $h(t) = 2(1 - \sqrt{1-t^2})/t$ is an increasing function on $[0,1)$ we have

$$\begin{aligned}
\|C_\phi - C_\psi\| &= \sup\{\|C_\phi(f) - C_\psi(f)\|_\infty : f \in H^\infty(B), \|f\|_\infty \le 1\} \\
&= \sup\{\|f\circ\phi - f\circ\psi\|_\infty : f \in H(B)\} \\
&= \sup\{\|f(\phi(z)) - f(\psi(z))\| : f \in H(B), z \in B\} \\
&= \sup_{z\in B}\, d_B(\phi(z),\psi(z)) \\
&= \sup_{z\in B} \frac{2 - 2\sqrt{1 - \beta(\phi(z),\psi(z))^2}}{\beta(\phi(z),\psi(z))} \\
&= \frac{2 - 2\sqrt{1 - d_\beta(\phi,\psi)^2}}{d_\beta(\phi,\psi)}.
\end{aligned}$$

□

Our aim is to determine the connected components of the space of composition operators on $H^\infty(B)$. The above result means that we can now do this by examining the space $(H(B), d_\beta)$. In order to achieve this, we use JB^*-triple tools such as Bergman operators and the quasi-inverse map as a substitute for the algebra structure used when $B = \Delta$ [12] or B is the unit ball of $\mathcal{C}_0(X)$ [2] and as a substitute for the inner product used when B is a Hilbert ball [2].

To begin with we note that the d_β-topology on $H(B)$ is stronger than the $\|\cdot\|_\infty$ topology. Indeed from (8) we have that

$$(g_{-w}(z))^{-w} = B(w,w)^{-\frac{1}{2}}(z-w)$$

for $z, w \in B$ and hence we may write

$$z - w = B(w,w)^{\frac{1}{2}}(g_{-w}(z))^{-w}.$$

Since $\|B(w,w)^{\frac{1}{2}}\| \leq 1$ and

$$\left\|(g_{-w}(z))^{-w}\right\| \leq \frac{\|g_{-w}(z)\|}{1-\|w\|\|g_{-w}(z)\|} \leq \frac{\|g_{-w}(z)\|}{1-\|g_{-w}(z)\|}$$

this gives that

$$\|z-w\| \leq \frac{\beta(z,w)}{1-\beta(z,w)}$$

for all $z, w \in B$. Therefore for $\phi, \psi \in H(B)$ we have

$$\sup_{z\in B}\|\phi(z)-\psi(z)\| \leq \sup_{z\in B}\frac{\beta(\phi(z),\psi(z))}{1-\beta(\phi(z),\psi(z))}$$

and hence $\|\phi-\psi\|_\infty \leq \frac{d_\beta(\phi,\psi)}{1-d_\beta(\phi,\psi)}$. In particular, if $d_\beta(\phi,\psi_t) \xrightarrow{t} 0$ then $\|\phi-\psi_t\|_\infty \to 0$. The converse however is not true. For example, in Δ, $\beta(a, e^{it}a) = |g_{-a}(e^{it}a)| = \left|\frac{(e^{it}-1)a}{1-e^{it}|a|^2}\right|$ which implies $d_\beta(\mathsf{id}, e^{it}\mathsf{id}) = 1$ for all $t \in (0, 2\pi)$, even though $\|\mathsf{id} - e^{it}\mathsf{id}\|_\infty \to 0$ as $t \to 0$.

However, the two topologies do agree on the set of holomorphic functions which map B strictly inside B. In other words, if $\|\phi\|_\infty < 1$ then $\|\phi-\psi_t\|_\infty \xrightarrow{t} 0$ if and only if $d_\beta(\phi,\psi_t) \to 0$. To see this, we note from (8) that

$$\begin{aligned}\|g_{-a}(b)\| &\leq \|(B(a,a)^{-\frac{1}{2}}(b-a))^a\| \\ &\leq \frac{\|b-a\|}{1-\|a\|^2-\|a\|\|b-a\|}\end{aligned}$$

from repeated use of (1) and (2) when $\|b-a\|$ is sufficiently small . Therefore if $\|\phi\|_\infty < 1$ we have

$$\begin{aligned}d_\beta(\phi,\psi_t) &= \sup_{z\in B}\|g_{-\phi(z)}(\psi_t(z))\| \\ &\leq \sup_{z\in B}\frac{\|\phi(z)-\psi_t(z)\|}{1-\|\phi(z)\|^2-\|\phi(z)\|\|\phi(z)-\psi_t(z)\|} \\ &\leq \frac{\|\phi-\psi_t\|_\infty}{1-\|\phi\|_\infty^2-\|\phi\|_\infty\|\phi-\psi_t\|_\infty}\end{aligned}$$

and hence $\|\phi-\psi_t\|_\infty \xrightarrow{t} 0$ implies that $d_\beta(\phi,\psi_t) \xrightarrow{t} 0$ as well.

Given $\phi, \psi \in H(B)$, it is obvious from the definition that $d_\beta(\phi,\psi) \leq 1$. Later results will show the importance of determining whether $d_\beta(\phi,\psi) < 1$. We remark therefore that if ϕ

maps B strictly inside B, then the condition $d_\beta(\phi,\psi)<1$ is satisfied for every $\psi\in H(B)$ which also maps B strictly inside B. To see this, we use (6) to write

$$\frac{1}{1-\|g_{-\phi(z)}(\psi(z))\|^2}=\|B^{-1}_{\phi(z)}B(\phi(z),\psi(z))B^{-1}_{\psi(z)}\|$$

and hence $d_\beta(\phi,\psi)=\sup_{z\in B}\|g_{-\phi(z)}(\psi(z))\|<1$ if and only if

$$\sup_{z\in B}\|B^{-1}_{\phi(z)}B(\phi(z),\psi(z))B^{-1}_{\psi(z)}\|<\infty.$$

Since from (1)

$$\|B^{-1}_{\phi(z)}B(\phi(z),\psi(z))B^{-1}_{\psi(z)}\|\le\frac{\|B(\phi(z),\psi(z))\|}{(1-\|\psi(z)\|^2)(1-\|\phi(z)\|^2)}$$

and $\|B(\phi(z),\psi(z))\|\le(1+\|\phi(z)\|\|\psi(z)\|)^2$ for all $z\in B$, it follows that if $\|\phi\|_\infty<1$ and $\|\psi\|_\infty<1$ then

$$\sup_{z\in B}\|B^{-1}_{\phi(z)}B(\phi(z),\psi(z))B^{-1}_{\psi(z)}\|\le\frac{4}{1-\|\phi\|^2_\infty}\frac{1}{1-\|\psi\|^2_\infty}<\infty$$

and hence $d_\beta(\phi,\psi)<1$.
Since $d_\beta(\phi,0)=\|\phi\|$, the converse is also true. In other words, for $\phi\in H(B)$

$$d_\beta(\phi,\psi)<1 \text{ for all } \psi \text{ with } \|\psi\|_\infty<1 \text{ if and only if } \|\phi\|_\infty<1. \tag{9}$$

Theorem 3.4. *Let $\phi,\psi\in H(B)$ with $d_\beta(\phi,\psi)<1$. Then the map*

$$t\mapsto\phi_t:=t\phi+(1-t)\psi$$

is a d_β-continuous path joining ϕ to ψ.

The proof breaks into two parts proved below. The first part (Lemma 3.6) shows that any convex combination ϕ_t of ϕ and ψ satisfies $d_\beta(\phi_t,\psi)\le d_\beta(\phi,\psi)$. The second part (Lemma 3.7) then uses this to show that the map $t\mapsto\phi_t$ is d_β-continuous which proves the theorem. As a result of the homeomorphism $\phi\mapsto C_\phi$ guaranteed by Proposition 3.3, this theorem immediately implies the following.

Corollary 3.5. *Let $\phi,\psi\in H(B)$ with $d_\beta(\phi,\psi)<1$. Then C_ϕ and C_ψ are in the same path connected component in the space of composition operators on $H^\infty(B)$.*

In particular, from (9), we have that the set of composition operators C_ϕ with $\|\phi\|_\infty<1$ is path connected. This is proved in a more general setting in [2]

Lemma 3.6. *Let $\phi,\psi\in H(B)$ satisfy $d_\beta(\phi,\psi)=\lambda<1$. Then for any $t\in[0,1]$ we have $d_\beta(\phi_t,\psi)\le\lambda$, where $\phi_t=t\phi+(1-t)\psi$.*

Proof. Note first that $\phi_t = \psi + t(\phi - \psi)$. Fix $z \in B$. For ease of notation we write $f(\phi)$ rather than $f(\phi(z))$ whenever f is a function on B. Now we use (7) with $a = \psi(z)$ and $b = t(\phi(z) - \psi(z))$ to obtain

$$\begin{aligned}
\beta(\psi(z), \phi_t(z)) &= \|g_{-\psi}(\psi + t(\phi - \psi))\| \\
&= \|\Big[B(\psi,\psi)^{-\frac{1}{2}}(t(\phi-\psi))\Big]^{\psi}\| \\
&= t\|\Big[B(\psi,\psi)^{-\frac{1}{2}}(\phi-\psi)\Big]^{t\psi}\| \qquad (\text{since } (tx)^y = tx^{ty}) \\
&= t\|\left([B(\psi,\psi)^{-\frac{1}{2}}(\phi-\psi)]^{\psi}\right)^{(t-1)\psi}\| \qquad (\text{since } (x^y)^z = x^{y+z}) \\
&= t\|(g_{-\psi}(\phi))^{(t-1)\psi}(z)\| \qquad \text{from (8).}
\end{aligned}$$

Since $\sup_{z\in B}\|g_{-\psi(z)}(\phi(z))\| = d_\beta(\phi,\psi) = \lambda < 1$ and $\|(t-1)\psi\|_\infty \le 1$ we use (2) to get

$$d_\beta(\psi, \phi_t) \le \frac{t\lambda}{1-(1-t)\lambda} \le \lambda. \tag{10}$$

□

Lemma 3.7. *Let $\phi, \psi \in H(B)$ satisfy $d_\beta(\phi,\psi) = \lambda < 1$. Then for $t, t+\delta \in [0,1]$ we have*

$$\lim_{|\delta|\to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0.$$

Proof. Assume firstly that $t > 0$. (Again, we write f for $f(z)$ where convenient.) Notice that $\phi_{t+\delta} = (t+\delta)\phi + (1-t-\delta)\psi = \phi_t + \delta(\phi - \psi)$. We apply (7) with $a = \phi_t(z)$ and $b = \delta(\phi(z) - \psi(z)) = \frac{\delta}{t}(\phi_t - \psi)$ to get

$$\begin{aligned}
g_{-\phi_t}(\phi_{t+\delta}) &= (B(\phi_t,\phi_t)^{-\frac{1}{2}}(\frac{\delta}{t}(\phi_t - \psi)))^{\phi_t} \\
&= (\varepsilon B(\phi_t,\phi_t)^{-\frac{1}{2}}(\psi - \phi_t))^{\phi_t} \qquad \text{where } \varepsilon := -\frac{\delta}{t} \\
&= \varepsilon\Big([B(\phi_t,\phi_t)^{-\frac{1}{2}}(\psi-\phi_t)]^{\varepsilon\phi_t}\Big)
\end{aligned}$$

since $(\alpha x)^y = \alpha x^{\alpha y}$ for $\alpha \in \mathbb{R}$. Now as $(x^y)^z = x^{y+z}$ we have

$$\begin{aligned}
g_{-\phi_t}(\phi_{t+\delta}) &= \varepsilon\left([B(\phi_t,\phi_t)^{-\frac{1}{2}}(\psi-\phi_t)]^{\phi_t}\right)^{(\varepsilon-1)\phi_t} \qquad \text{which from (8)} \\
&= \varepsilon(g_{-\phi_t}(\psi))^{(\varepsilon-1)\phi_t}.
\end{aligned}\tag{11}$$

From Lemma 3.6 we have that $d_\beta(\phi_t,\psi) \le d_\beta(\phi,\psi)$ and hence $\|g_{-\phi_t(z)}(\psi(z))\| \le \lambda < 1$ for all $z \in B$ and we can choose δ, and hence ε, sufficiently small so that $|\lambda(\varepsilon - 1)| \le \lambda' < 1$. In particular,

$$\|g_{-\phi_t(z)}(\psi(z))\|\|(\varepsilon-1)\phi_t(z)\| \le \lambda' < 1$$

for all $z \in B$. We then have

$$\begin{aligned}
d_\beta(\phi_t, \phi_{t+\delta}) &= \sup_{z\in B}\|g_{-\phi_t(z)}(\phi_{t+\delta}(z))\| \\
&= \sup_{z\in B}\|\varepsilon(g_{-\phi_t(z)}(\psi(z)))^{(\varepsilon-1)\phi_t(z)}\| \qquad (\text{from (11)})
\end{aligned}$$

$$\leq |\varepsilon| \frac{\lambda}{1-\lambda'} = \frac{|\delta\lambda|}{t(1-\lambda')} \qquad \text{(from (2))}.$$

In particular, $\lim_{|\delta|\to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0$.

If $t = 0$ then $\phi_t = \phi_0 = \psi$ and $\phi_{t+\delta} = \phi_\delta$. Then (10) shows that $d_\beta(\phi_t, \phi_{t+\delta}) \longrightarrow 0$ as $|\delta| \to 0$. □

To get the following result we can now adapt the one dimensional proof in [12] to the infinite dimensional setting of a bounded symmetric domain.

Theorem 3.8. *Let B be a bounded symmetric domain and let $\phi, \psi \in H(B)$. The following are equivalent.*

(i) *C_ϕ and C_ψ are in the same path connected component of the space of composition operators on $H^\infty(B)$,*
(ii) *$d_\beta(\phi, \psi) < 1$,*
(iii) *$\|C_\phi - C_\psi\| < 2$.*

Proof. (ii) $\iff$ (iii) is immediate from Proposition 3.3. Corollary 3.5 is precisely the statement that (ii) implies (i).

To show (i) implies (ii), let $\phi \in H(B)$ and let $[\phi] = \{\psi \in H(B), d_\beta(\phi, \psi) < 1\}$. We recall that the map $\phi \mapsto C_\phi$ is a homeomorphism. Then, since (ii) implies (i), $[\phi]$ is contained in the path connected component of ϕ in $H(B)$. In fact $[\phi]$ is the path connected component of ϕ. To see this, choose $\omega \in H(B)$ which is not in $[\phi]$. Then $d_\beta(\phi, \omega) = \sup_{z\in B} \beta(\phi(z), \omega(z)) = 1$ which implies that

$$\sup_{z\in B} C_B(\phi(z), \omega(z)) = \sup_{z\in B} \tanh^{-1} \beta(\phi(z), \omega(z)) = \infty.$$

For $\psi \in [\phi]$, $\sup_{z\in B} C_B(\phi(z), \psi(z)) < \infty$ and so the triangle inequality for the Carathéodory distance C_B implies that

$$\sup_{z\in B} C_B(\psi(z), \omega(z)) \geq \sup_{z\in B} \big(C_B(\phi(z), \omega(z)) - C_B(\phi(z), \psi(z))\big) = \infty.$$

Thus $d_\beta(\psi, \omega) = 1$. In other words, the d_β-distance between any element of $[\phi]$ and any element not in $[\phi]$ is equal to 1. In particular, there can be no d_β-continuous path from an element of $[\phi]$ to an element not in $[\phi]$. We conclude $[\phi]$ is the path-connected component of ϕ in $H(B)$. Again by the homeomorphism of Proposition 3.3 the path component of C_ϕ in the space of composition operators is $\{C_\psi : \psi \in [\phi]\}$ and thus (i) implies (ii). □

References

[1] R. Aron, P. Galindo and M. Lindström, *Compact homomorphisms between algebras of analytic functions*, Studia Math., **123**(3), 235–247 (1997).
[2] R. Aron, P. Galindo and M. Lindström, *Connected components in the space of composition operators*, Integral Equations and Operator Theory, **45**, 1–14 (2003).

[3] C. H. Chu and P. Mellon, *Jordan structures in Banach spaces and symmetric manifolds*, Exposition. Math., **16**(2), 157–180 (1998).
[4] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics. CRC Press, Boca Raton (1995).
[5] S. Dineen, The Schwarz Lemma. Oxford University Press (1989).
[6] S. Dineen, M. Mackey and P. Mellon, *The density property for JB^*-triples*, Studia Math., **137**(4), 143–160 (1999).
[7] P. Galindo, M. Lindström and R. Ryan, *Weakly compact composition operators between algebras of bounded analytic functions*, Proc. Amer. Math. Soc., **128**(1), 149–155 (2000).
[8] W. Kaup, *Algebraic characterization of symmetric complex Banach manifolds*, Math. Ann., **228**, 39–64 (1977).
[9] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z., **138**, 503–529 (1983).
[10] W. Kaup, *Hermitian Jordan triple systems and the automorphisms of bounded symmetric domains*, in *Third International Conference on non Associative Algebra and its Applications*, pp. 12–17, Oviedo, Spain (1993).
[11] O. Loos, Bounded symmetric domains and Jordan pairs. University of California at Irvine, Lecture Notes (1977).
[12] B. MacCluer, S. Ohno and R. Zhao, *Topological structure of the space of composition operators on H^∞*, Integral Equations Operator Theory, **40**(4), 481–494 (2001).
[13] K. Madigan, Composition operators into Lipschitz type spaces, Ph.D. thesis, SUNY Albany (1993).
[14] P. Mellon, *Holomorphic invariance on bounded symmetric domains*, J. Reine Angew. Math., **523**, 199–223 (2000).
[15] B. Russo, *Holomorphic composition operators in several complex variables*, in *Studies on composition operators (Laramie, WY, 1996)*, pp. 191–212. Amer. Math. Soc., Providence, RI (1998).
[16] J. H. Shapiro, Composition operators and classical function theory. Springer–Verlag (1993).
[17] C. Toews, *Topological components of the set of composition operators on $H^\infty(B_N)$*, Integral Equatioons and Operator Theory, (to appear) (2003).

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