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Topologies on the set of iterates of a holomorphic function in infinite dimensions

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Abstract

Let $f : B \mapsto B$ be a compact holomorphic map on the open unit ball B of a complex Banach space Z in possibly infinite dimensions, where f compact means $f(B)$ is relatively compact. The sequence of iterates $(f^n)_n$ of f (where $f^n := f \circ f^{n-1}, f^1 := f$) is of much interest and, since it generally does not converge, the set of all its subsequential limits for a particular topology have been studied instead.

We prove that the pointwise limit of any subsequence of $(f^n)_n$ is itself a holomorphic function. We show, in fact, that on the set of iterates $\{f^n : n \in \mathbb{N}\}$ the topology of pointwise convergence on B coincides with any finer topology on the space, $H(B, Z)$, of holomorphic functions from B to Z. In particular, it coincides with both the compact open topology and the topology of local uniform convergence on B. Despite the fact that these topologies are not first countable, we prove that the set of accumulation points of $(f^n)_n$ coincides with the set of all its subsequential limits.

1 Introduction

Let B be the open unit ball of an arbitrary complex Banach space Z, in finite or infinite dimensions and let $f : B \mapsto B$ be holomorphic. The iterates of f are the maps f^n, where f^n is the n fold composition $f \circ \cdots \circ f$ of f, $n \geq 1$. The convergence, or otherwise, of the sequence of iterates $(f^n)_n$ is closely related to the existence of fixed points of f in B. There is a century long history of the study of iterates and how these relate to fixed points by,

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among many others, on the unit disc Δ in $\mathbb{C}$ Julia [13], Denjoy [8], Wolff [22, 23, 24], the polydisc [11], the finite dimensional Hilbert ball [12] and other finite dimensional domains [1, 2, 3]. For a small sample of more recent work on infinite dimensional domains we refer to [4, 5, 6, 9, 10, 14, 15, 7, 19, 16, 17].

The classical theory tells us that if f has a fixed point in Δ, Schwarz's lemma implies that either f is effectively a rotation or $(f^n)_n$ converges to the fixed point; whereas if f has no fixed point in Δ then $(f^n)_n$ converges uniformly on compact subsets of Δ to $\xi \in \partial\Delta$. While the fixed point free theory extends to a finite dimensional Hilbert ball [12], it fails for the infinite dimensional Hilbert ball [20], due to lack of a Montel type theorem ensuring that $\{f^n : n \in \mathbb{N}\}$ is a normal family. If, however, f itself is chosen to be compact, that is, if $f(B)$ is relatively compact, then a Denjoy-Wolff theorem exists for the infinite dimensional Hilbert ball [7]. The key property of the Hilbert ball required for this is its strict convexity and, indeed, $(f^n)_n$ generally fails to converge even for finite dimensional non-strictly convex balls [18, Corollary 2.10] or [7, Example 2].

In infinite dimensions then, for an arbitrary compact holomorphic self map f, $(f^n)_n$ does not generally converge, so we ask instead what are the accumulation points of $(f^n)_n$, with respect to some convenient topology on the space of holomorphic maps $H(B, Z)$. As the plan is to subsequently use holomorphic techniques to locate the images $g(B)$, for any accumulation point g of $(f^n)_n$ with respect to the topology in question, we ultimately need such accumulation points g to be holomorphic. As the limit of a sequence of holomorphic functions for the topology of local uniform convergence (denoted here by τ) is again holomorphic this is usually the preferred topology. We note that in finite dimensions, the topology of local uniform convergence coincides with the compact open topology and is determined by uniform convergence on compact subsets of B.

In infinite dimensions however, topologies on the space of holomorphic functions $H(B, Z)$ are generally not determined by sequences but instead require nets. This is the case for the local uniform topology. For this reason, most authors restrict themselves to looking at limits of convergent subsequences of $(f^n)_n$ in the τ topology, henceforth called τ subsequential limits, since this allows use of the key fact that every subsequence of $(f^n)_n$ has itself a τ subsequential limit [7, Lemma 1 and Remark 1].

We prove that the set of τ accumulation points of $(f^n)_n$ coincides with the set of all its subsequential limits and that on $(f^n)_n$, the topologies of local uniform convergence and pointwise convergence coincide. This extends, in fact, to any topology ν on $H(B, Z)$ that is finer than the topology of pointwise convergence on B. A consequence is that $(f^n)_n$ can be considered a relatively compact topological semigroup, for any topology that is finer than the topology of pointwise convergence on B. This is the case, in particular, for both the compact-open topology and the topology of local uniform convergence. Similar results were initially observed by the authors in a slightly different setting, where convergence of $(f^{n_k}(z))_k$ at a single point $z \in B$ to an extreme point $e \in \overline{B}$ actually forces local uniform convergence of $(f^{n_k})_k$ to the constant map e. This early result, Theorem 3.3 below, motivated the rest of this paper, so we include it for completeness.

2 Background

Throughout let Z be a complex Banach space with open unit ball B and denote by $H(B,Z)$ the space of holomorphic maps from B to Z. Several useful topologies arise on $H(B,Z)$, in addition to the topology, ρ, of pointwise convergence on B. The compact open topology τ_0 coincides here with the topology of uniform convergence on compact subsets of B. We denote by τ the topology of local uniform convergence on B, where τ convergence means that for any $x \in B$, there is a neighbourhood U of x on which we have uniform convergence. Limits of sequences or nets in τ_0 or ρ are not necessarily holomorphic functions. The key advantage of the local uniform topology is that such limit points are automatically holomorphic. Of course, if Z is finite dimensional then $\tau = \tau_0$. For convenience, we will refer to ρ, τ_0 and τ as the pointwise, compact-open and local uniform topologies throughout.

We note for infinite dimensional Z these topologies are not apriori necessarily determined by sequences, instead nets (or filters) are required to determine whether or not sets in $H(B,Z)$ are open or closed, and hence when referring to convergence above we mean convergence of nets. In such a case we say that the topology is non-sequential. Despite these topologies being non-sequential, we prove below that for compact f all accumulation points of $(f^n)_n$ are already sub-sequential limits.

Recall that if g is a holomorphic function on an open set Ω in a complex Banach space X then for differentiable $\gamma : [a,b] \to \Omega$ we may define the path integral $\int_a^b g(\gamma(t))\gamma'(t)dt$ in terms of limits of Riemann sums, entirely analogously to the scalar situation. Linearity of the integral and the Hahn-Banach Theorem allow much of the scalar theory to be duplicated in this situation. For example, we have the mean value property:

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} g(z + we^{i\theta})d\theta \text{ if } B(z, \|w\|) \subset \Omega.$$

We recall also that $e \in \overline{B}$ is a complex extreme point if, for $y \in Z$, $e + \Delta y \subset \overline{B}$ implies $y = 0$.

3 New Results

We quote the following key result for convenience [7, Lemma 1 and Remark 1]. Recall that a sequence is said to be relatively sequentially compact if every subsequence itself admits a convergent subsequence.

3.1 Lemma. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a Banach space. Then $(f^n)_n$ has a subsequence which is τ-convergent to a function in* $\mathrm{Hol}(B, \overline{B})$. *Moreover, every subsequence of $(f^n)_n$ has itself a τ-convergent subsequence (and hence $(f^n)_n$ is relatively sequentially compact in $H(B,Z)$ for the τ topology).*

The following result will be used several times.

3.2 Proposition. *Let X be a topological space and let $(x_n)_n$ be a relatively sequentially compact sequence in X. If $(x_n)_n$ has a unique subsequential limit $\overline{x} \in X$ then $(x_n)_n$ converges to $\overline{x}$.*

Proof. Suppose for the sake of contradiction that $(x_n)_n$ does not converge to $\overline{x}$. Then there exists an open neighbourhood U of $\overline{x}$ such that $(x_n)_n$ does not eventually lie in U. In particular, there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which lies in $X \setminus U$. Relative sequential compactness of $(x_n)_n$ guarantees a convergent subsequence $(x_{n_{k_l}})_l$ of $(x_{n_k})_k$, converging to $\overline{y}$, say. Closure of $X \setminus U$ implies $\overline{y} \in X \setminus U$ and, since $\overline{x} \in U$, then $\overline{y} \neq \overline{x}$. This contradicts our hypothesis of a unique limit to all convergent subsequences, proving the result.

■

Our next result shows that if a subsequence of $(f^n)_n$ converges at a single point $z \in B$ to an extreme point e then it actually converges locally uniformly to e, where we will identify e with the constant mapping $z \mapsto e$ on B. This provides initial evidence that, for a sequence of iterates of a compact holomorphic map, the pointwise topology actually coincides with more general topologies. An alternative proof may be given in terms of the concept of holomorphic boundary components.

3.3 Theorem. *Let $f : B \to B$ be a compact holomorphic map. Then for any sequence $(n_k)_k$ of $\mathbb{N}$ and any complex extreme point e of $\overline{B}$, the following are equivalent:*

(i) *there exists $z \in B$ such that $\lim_k f^{n_k}(z) = e$;*

(ii) $\lim_k f^{n_k}(w) = e$, *for all $w \in B$;*

(iii) $(f^{n_k})_k$ *converges locally uniformly on B to e.*

Proof. *(i)* ⇒ *(iii)*. Assume (i) and denote by Ω the set of limits of τ-convergent subsequences (the τ subsequential limits) of $(f^{n_k})_k$. This set is non-empty by Lemma 3.1. Fix $g \in \Omega$. Then there is a subsequence $(f^{n_{k_j}})_j$ which is τ-convergent to g. By hypothesis, $g(z) = e$. Choosing $r > 0$ such that $\overline{B}(z, r) \subset B$ and $w \in B$, the mean value property gives

$$e = g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(z + re^{i\theta} w) d\theta.$$

As e is an extreme point of $\overline{B}$, this guarantees (by an application of [21, Theorem 3.2] for example) that the integrand is constant and that $g|_{B(z,r)} = e$. The identity principle then gives $g = e$. In other words, all τ-convergent subsequences of $(f^{n_k})_k$ must converge to the constant map e and thus $\Omega = \{e\}$. The hypotheses of Proposition 3.2 are therefore satisfied and we can conclude that $(f^{n_k})_k$ is τ-convergent to e. The *(iii)* ⇒ *(ii)* ⇒ *(i)* implications are, of course, immediate.

■

3.4 Proposition. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Then the following are equivalent for a subsequence $(f^{n_k})_k$ of $(f^n)_n$:*

(i) $(f^{n_k})_k$ *converges pointwise;*

(ii) $(f^{n_k})_k$ *converges locally uniformly.*

Proof. Assuming (i) holds, we may define a function $l : B \to \overline{B}$ by $l(z) = \lim_k f^{n_k}(z)$. Let Ω be the set of all τ-subsequential limits of the subsequence $(f^{n_k})_k$. This set is non-empty by Lemma 3.1. Let $g \in \Omega$. Then there is a subsequence $(f^{n_{k_j}})_j$ of $(f^{n_k})_k$ such that $f^{n_{k_j}} \xrightarrow{j} g$ locally uniformly. For any $z \in B$ we have $g(z) = \lim_j f^{n_{k_j}}(z) = \lim_k f^{n_k}(z) = l(z)$. Thus $l = g \in \Omega$ and, in particular, l is holomorphic and $\Omega = \{l\}$. Again by Proposition 3.2, we conclude that $(f^{n_k})_k$ is τ-convergent to l giving (ii). The reverse implication is immediate.

■

3.5 Corollary. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Then the pointwise limit of any convergent subsequence $(f^{n_k})_k$ of the sequence of iterates $(f^n)_n$ is a holomorphic function.*

3.6 Remark. We note that a subnet of a sequence itself contains a subsequence, giving, in particular, that every subnet of $(f^n)_n$ itself contains a subsequence. Indeed, let $(a_n)_n$ be any sequence and $(a_\lambda)_\lambda$ be a subnet of $(a_n)_n$. Thus there is a directed set D and a monotone function $h : D \to \mathbb{N}$ such that for any $k \in \mathbb{N}$ there exists $\lambda \in D$ so that $h(\lambda) \geq k$. We construct a subsequence as follows. Choose λ_1 so that $h(\lambda_1) \geq 1$. Given $\lambda_1, \dots, \lambda_{n-1}$, we choose λ_n so that $h(\lambda_n) \geq h(\lambda_{n-1}) + 1$. Then $(a_{\lambda_k})_k$ is a subsequence of $(a_\lambda)_\lambda$ and satisfies $\lim_k h(\lambda_k) = \infty$.

3.7 Remark. From above, the set of accumulation points of any sequence, that is the set of limits of all convergent subnets, coincides with the set of all its subsequential limits irrespective of topology. In particular, the set of subsequential limits and the set of accumulation points of $(f^n)_n$ coincide for any topology. By Corollary 3.5 therefore, the pointwise limit of any convergent subnet of $(f^n)_n$ is holomorphic.

3.8 Corollary. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Let $(f^{n_\alpha})_\alpha$ be any subnet of $(f^n)_n$. Then the following are equivalent.*

(i) $(f^{n_\alpha})_\alpha$ *is pointwise convergent;*

(ii) $(f^{n_\alpha})_\alpha$ *is locally uniformally convergent.*

Proof. Clearly (ii) implies (i) so it suffices to prove (i) implies (ii). Let $(f^{n_\alpha})_\alpha$ be ρ convergent to h. Suppose that $(f^{n_\alpha})_\alpha$ is not τ convergent to h. By adapting the proof of Proposition 3.4, and use of Remark 3.7, we reach a contradiction.

■

The next two results follow immediately from Corollary 3.8 and Corollary 3.5.

3.9 Corollary. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Then the pointwise limit of any subnet $(f^{n_\alpha})_\alpha$ of the sequence of iterates $(f^n)_n$ is a holomorphic function.*

3.10 Corollary. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Let ν be any topology on $H(B, Z)$ finer than the pointwise topology and let $(f^{n_\alpha})_\alpha$ be any subnet of the sequence of iterates. Then the following are equivalent.*

(i) $(f^{n_\alpha})_\alpha$ *is pointwise convergent;*

(ii) $(f^{n_\alpha})_\alpha$ *is ν convergent;*

(iii) $(f^{n_\alpha})_\alpha$ *is locally uniformly convergent.*

3.11 Example. The limit of any convergent subnet of $(f^n)_n$ for the compact open topology is holomorphic and it is also a subsequential limit.

We may now use Lemma 3.1 again to rephrase Corollary 3.10 in the language of topological semi-groups as follows. We recall that for semi-groups relatively compact is often called pre-compact.

3.12 Corollary. *Let $f : B \to B$ be a compact holomorphic map on the open unit ball B of a complex Banach space. Let ν be any topology on $H(B, Z)$ finer than the topology of pointwise convergence on B. Then $(f^n)_n$ is a pre-compact topological semi-group.*

Proof. Let ν be any topology finer than the pointwise topology on B. By Remark 3.6 above, every subnet of $(f^n)_n$ itself contains a subsequence $(f^{n_k})_k$, which by Lemma 3.1 admits a τ convergent subsequence and hence, by Corollary 3.10, also admits a ν convergent subsequence. In other words, every subnet of $(f^n)_n$ has a ν convergent subnet, that is, $(f^n)_n$ is ν pre-compact. It is clear that $(f^n)_n$ is a semi-group. For the pointwise topology, ρ, it is easy to see that $(f^n)_n$ is a topological semi-group and therefore by Corollary 3.10 the same is true for ν.

■

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