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ISS Property with Respect to Boundary Disturbances for a Class of Riesz-Spectral Boundary Control Systems [★]

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Abstract

This paper deals with the establishment of Input-to-State Stability (ISS) estimates for infinite dimensional systems with respect to both boundary and distributed disturbances. First, a new approach is developed for the establishment of ISS estimates for a class of Riesz-spectral boundary control systems satisfying certain eigenvalue constraints. Second, a concept of weak solutions is introduced in order to relax the disturbances regularity assumptions required to ensure the existence of classical solutions. The proposed concept of weak solutions, that applies to a large class of boundary control systems which is not limited to the Riesz-spectral ones, provides a natural extension of the concept of both classical and mild solutions. Assuming that an ISS estimate holds true for classical solutions, we show the existence, the uniqueness, and the ISS property of the weak solutions.

Key words: Distributed parameter systems; Boundary control systems; Boundary disturbances; Input-to-state stability; Weak solutions.

1 Introduction

The concept of Input-to-State Stability (ISS), originally introduced by Sontag for finite dimensional systems [33], is one of the main tools for assessing the robustness of a system with respect to external disturbances. This notion has been extensively investigated for finite dimensional nonlinear systems during the last three decades. More recently, the possible extension of ISS properties to both linear [2,18–20,34] and nonlinear [36,37] Partial Differential Equations (PDEs), and more generally to infinite dimensional systems, has attracted much attention [21,29,30].

For infinite dimensional systems, there exist essentially two distinct types of perturbations. The first type includes distributed perturbations; namely, perturbations acting directly in the state equation. The second type concerns boundary perturbations; namely, perturbations acting on the system state through an algebraic

constraint by the means of an unbounded operator. In the case of PDEs, the distributed perturbations are also called in-domain perturbations as they appear directly in the PDEs. In contrast, the boundary perturbations appear in the boundary conditions of the PDEs. This second type of perturbation naturally appears in numerous boundary control problems such as heat equations [8], transport equations [19], diffusion or diffusive equations [2], and vibration of structures [8] with numerous practical applications, e.g., in robotics [12,13] and aerospace engineering [22].

While many results have been reported regarding the ISS property with respect to distributed disturbances [1,9,10,24–27,31], the establishment of ISS properties with respect to boundary disturbances remains challenging [2,18–20]. The traditional method to study abstract boundary control systems consists of transferring the boundary disturbances into a distributed one by means of a bounded lifting operator. By doing so, the original boundary control system is made equivalent to a standard evolution equation for which efficient analysis tools exist. The main issue with such an approach is that the resulting distributed perturbation involves the time derivative of the boundary perturbation [8]. In particular, this induces two main difficulties. First, this approach fails, in general, to establish the ISS property with respect to the boundary disturbance, but can only show the ISS property with respect to the first

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time derivative of the boundary disturbance. A notable exception is the case of monotone control systems for which the lifting approach can be successfully used to establish ISS estimates, see [28] for details. Second, in order to ensure the existence of classical solutions, one has to assume that the boundary disturbance is twice continuously differentiable. The relaxation of this regularity assumption requires the introduction of a concept of mild or weak solutions extending the one of classical solutions. However, the explicit occurrence of the time derivative of the boundary perturbation in the evolution equation does not allow, in a general setting, a straightforward introduction of such a concept of mild or weak solutions for boundary disturbances that are only assumed, e.g., to be continuous [11].

Inspired by well established finite-dimensional techniques, it has been proposed to resort to Lyapunov functions to establish the ISS properties of PDEs [2,34,36,37]. An other approach, based on functional analysis tools, has been proposed in [19] for the study of 1-D parabolic equations. In the problem therein, (the negative of) the underlying disturbance free operator belongs to the class of Sturm-Liouville operators. Thus, its eigenvectors can be selected such that they form a Hilbert basis of the underlying Hilbert space. By projecting the system trajectories onto this Hilbert basis and using the self-adjoint nature of the disturbance free operator, it was shown that the analysis of the system trajectories reduces to the study of a countably infinite number of Ordinary Differential Equations (ODEs). Each of these ODEs characterizes the time domain evolution of one coefficient of the system trajectory when expressed in the aforementioned Hilbert basis. Then, the ISS property was obtained by solving these ODEs and by resorting to Parseval's identity. On one hand, this idea of projecting the system trajectories over a Hilbert basis was then investigated in [17] for the study of the asymptotic gains of a damped wave equation. On the other hand, this approach was further investigated in [23] for the study of the ISS properties of a clamped-free damped string. Specifically, the system trajectories were projected over a Riesz basis [5] formed by eigenvectors of the disturbance free operator. Then, the property connecting the norm of a vector and its coefficients in a Riesz basis was used to obtain the desired ISS estimates.

The contribution of this paper is twofold. First, we develop a new approach for the establishment of ISS estimates with respect to boundary disturbances for a class of Riesz-spectral boundary control systems satisfying certain eigenvalue constraints. The ISS property of such analytic semigroups was investigated first in [14–16] for L^∞ boundary inputs. The problem was embedded into the extrapolation space \mathcal{H}_{-1} while invoking admissibility conditions for returning to the original Hilbert space \mathcal{H} . The approach adopted in this paper differs by generalizing the ideas developed first in [19] and then in [23]. Assuming boundary and distributed disturbances of

class \mathcal{C}^2 and \mathcal{C}^1 , respectively, the ISS property is established for classical solutions by taking advantage of the projection of the system trajectories over a Riesz basis formed by the eigenvectors of the disturbance free operator. This new tool can be used either to derive general (see proof of Theorem 1) or system specific (see Subsection 5.3) versions of the ISS estimates. The advantage of this approach is fourfold. First, all the computations are performed within the original Hilbert space \mathcal{H} while avoiding the embedding of the problem into the extrapolation space \mathcal{H}_{-1} . Second, it provides a spectral decomposition of the Riesz-spectral boundary control system under the form of a countable infinite number of ODEs describing the time domain evolution of the coefficients of the system trajectory in the Riesz basis. Such spectral decompositions have been used, e.g., for control design; see [6,7] for the feedback stabilization of heat and wave equations. However, these classical spectral decompositions involve the time derivative \dot{d} of the boundary disturbance d . In sharp contrast, the one provided in this paper only involves the boundary perturbation d . Third, we show that the d -term of the aforementioned spectral decomposition can be related to stationary solutions of the studied Riesz-spectral boundary control system. Thus, the obtained constant related to the boundary perturbation in the ISS estimate is expressed as a function of the energy of stationary solutions of the abstract boundary control system. Finally, it is shown that the approach used for deriving the ISS estimate also allows the direct establishment of fading memory estimates without invoking a conversion lemma [21, Chap. 7]. Such fading memory estimates are useful for the establishment of small gain conditions ensuring the stability of interconnected systems [21].

The second contribution of this paper deals with the introduction of a concept of weak solutions for abstract boundary control systems that allows the relaxation of the perturbations regularity assumptions from \mathcal{C}^2 and \mathcal{C}^1 to \mathcal{C}^0 . This approach applies to a general class of boundary control systems that is not limited to Riesz-spectral ones. The first concept of weak solutions for infinite dimensional nonhomogeneous Cauchy problems was originally introduced in [3]. In the case of Hilbert spaces, an alternative version of such a concept was proposed under a variational form, see, e.g., [8, Def. 3.1.6]. The advantage of such a formulation is that it bridges the gap with the concept of solutions from distribution theory [8, A.5.29]. The purpose of this paper is to extend such a concept to the case of abstract boundary control systems. The proposed approach is an extension of the concept of classical solutions that presents three main advantages. First, the concept of weak solution is introduced directly within the original Hilbert space \mathcal{H} under a variational formulation. In particular, it provides an alternative framework that avoids either the traditional embedding of the original problem into the extrapolation space \mathcal{H}_{-1} [11,35] or the abstract extension of the

mild solutions by density arguments [32]. Second, the proposed definition of weak solutions only depends on the operators of the abstract boundary control system; in particular, it does not make explicitly appear the C_0 -semigroup generated by the disturbance free operator. Third, by its variational formulation and the introduction of test functions, the concept of weak solution is inspired by distribution theory. Assuming that an ISS estimate holds true with respect to the classical solutions of the abstract boundary control system, we show the existence and the uniqueness of the weak solutions, as well as their compatibility with the concept of mild solutions. It is also shown that the ISS estimate satisfied by classical solutions also holds true for weak solutions.

The remainder of this paper is organized as follows. Notations and definitions are introduced in Section 2. The establishment of an ISS estimate for a class of Riesz-spectral boundary control systems with respect to classical solutions is presented in Section 3. Then, a concept of weak solutions and its properties are presented in Section 4 for a large class of boundary control systems. The obtained results are applied on an illustrative example in Section 5. Concluding remarks are provided in Section 6.

2 Notations and Definitions

2.1 Notations

The sets of non-negative integers, positive integers, integers, real, non-negative real, positive real, negative real, and complex numbers are denoted by \mathbb{N} , \mathbb{N}^* , \mathbb{Z} , \mathbb{R} , \mathbb{R}_+ , \mathbb{R}_+^* , \mathbb{R}_-^* , and \mathbb{C} , respectively. For any $z \in \mathbb{C}$, $\operatorname{Re} z$ denotes the real part of z . Throughout the paper, the field \mathbb{K} is either \mathbb{R} or \mathbb{C} . We consider the following classical classes of comparison function:

$$\begin{aligned} \mathcal{K} &= \{ \gamma \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+) : \gamma \text{ strictly increasing, } \gamma(0) = 0 \}, \\ \mathcal{L} &= \{ \gamma \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+) : \gamma \text{ strictly decreasing,} \\ &\quad \lim_{t \rightarrow +\infty} \gamma(t) = 0 \}, \\ \mathcal{KL} &= \{ \beta \in \mathcal{C}^0(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R}_+) : \beta(\cdot, t) \in \mathcal{K} \text{ for } t \geq 0, \\ &\quad \beta(x, \cdot) \in \mathcal{L} \text{ for } x > 0 \}. \end{aligned}$$

For an interval $I \subset \mathbb{R}$ and a \mathbb{K} -normed linear space $(E, \|\cdot\|_E)$, $\mathcal{C}^n(I; E)$ denotes the set of functions $f : I \rightarrow E$ that are n times continuously differentiable. For any $a < b$, we endow $\mathcal{C}^0([a, b]; E)$ with the norm $\|\cdot\|_{\mathcal{C}^0([a, b]; E)}$ defined for any $f \in \mathcal{C}^0([a, b]; E)$ by

$$\|f\|_{\mathcal{C}^0([a, b]; E)} \triangleq \sup_{t \in [a, b]} \|f(t)\|_E.$$

For a given linear operator L , $\mathcal{R}(L)$, $\ker(L)$, and $\rho(L)$ denote its range, its kernel, and its resolvent set, respectively. $\mathcal{L}(E, F)$ denotes the set of bounded linear operators from E to F . Let $(\mathbb{K}^m, \|\cdot\|_{\mathbb{K}^m})$ be a normed space

with $m \in \mathbb{N}^*$. For a given basis $\mathcal{E} = (e_1, e_2, \dots, e_m)$ of \mathbb{K}^m , we denote by $\|\cdot\|_{\infty, \mathcal{E}}$ the infinity norm¹ in \mathcal{E} . By virtue of the equivalence of the norms in finite dimension, we denote by $c(\mathcal{E}) \in \mathbb{R}_+^*$ the smallest constant such that $\|\cdot\|_{\infty, \mathcal{E}} \leq c(\mathcal{E}) \|\cdot\|_{\mathbb{K}^m}$.

Finally, we introduce the Kronecker notation: $\delta_{a,b} = 1$ if $a = b$, 0 otherwise. The time derivative of a real-valued differentiable function $f : I \rightarrow \mathbb{R}$ is denoted by \dot{f} . If \mathcal{H} is a Hilbert space, the time derivative of an \mathcal{H} -valued differentiable function $f : I \rightarrow \mathcal{H}$ is denoted by df/dt .

2.2 Definitions and related properties

In this paper, we consider the following definition of *boundary control systems* [8, Def. 3.3.2].

Definition 1 (Boundary control system) *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a separable Hilbert space over \mathbb{K} and $(\mathbb{K}^m, \|\cdot\|_{\mathbb{K}^m})$ be a \mathbb{K} -normed space with $m \in \mathbb{N}^*$. Consider the abstract system taking the form:*

$$\begin{cases} \frac{dX}{dt}(t) = AX(t) + U(t), & t > 0 \\ \mathcal{B}X(t) = d(t), & t > 0 \\ X(0) = X_0 \end{cases} \quad (1)$$

with

- $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ a linear (unbounded) operator;
- $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \rightarrow \mathbb{K}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ a linear boundary operator;
- $U : \mathbb{R}_+ \rightarrow \mathcal{H}$ the distributed disturbance;
- $d : \mathbb{R}_+ \rightarrow \mathbb{K}^m$ the lumped disturbance.

We say that $(\mathcal{A}, \mathcal{B})$ is a boundary control system, with associated abstract system (1), if

- (1) the disturbance free operator \mathcal{A}_0 , defined over the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$, is the generator of a C_0 -semigroup S on \mathcal{H} ;
- (2) there exists a bounded operator $B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, called a lifting operator, such that $\mathcal{R}(B) \subset D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, and $\mathcal{B}B = I_{\mathbb{K}^m}$.

Remark 1 *As the boundary input space \mathbb{K}^m is finite dimensional, the bounded nature of B and $\mathcal{A}B$ is immediate. Thus, the existence of the lifting operator B reduces to the surjectivity of $\mathcal{B}|_{D(\mathcal{A})}$.*

We then introduce the concept of Riesz-spectral operators [8, Def. 2.3.4].

¹ Defined as the maximum of the absolute values of the vector components when written in the basis \mathcal{E} .

Definition 2 (Riesz spectral operator) Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a linear and closed operator with simple eigenvalues λ_n and corresponding eigenvectors $\phi_n \in D(\mathcal{A}_0)$, $n \in \mathbb{N}$. \mathcal{A}_0 is a Riesz-spectral operator if

- (1) $\{\phi_n, n \in \mathbb{N}\}$ is a Riesz basis [5]:
 (a) $\{\phi_n, n \in \mathbb{N}\}$ is maximal, i.e., $\overline{\text{span}_{\mathbb{K}} \phi_n} = \mathcal{H}$;
 (b) there exist constants $m_R, M_R \in \mathbb{R}_+^*$ such that for all $N \in \mathbb{N}$ and all $\alpha_0, \dots, \alpha_N \in \mathbb{K}$,

$$m_R \sum_{n=0}^N |\alpha_n|^2 \leq \left\| \sum_{n=0}^N \alpha_n \phi_n \right\|_{\mathcal{H}}^2 \leq M_R \sum_{n=0}^N |\alpha_n|^2; \quad (2)$$

- (2) the closure of $\{\lambda_n, n \in \mathbb{N}\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n, n \in \mathbb{N}\}}$, $[a, b] \not\subset \overline{\{\lambda_n, n \in \mathbb{N}\}}$.

A subset of the properties satisfied by Riesz-spectral operators are gathered in the following lemma [8, Lemmas 2.3.2 and 2.3.3].

Lemma 1 Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a Riesz-spectral operator. With the notations of Definition 2, the following are true.

- The eigenvalues of the adjoint operator \mathcal{A}_0^* are provided for $n \in \mathbb{N}$ by $\mu_n \triangleq \overline{\lambda_n}$ and the associated eigenvectors $\psi_n \in D(\mathcal{A}_0^*)$ can be selected such that $\{\phi_n, n \in \mathbb{N}\}$ and $\{\psi_n, n \in \mathbb{N}\}$ are biorthogonal, i.e., for all $n, m \in \mathbb{N}$, $\langle \phi_n, \psi_m \rangle_{\mathcal{H}} = \delta_{n,m}$.
- The sequence of vectors $\{\psi_n, n \in \mathbb{N}\}$ is a Riesz basis.
- For all $(\alpha_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$,

$$\sum_{n \in \mathbb{N}} |\alpha_n|^2 < \infty \Leftrightarrow \sum_{n \in \mathbb{N}} \alpha_n \phi_n \in \mathcal{H}. \quad (3)$$

- For all $z \in \mathcal{H}$,

$$z = \sum_{n \in \mathbb{N}} \langle z, \psi_n \rangle_{\mathcal{H}} \phi_n = \sum_{n \in \mathbb{N}} \langle z, \phi_n \rangle_{\mathcal{H}} \psi_n. \quad (4)$$

- \mathcal{A}_0 is the generator of a C_0 -semigroup S if and only if $\sup_{n \in \mathbb{N}} \text{Re } \lambda_n < \infty$. In this case,

$$\forall t \in \mathbb{R}_+, \forall z \in \mathcal{H}, S(t)z = \sum_{n \in \mathbb{N}} e^{\lambda_n t} \langle z, \psi_n \rangle_{\mathcal{H}} \phi_n, \quad (5)$$

and its growth-bound satisfies $\omega_0 = \sup_{n \in \mathbb{N}} \text{Re } \lambda_n$.

Finally, we introduce the following definition.

Definition 3 (Riesz-spectral boundary control system) We say that $(\mathcal{A}, \mathcal{B})$ is a Riesz-spectral boundary control system, with associated abstract system (1), if

- (1) $(\mathcal{A}, \mathcal{B})$ is a boundary control system with associated abstract system (1);
- (2) the underlying disturbance free operator \mathcal{A}_0 is a Riesz-spectral operator.

In particular, we consider the class of Riesz-spectral boundary control systems such that:

$$\omega_0 = \sup_{n \in \mathbb{N}} \text{Re } \lambda_n < 0, \quad \zeta \triangleq \sup_{n \in \mathbb{N}} \frac{|\lambda_n|}{|\text{Re } \lambda_n|} < \infty. \quad (6)$$

The constraint $\omega_0 < 0$ guarantees the exponential stability of the C_0 -semigroup S while $\zeta < +\infty$ ensures that the system is of parabolic type. The slight relaxation of the latter constraint is discussed in Remark 3.

3 ISS Estimate for Classical Solutions

We investigate in this section the case of classical solutions [8, Def. 3.1.1] for (1).

Definition 4 (Classical solutions) Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system. Let $X_0 \in D(\mathcal{A})$, $d \in C^0(\mathbb{R}_+; \mathbb{K}^m)$ such that $\mathcal{B}X_0 = d(0)$, and $U \in C^0(\mathbb{R}_+; \mathcal{H})$ be given. We say that X is a classical solution of (1) associated with (X_0, d, U) if $X \in C^0(\mathbb{R}_+; D(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H})$, $X(0) = X_0$, and for all $t \geq 0$, $(dX/dt)(t) = \mathcal{A}X(t) + U(t)$ and $\mathcal{B}X(t) = d(t)$.

3.1 ISS for classical solutions

The ISS property of the studied class of Riesz-spectral boundary control systems was established in [14–16] for L^∞ boundary inputs by using the usual concept of mild solutions within the extrapolation space. In this Subsection 3.1, we develop a different approach for assessing such a result for classical solutions. The proposed approach generalizes the ideas developed in [19,23] consisting in the projection of the system trajectories over adequate Riesz bases. This approach relies on a novel spectral decomposition (see (9) and Remark 2) and can be used either to derive general (see proof of Theorem 1) or system specific (see Subsection 5.3) versions of the ISS estimates. Such an approach is further investigated in Subsection 3.2 for the derivation of a novel energy-based ISS estimate (see Theorem 2), as well as the derivation of fading memory estimates (see Remark 6).

Theorem 1 Let $(\mathcal{A}, \mathcal{B})$ be a Riesz-spectral boundary control system such that the eigenvalue constraints (6) hold. For every initial condition $X_0 \in D(\mathcal{A})$, and every disturbance $d \in C^2(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in C^1(\mathbb{R}_+; \mathcal{H})$ such that $\mathcal{B}X_0 = d(0)$, the abstract system (1) has a unique classical solution $X \in C^0(\mathbb{R}_+; D(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H})$ associated with (X_0, d, U) . Furthermore, the system is exponentially ISS in the sense that there exist $C_0, C_1, C_2 \in \mathbb{R}_+^*$,

independent of X_0 , d , and U , such that for all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}} \leq C_0 e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}} + C_1 \|d\|_{C^0([0,t];\mathbb{K}^m)} + C_2 \|U\|_{C^0([0,t];\mathcal{H})} \quad (7)$$

with $\kappa_0 = -\omega_0 > 0$, where ω_0 is the growth bound of the C_0 -semigroup S generated by the disturbance free operator \mathcal{A}_0 .

Proof of Theorem 1. The existence and uniqueness of the classical solutions under the assumed regularity assumptions directly follows from classical results for abstract boundary control systems (see, e.g., [8, Thm 3.1.3 and Thm 3.3.3]). Thus, the proof is devoted to the derivation of the ISS estimate (7). Let B be a lifting operator associated with the boundary control system $(\mathcal{A}, \mathcal{B})$ as provided by Definition 1. Let $X_0 \in D(\mathcal{A})$, $d \in C^2(\mathbb{R}_+; \mathbb{K}^m)$, and $U \in C^1(\mathbb{R}_+; \mathcal{H})$ such that $\mathcal{B}X_0 = d(0)$ be given. Let X be the classical solution of (1) associated with (X_0, d, u) . Adopting the notations of Definition 2 and Lemma 1, we have from (4) that for all $t \geq 0$,

$$X(t) = \sum_{n \in \mathbb{N}} \langle X(t), \psi_n \rangle_{\mathcal{H}} \phi_n. \quad (8)$$

Introducing for all $n \in \mathbb{N}$ and all $t \geq 0$, $c_n(t) \triangleq \langle X(t), \psi_n \rangle_{\mathcal{H}}$, then $c_n \in C^1(\mathbb{R}_+; \mathbb{K})$ and its time derivative is given by

$$\begin{aligned} \dot{c}_n(t) &= \left\langle \frac{dX}{dt}(t), \psi_n \right\rangle_{\mathcal{H}} \\ &= \langle \mathcal{A}X(t) + U(t), \psi_n \rangle_{\mathcal{H}} \\ &= \langle \mathcal{A}\{X(t) - Bd(t)\}, \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Bd(t), \psi_n \rangle_{\mathcal{H}} \\ &\quad + \langle U(t), \psi_n \rangle_{\mathcal{H}}, \end{aligned}$$

where the last equality holds true because $X(t) \in D(\mathcal{A})$ and $R(B) \subset D(\mathcal{A})$, providing $X(t) - Bd(t) \in D(\mathcal{A})$. Furthermore, $\mathcal{B}\{X(t) - Bd(t)\} = d(t) - d(t) = 0$. Thus $X(t) - Bd(t) \in D(\mathcal{A}_0)$, yielding

$$\begin{aligned} \langle \mathcal{A}\{X(t) - Bd(t)\}, \psi_n \rangle_{\mathcal{H}} &= \langle \mathcal{A}_0\{X(t) - Bd(t)\}, \psi_n \rangle_{\mathcal{H}} \\ &= \langle X(t) - Bd(t), \mathcal{A}_0^* \psi_n \rangle_{\mathcal{H}} \\ &= \langle X(t) - Bd(t), \overline{\lambda_n} \psi_n \rangle_{\mathcal{H}} \\ &= \lambda_n \langle X(t) - Bd(t), \psi_n \rangle_{\mathcal{H}}. \end{aligned}$$

We get for all $t \geq 0$,

$$\dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n \langle Bd(t), \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Bd(t), \psi_n \rangle_{\mathcal{H}} + \langle U(t), \psi_n \rangle_{\mathcal{H}}. \quad (9)$$

As all the terms involved in (9) are continuous over \mathbb{R}_+ , a straightforward integration gives for all $t \geq 0$,

$$c_n(t) = e^{\lambda_n t} c_n(0) - \lambda_n \int_0^t e^{\lambda_n(t-\tau)} \langle Bd(\tau), \psi_n \rangle_{\mathcal{H}} d\tau$$

$$\begin{aligned} &+ \int_0^t e^{\lambda_n(t-\tau)} \langle \mathcal{A}Bd(\tau), \psi_n \rangle_{\mathcal{H}} d\tau \\ &+ \int_0^t e^{\lambda_n(t-\tau)} \langle U(\tau), \psi_n \rangle_{\mathcal{H}} d\tau. \end{aligned} \quad (10)$$

Note that

$$X(t) = \sum_{n \in \mathbb{N}} c_n(t) \phi_n, \quad S(t)X_0 = \sum_{n \in \mathbb{N}} e^{\lambda_n t} c_n(0) \phi_n,$$

$$\begin{aligned} &\int_0^t S(t-\tau) \mathcal{A}Bd(\tau) d\tau \\ &= \sum_{n \in \mathbb{N}} \left\langle \int_0^t S(t-\tau) \mathcal{A}Bd(\tau) d\tau, \psi_n \right\rangle_{\mathcal{H}} \phi_n \\ &= \sum_{n \in \mathbb{N}} \int_0^t \langle S(t-\tau) \mathcal{A}Bd(\tau), \psi_n \rangle_{\mathcal{H}} d\tau \phi_n \\ &= \sum_{n \in \mathbb{N}} \int_0^t e^{\lambda_n(t-\tau)} \langle \mathcal{A}Bd(\tau), \psi_n \rangle_{\mathcal{H}} d\tau \phi_n, \end{aligned}$$

and, similarly,

$$\int_0^t S(t-\tau) U(\tau) d\tau = \sum_{n \in \mathbb{N}} \int_0^t e^{\lambda_n(t-\tau)} \langle U(\tau), \psi_n \rangle_{\mathcal{H}} d\tau \phi_n. \quad (11)$$

Thus, introducing

$$\alpha_n(t) \triangleq \lambda_n \int_0^t e^{\lambda_n(t-\tau)} \langle Bd(\tau), \psi_n \rangle_{\mathcal{H}} d\tau, \quad (12)$$

we deduce from (3) and (10) that $(\alpha_n(t))_{n \in \mathbb{N}}$ is a square summable sequence for all $t \geq 0$ and that

$$\alpha(t) \triangleq \sum_{n \in \mathbb{N}} \alpha_n(t) \phi_n \in \mathcal{H}.$$

Therefore, multiplying both sides of (10) by ϕ_n and summing over $n \in \mathbb{N}$ yields

$$\begin{aligned} X(t) &= S(t)X_0 - \alpha(t) \\ &\quad + \int_0^t S(t-\tau) \{\mathcal{A}Bd(\tau) + U(\tau)\} d\tau. \end{aligned}$$

Thus, for all $t \geq 0$, we have

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} &\leq \|S(t)X_0\|_{\mathcal{H}} + \|\alpha(t)\|_{\mathcal{H}} \\ &\quad + \left\| \int_0^t S(t-\tau) \{\mathcal{A}Bd(\tau) + U(\tau)\} d\tau \right\|_{\mathcal{H}}. \end{aligned} \quad (13)$$

Let $m_r, M_R \in \mathbb{R}_+^*$ be the constants associated with the inequality (2) for the Riesz basis formed by the eigenvectors ϕ_n of \mathcal{A}_0 . Introducing $\kappa_0 = -\omega_0 > 0$ where ω_0

is the growth bound of S , it is easy to see based on (2) and (5) that

$$\|S(t)X_0\|_{\mathcal{H}} \leq \sqrt{\frac{M_R}{m_R}} e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}}. \quad (14)$$

Similarly,

$$\begin{aligned} & \left\| \int_0^t S(t-\tau) \{ABd(\tau) + U(\tau)\} d\tau \right\|_{\mathcal{H}} \\ & \leq \sqrt{\frac{M_R}{m_R}} \int_0^t e^{-\kappa_0(t-\tau)} \|ABd(\tau) + U(\tau)\|_{\mathcal{H}} d\tau \\ & \leq \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}} \{ \|AB\|_{\mathcal{L}(\mathbb{K}^m, \mathcal{H})} \|d\|_{C^0([0,t], \mathbb{K}^m)} + \|U\|_{C^0([0,t], \mathcal{H})} \}. \end{aligned} \quad (15)$$

It remains to evaluate $\|\alpha(t)\|_{\mathcal{H}}$. To do so, consider a basis $\mathcal{E} = (e_1, e_2, \dots, e_m)$ of \mathbb{K}^m . We introduce $d_1, d_2, \dots, d_m \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{K})$ such that

$$d = \sum_{k=1}^m d_k e_k.$$

Based on this projection, one can get for all $\tau \in [0, t]$,

$$\begin{aligned} |\langle Bd(\tau), \psi_n \rangle_{\mathcal{H}}| &= \left| \sum_{k=1}^m d_k(\tau) \langle Be_k, \psi_n \rangle_{\mathcal{H}} \right| \\ &\leq \|d(\tau)\|_{\infty, \mathcal{E}} \sum_{k=1}^m |\langle Be_k, \psi_n \rangle_{\mathcal{H}}| \\ &\leq c(\mathcal{E}) \|d(\tau)\|_{\mathbb{K}^m} \sum_{k=1}^m |\langle Be_k, \psi_n \rangle_{\mathcal{H}}| \\ &\leq c(\mathcal{E}) \|d\|_{C^0([0,t], \mathbb{K}^m)} \sum_{k=1}^m |\langle Be_k, \psi_n \rangle_{\mathcal{H}}|. \end{aligned}$$

Thus, for all $t \geq 0$, we have

$$\begin{aligned} |\alpha_n(t)| &\leq |\lambda_n| \int_0^t e^{\operatorname{Re} \lambda_n(t-\tau)} |\langle Bd(\tau), \psi_n \rangle_{\mathcal{H}}| d\tau \quad (16) \\ &\leq \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n} \right| c(\mathcal{E}) \|d\|_{C^0([0,t], \mathbb{K}^m)} \sum_{k=1}^m |\langle Be_k, \psi_n \rangle_{\mathcal{H}}| \\ &\quad \times \int_0^t -\operatorname{Re} \lambda_n \cdot e^{\operatorname{Re} \lambda_n(t-\tau)} d\tau \\ &\leq \zeta c(\mathcal{E}) \|d\|_{C^0([0,t], \mathbb{K}^m)} \sum_{k=1}^m |\langle Be_k, \psi_n \rangle_{\mathcal{H}}|, \end{aligned}$$

where the eigenvalue constraints (6) have been used. We deduce that, for all $t \geq 0$,

$$|\alpha_n(t)|^2 \leq m \zeta^2 c(\mathcal{E})^2 \|d\|_{C^0([0,t], \mathbb{K}^m)}^2 \sum_{k=1}^m |\langle Be_k, \psi_n \rangle_{\mathcal{H}}|^2.$$

From (2), we deduce first that

$$\sum_{n \in \mathbb{N}} |\alpha_n(t)|^2 \leq \frac{m}{m_R} \zeta^2 c(\mathcal{E})^2 \|d\|_{C^0([0,t], \mathbb{K}^m)}^2 \sum_{k=1}^m \|Be_k\|_{\mathcal{H}}^2,$$

and then, for all $t \geq 0$,

$$\|\alpha(t)\|_{\mathcal{H}} \leq \zeta c(\mathcal{E}) \sqrt{m \frac{M_R}{m_R} \sum_{k=1}^m \|Be_k\|_{\mathcal{H}}^2} \|d\|_{C^0([0,t], \mathbb{K}^m)}. \quad (17)$$

Substituting inequalities (14-15) and (17) into (13), we obtain the desired result (7) with

$$C_0 = \sqrt{\frac{M_R}{m_R}}, \quad C_2 = \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}},$$

$$C_1 = \sqrt{\frac{M_R}{m_R}} \left\{ \frac{1}{\kappa_0} \|AB\|_{\mathcal{L}(\mathbb{K}^m, \mathcal{H})} + \zeta c(\mathcal{E}) \sqrt{m \sum_{k=1}^m \|Be_k\|_{\mathcal{H}}^2} \right\}. \quad (18)$$

This concludes the proof. \square

Remark 2 Equations (8-9) actually hold true for the classical solutions associated with any Riesz-spectral boundary control system. Thus, the original Riesz-spectral boundary control system is equivalent to a countably infinite number of uncoupled ODEs describing the time domain evolution of the coefficients c_n corresponding to the projection of the system trajectories onto the Riesz basis $\{\phi_n, n \in \mathbb{N}\}$. The notable feature of the obtained ODEs (9) is that they provide a spectral decomposition that involves the boundary disturbance d but avoids the occurrence of its time derivative \dot{d} . This important feature offers opportunities for feedback control design. We refer, e.g., to [6, 7] for the global stabilization of heat and wave equations based on such a spectral decomposition but in the presence of a \dot{d} term.

Remark 3 In the proof of Theorem 1, the eigenvalue constraint $\zeta < \infty$ can be weakened to

$$\forall k \in \{1, \dots, m\}, \quad \sum_{n \geq 0} \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n} \right|^2 |\langle Be_k, \psi_n \rangle_{\mathcal{H}}|^2 < \infty. \quad (19)$$

It is easy to see that the condition above does not depend on a specific selection of either the lifting operator B (when $\omega_0 < 0$) or the basis $\mathcal{E} = (e_1, e_2, \dots, e_m)$ of \mathbb{K}^m . In this case, the constant C_1 is given by

$$\begin{aligned} C_1 &= \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}} \|AB\|_{\mathcal{L}(\mathbb{K}^m, \mathcal{H})} \quad (20) \\ &\quad + c(\mathcal{E}) \sqrt{m M_R \sum_{k=1}^m \sum_{n \geq 0} \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n} \right|^2 |\langle Be_k, \psi_n \rangle_{\mathcal{H}}|^2}. \end{aligned}$$

3.2 An energy-based interpretation for the constant related to the boundary perturbation in the ISS estimate

The obtained expression (18) of the constant C_1 depends on the selected lifting operator B . However, the lifting operator provided by Definition 1 is not unique. The objective of this subsection is to provide a constructive definition of a constant C_1 , independent of a specific selection of the lifting operator B , such that the ISS estimate (7) holds true.

Lemma 2 *Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system such that $0 \in \rho(\mathcal{A}_0)$. For any $e \in \mathbb{K}^m$, there exists a unique $X_e \in D(\mathcal{A})$ such that $\mathcal{A}X_e = 0$ and $\mathcal{B}X_e = e$. Furthermore, if B is a lifting operator associated with $(\mathcal{A}, \mathcal{B})$, then*

$$X_e = Be - \mathcal{A}_0^{-1} \mathcal{A}Be.$$

Proof of Lemma 2. For the uniqueness part, by linearity, it is sufficient to check that $\mathcal{A}X = 0$ and $\mathcal{B}X = 0$ implies $X = 0$. But $\mathcal{B}X = 0$ implies $X \in D(\mathcal{A}_0)$ whence $\mathcal{A}_0 X = 0$. As \mathcal{A}_0 is injective, this yields $X = 0$. For the existence part, let B be a lifting operator associated with the boundary control system $(\mathcal{A}, \mathcal{B})$ as provided by Definition 1. Consider $\tilde{X}_e = X_e - Be$. Then for $X_e \in D(\mathcal{A})$,

$$\begin{aligned} \mathcal{A}X_e &= 0 \text{ and } \mathcal{B}X_e = e \\ \Leftrightarrow \mathcal{A}\tilde{X}_e &= -\mathcal{A}Be \text{ and } \mathcal{B}\tilde{X}_e = 0 \\ \Leftrightarrow \mathcal{A}_0\tilde{X}_e &= -\mathcal{A}Be \text{ and } \tilde{X}_e \in D(\mathcal{A}_0) \\ \Leftrightarrow \tilde{X}_e &= -\mathcal{A}_0^{-1} \mathcal{A}Be. \end{aligned}$$

Thus, $X_e = Be - \mathcal{A}_0^{-1} \mathcal{A}Be$ is the unique solution. \square

Remark 4 *The stationary trajectory $X(t) = X_e$ provided by Lemma 2 is the classical solution of the abstract boundary control system (1) associated with the initial condition $X_0 = X_e$, the constant boundary disturbance $d(t) = e$, and the zero distributed disturbance $U = 0$.*

We can now introduce the main result of this section.

Theorem 2 *Let $\mathcal{E} = (e_1, e_2, \dots, e_m)$ be a basis of \mathbb{K}^m . Under the assumptions of Theorem 1, the conclusion of the theorem holds true with constants $C_0, C_1, C_2 > 0$ involved in the ISS estimate (7) given by*

$$C_0 = \sqrt{\frac{M_R}{m_R}}, \quad C_2 = \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}},$$

$$C_1 = \zeta c(\mathcal{E}) \sqrt{m \frac{M_R}{m_R} \sum_{k=1}^m \|X_{e,k}\|_{\mathcal{H}}^2},$$

where $X_{e,k} \in D(\mathcal{A})$ is, for all $k \in \{1, \dots, m\}$, the unique solution of $\mathcal{A}X_{e,k} = 0$ and $\mathcal{B}X_{e,k} = e_k$.

Proof of Theorem 2. Consider the proof of Theorem 1 up to Equation (9) included. For the basis $\mathcal{E} = (e_1, e_2, \dots, e_m)$ of \mathbb{K}^m , let $d_1, d_2, \dots, d_m \in C^2(\mathbb{R}_+; \mathbb{K})$ be such that

$$d = \sum_{k=1}^m d_k e_k.$$

As \mathcal{A}_0 is a Riesz-spectral operator with $\omega_0 < 0$, we have $0 \in \rho(\mathcal{A}_0)$. Based on Lemma 2, let, for any $k \in \{1, \dots, m\}$, $X_{e,k} \in D(\mathcal{A})$ be the unique solution of $\mathcal{A}X_{e,k} = 0$ and $\mathcal{B}X_{e,k} = e_k$. In particular, an explicit expression is given by $X_{e,k} = Be_k - \mathcal{A}_0^{-1} \mathcal{A}Be_k$. Introducing $\tilde{X}_{e,k} \triangleq X_{e,k} - Be_k = -\mathcal{A}_0^{-1} \mathcal{A}Be_k$, this yields $\tilde{X}_{e,k} \in D(\mathcal{A}_0)$. Thus we have,

$$\begin{aligned} \lambda_n \langle X_{e,k}, \psi_n \rangle_{\mathcal{H}} &= \langle X_{e,k}, \overline{\lambda_n} \psi_n \rangle_{\mathcal{H}} \\ &= \langle X_{e,k}, \mathcal{A}_0^* \psi_n \rangle_{\mathcal{H}} \\ &= \langle \tilde{X}_{e,k}, \mathcal{A}_0^* \psi_n \rangle_{\mathcal{H}} + \langle Be_k, \mathcal{A}_0^* \psi_n \rangle_{\mathcal{H}} \\ &= \langle \mathcal{A}_0 \tilde{X}_{e,k}, \psi_n \rangle_{\mathcal{H}} + \lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}} \\ &= -\langle \mathcal{A}Be_k, \psi_n \rangle_{\mathcal{H}} + \lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}}. \end{aligned}$$

Introducing that

$$D = \sum_{k=1}^m d_k X_{e,k} \in C^0(\mathbb{R}_+; D(\mathcal{A})) \cap C^2(\mathbb{R}_+; \mathcal{H}), \quad (21)$$

it follows that

$$\begin{aligned} &-\lambda_n \langle Bd(t), \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Bd(t), \psi_n \rangle_{\mathcal{H}} \\ &= \sum_{k=1}^m d_k(t) \{-\lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}} + \langle \mathcal{A}Be_k, \psi_n \rangle_{\mathcal{H}}\} \\ &= -\lambda_n \sum_{k=1}^m d_k(t) \langle X_{e,k}, \psi_n \rangle_{\mathcal{H}} \\ &= -\lambda_n \langle D(t), \psi_n \rangle_{\mathcal{H}}. \end{aligned}$$

Thus, we get from (9) that for all $t \geq 0$,

$$\dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n \langle D(t), \psi_n \rangle_{\mathcal{H}} + \langle U(t), \psi_n \rangle_{\mathcal{H}},$$

and that for all $t \geq 0$,

$$\begin{aligned} c_n(t) &= e^{\lambda_n t} c_n(0) - \lambda_n \int_0^t e^{\lambda_n(t-\tau)} \langle D(\tau), \psi_n \rangle_{\mathcal{H}} d\tau \\ &\quad + \int_0^t e^{\lambda_n(t-\tau)} \langle U(\tau), \psi_n \rangle_{\mathcal{H}} d\tau. \end{aligned}$$

The first and third terms on the right hand side of the equation above have been estimated in the proof of Theorem 1. This procedure yields the constants C_0 and C_2 . The second term can be treated with the same procedure that one employed for (12) via the projection (21). This also provides the estimate of C_1 . \square

Remark 5 The constant C_1 given by Theorem 2 depends on the energy $\|X_{e,k}\|_{\mathcal{H}}$ of m linearly independent² stationary solutions $X_{e,k}$ of the abstract boundary control system that are associated with the constant boundary perturbations $d(t) = e_k$ and the zero distributed disturbance $U = 0$.

Remark 6 One of the main applications of ISS estimates relies in the establishment of small gain conditions for assessing the stability of interconnected systems [21, Sec. 8.1]. This requires the conversion of the ISS estimates into fading memory estimates, e.g., by means of a conversion lemma [21, Chap. 7]. Nevertheless, in the context of Theorems 1 and 2, it is actually not necessary to resort to such a conversion lemma because a slight adaptation of the associated proofs (specifically, estimates (15-16)) directly shows that the following fading memory estimate holds true for all $\epsilon \in [0, 1)$ and all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}} \leq C_0 e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}} + \frac{C_1}{1-\epsilon} \|e^{-\epsilon \kappa_0(t-\cdot)} d\|_{\mathcal{C}^0([0,t]; \mathbb{K}^m)} + \frac{C_2}{1-\epsilon} \|e^{-\epsilon \kappa_0(t-\cdot)} U\|_{\mathcal{C}^0([0,t]; \mathcal{H})}.$$

4 Concept of Weak Solutions and ISS Property

Assume that a given boundary control system³ $(\mathcal{A}, \mathcal{B})$ satisfies an ISS estimate⁴ with respect to classical solutions. The objective of this section is to extend such an ISS estimate to every initial condition $X_0 \in \mathcal{H}$ and every disturbance $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$. To do so, we introduce a concept of weak solutions that extends to abstract boundary control systems the concept of weak solutions originally introduced for infinite dimensional nonhomogeneous Cauchy problems in [3] and further investigated in [8, Def. 3.1.6, A.5.29]. This represents an alternative to the traditional concept of mild solutions defined within the extrapolation space [11].

4.1 Definition of weak solutions

To motivate the definition of weak solutions, let us consider first X a classical solution associated with (X_0, d, U) . Let $T > 0$ and a function $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*))$ be arbitrarily given. As $X(t) - Bd(t) \in D(\mathcal{A}_0)$, we obtain for all $t \in [0, T]$,

$$\begin{aligned} \left\langle \frac{dX}{dt}(t), z(t) \right\rangle_{\mathcal{H}} &= \langle \mathcal{A}X(t) + U(t), z(t) \rangle_{\mathcal{H}} \\ &= \langle \mathcal{A} \{X(t) - Bd(t)\}, z(t) \rangle_{\mathcal{H}} \\ &\quad + \langle \mathcal{A}Bd(t), z(t) \rangle_{\mathcal{H}} + \langle U(t), z(t) \rangle_{\mathcal{H}} \end{aligned}$$

² Directly follows from the definition of $X_{e,k}$ and the fact that \mathcal{E} is a basis.

³ Not necessarily a Riesz-spectral one.

⁴ In the sense provided later in Theorem 3.

$$\begin{aligned} &= \langle X(t), \mathcal{A}_0^* z(t) \rangle_{\mathcal{H}} - \langle Bd(t), \mathcal{A}_0^* z(t) \rangle_{\mathcal{H}} \\ &\quad + \langle \mathcal{A}Bd(t), z(t) \rangle_{\mathcal{H}} + \langle U(t), z(t) \rangle_{\mathcal{H}}. \end{aligned}$$

Assuming now that $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$, integration by parts gives

$$\begin{aligned} \int_0^T \left\langle \frac{dX}{dt}(t), z(t) \right\rangle_{\mathcal{H}} dt &= \langle X(T), z(T) \rangle_{\mathcal{H}} - \langle X_0, z(0) \rangle_{\mathcal{H}} \\ &\quad - \int_0^T \left\langle X(t), \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt. \end{aligned}$$

Based on the two identities above, we introduce the following definition.

Definition 5 (Weak solutions) Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system. For $X_0 \in \mathcal{H}$ and disturbances $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$, we say that $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ is a weak solution of the abstract boundary control system (1) associated with (X_0, d, U) if for all $T > 0$ and for all $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^* z \in \mathcal{C}^0([0, T]; \mathcal{H})$ and $z(T) = 0$ (such a function z is called a test function over $[0, T]$), the following equality holds true:

$$\begin{aligned} &\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt \\ &= - \langle X_0, z(0) \rangle_{\mathcal{H}} + \int_0^T \langle Bd(t), \mathcal{A}_0^* z(t) \rangle_{\mathcal{H}} dt \\ &\quad - \int_0^T \langle \mathcal{A}Bd(t), z(t) \rangle_{\mathcal{H}} dt - \int_0^T \langle U(t), z(t) \rangle_{\mathcal{H}} dt, \end{aligned} \quad (22)$$

where B is an arbitrary lifting operator associated with $(\mathcal{A}, \mathcal{B})$.

Remark 7 Definition 5 is relevant because for any $T > 0$, the set of test functions over $[0, T]$ is not reduced to only the zero function. Indeed, the function z defined for all $t \in [0, T]$ by $z(t) = (t - T)z_0$ with $z_0 \in D(\mathcal{A}_0^*) \setminus \{0\}$ is obviously a non zero test function over $[0, T]$.

Remark 8 The definition of a weak solution for the abstract system (1) is compatible with the notion of classical solution. Indeed, the developments preliminary to the introduction of Definition 5 show that a classical solution is also a weak solution.

Remark 9 At first sight, the right hand side of (22) depends on the selected lifting operator B , making it necessary to specify the selected lifting operator B when saying that a trajectory X is a weak solution associated with (X_0, d, U) . However, Definition 5 implicitly claims that a weak solution is actually independent of the selected lifting operator B . This directly follows from the fact that if

B and \tilde{B} are two lifting operators associated with $(\mathcal{A}, \mathcal{B})$, then $\hat{B} \triangleq B - \tilde{B}$ satisfies $R(\hat{B}) \subset D(\mathcal{A})$ and $\mathcal{B}\hat{B} = \mathcal{B}B - \mathcal{B}\tilde{B} = I_{\mathbb{K}^m} - I_{\mathbb{K}^m} = 0$, i.e., $R(\hat{B}) \subset D(\mathcal{A}_0)$. Thus we obtain, $\langle \hat{B}d(t), \mathcal{A}_0^*z(t) \rangle_{\mathcal{H}} = \langle \mathcal{A}_0\hat{B}d(t), z(t) \rangle_{\mathcal{H}} = \langle \mathcal{A}\hat{B}d(t), z(t) \rangle_{\mathcal{H}}$, from which we deduce that

$$\begin{aligned} & \langle Bd(t), \mathcal{A}_0^*z(t) \rangle_{\mathcal{H}} - \langle \mathcal{A}Bd(t), z(t) \rangle_{\mathcal{H}} \\ &= \langle \tilde{B}d(t), \mathcal{A}_0^*z(t) \rangle_{\mathcal{H}} - \langle \mathcal{A}\tilde{B}d(t), z(t) \rangle_{\mathcal{H}}. \end{aligned}$$

This equality shows that the right hand side of (22) remains unchanged when switching between different lifting operators B associated with $(\mathcal{A}, \mathcal{B})$. So, it indeed makes sense to discuss about weak solutions without mentioning a particular lifting operator B associated with $(\mathcal{A}, \mathcal{B})$.

Remark 10 Note that the concept of weak solution does not require that the initial condition satisfies the boundary condition $\mathcal{B}X_0 = d(0)$. Such an algebraic condition is not even well defined when $X_0 \notin D(\mathcal{B})$.

4.2 Properties of weak solutions

When defining a notion of a weak solution, it is generally desirable to preserve the uniqueness of the solution.

Lemma 3 Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system such that the disturbance-free operator \mathcal{A}_0 is injective. Then, for any given $X_0 \in \mathcal{H}$ and disturbances $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ and $u \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$, there exists at most one weak solution associated with (X_0, d, U) of the abstract system (1).

The proof of Lemma 3 is in Annex A. We deduce from Remark 8 and Lemma 3 the following result.

Corollary 1 Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system such that the disturbance-free operator \mathcal{A}_0 is injective. For any initial condition $X_0 \in D(\mathcal{A})$ and disturbances $d \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ such that $\mathcal{B}X_0 = d(0)$, the concepts of classical and weak solutions coincide. More specifically, the two following statements are equivalent:

- (1) X is a classical solution associated with (X_0, d, U) ;
- (2) X is a weak solution associated with (X_0, d, U) .

When X is a classical solution of the abstract boundary control system (1) associated with (X_0, d, U) , we have by Definition 4 that $X(0) = X_0$. Such an initial condition is not explicitly imposed in the Definition 5 of a weak solution. However, it is a consequence of (22) as shown by the following lemma.

Lemma 4 Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system. Under the terms of Definition 5, assume that X is a weak solution associated with (X_0, d, U) . Then $X(0) = X_0$.

Proof of Lemma 4. Let $z \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ such that $\mathcal{A}_0^*z \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$. Then for all $T > 0$, $\tilde{z}_T \triangleq z - z(T)$ is a test function over $[0, T]$. As $dz(T)/dt = 0$, (22) gives for all $T > 0$,

$$\begin{aligned} & \frac{1}{T} \int_0^T \langle X(t), \mathcal{A}_0^*z(t) - \mathcal{A}_0^*z(T) \rangle_{\mathcal{H}} dt \\ &+ \frac{1}{T} \int_0^T \left\langle X(t), \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt \\ &= \left\langle X_0, \frac{z(T) - z(0)}{T} \right\rangle_{\mathcal{H}} \\ &+ \frac{1}{T} \int_0^T \langle Bd(t), \{\mathcal{A}_0^*z(t) - \mathcal{A}_0^*z(T)\} \rangle_{\mathcal{H}} dt \\ &- \frac{1}{T} \int_0^T \langle \mathcal{A}Bd(t), z(t) - z(T) \rangle_{\mathcal{H}} dt \\ &- \frac{1}{T} \int_0^T \langle U(t), z(t) - z(T) \rangle_{\mathcal{H}} dt. \end{aligned} \tag{23}$$

It is straightforward to show that for any $f, g \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$,

$$\lim_{T \rightarrow 0^+} \frac{1}{T} \int_0^T \langle f(t), g(t) - g(T) \rangle_{\mathcal{H}} dt = 0.$$

Thus, based on the regularity assumptions given in Definition 5, we deduce, by letting $T \rightarrow 0^+$ in (23),

$$\left\langle X(0), \frac{dz}{dt}(0) \right\rangle_{\mathcal{H}} = \left\langle X_0, \frac{dz}{dt}(0) \right\rangle_{\mathcal{H}}.$$

Taking in particular $z(t) = tz_0$ with $z_0 \in D(\mathcal{A}_0^*)$, we get that $\langle X(0) - X_0, z_0 \rangle_{\mathcal{H}} = 0$ holds true for all $z_0 \in D(\mathcal{A}_0^*)$. As $\overline{D(\mathcal{A}_0^*)} = \mathcal{H}$, we deduce that $X(0) = X_0$. \square

Thus, for a weak solution X associated with (X_0, d, u) , it makes sense to say that X_0 is the initial condition of the system trajectory.

4.3 Existence of weak solutions and extension of ISS estimates

The proof of the following theorem is in Annex B.

Theorem 3 Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system. Assume that there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that for any initial condition $X_0 \in D(\mathcal{A})$ and any disturbances $d \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ such that $\mathcal{B}X_0 = d(0)$, the classical solution X of the abstract boundary control system (1) associated with (X_0, d, U) satisfies for all $t \geq 0$,

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} &\leq \beta(\|X_0\|_{\mathcal{H}}, t) + \gamma_1 (\|d\|_{\mathcal{C}^0([0,t], \mathbb{K}^m)}) \\ &+ \gamma_2 (\|U\|_{\mathcal{C}^0([0,t], \mathcal{H})}). \end{aligned} \tag{24}$$

Then, for any initial condition $X_0 \in \mathcal{H}$, and any disturbances $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$:

- (1) the abstract boundary control system (1) has a unique weak solution $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ associated with (X_0, d, U) ;
- (2) this weak solution satisfies the ISS estimate (24) for all $t \geq 0$.

Remark 11 Theorem 3 ensures that the ISS property holds for weak solutions if and only if it holds for classical solutions. Thus, in the study of a given abstract boundary control system, it is actually sufficient to study the ISS property for classical solutions associated with sufficiently smooth disturbances (from the density argument used in the proof of Theorem 3, the study can even be restricted to the only disturbances of class C^∞) to conclude that the ISS property holds for weak solutions.

We directly deduce from Theorem 3 the following extension of Theorems 1 and 2 for Riesz-spectral operators.

Corollary 2 Let $(\mathcal{A}, \mathcal{B})$ be a Riesz-spectral boundary control system such that the eigenvalue constraints (6) hold true⁵. For every initial condition $X_0 \in \mathcal{H}$, and every disturbance $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$, the abstract system (1) has a unique weak solution $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ associated with (X_0, d, U) . Furthermore, X satisfies the ISS estimate (7) with constants κ_0, C_0, C_1, C_2 given by either Theorem 1 or 2.

Based on the existence and uniqueness of weak solutions, and by linearity of (22), we can state the following result.

Corollary 3 Assume that the assumptions of Theorem 3 hold true. For $i \in \{1, 2\}$, let X_i be the weak solution associated with $(X_{i,0}, d_i, U_i)$. Then, for all $\alpha, \beta \in \mathbb{K}$, $\alpha X_1 + \beta X_2$ is the unique weak solution associated with $(\alpha X_{1,0} + \beta X_{2,0}, \alpha d_1 + \beta d_2, \alpha U_1 + \beta U_2)$.

4.4 Mild solutions and semigroup property

4.4.1 Compatibility with the concept of mild solutions

When the boundary disturbance satisfies the additional regularity assumption $d \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K}^m)$, a classical approach for extending the concept of classical solutions is to define X provided by

$$X(t) = S(t)(X_0 - Bd(0)) + Bd(t) + \int_0^t S(t-\tau) \left\{ -B\dot{d}(\tau) + ABd(\tau) + U(\tau) \right\} d\tau \quad (25)$$

as the mild solution associated with (X_0, d, U) . In particular, any classical solution X satisfies the identity (25).

⁵ The constraint $\zeta < \infty$ can be relaxed to (19). In that case, the constant C_1 is given by (20).

The following result shows that such an approach is compatible with the concept of weak solutions.

Theorem 4 Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system such that the assumptions of Theorem 3 hold true. Under the terms of Definition 5, let X be the weak solution associated with (X_0, d, U) . Assume that the boundary disturbance satisfies the extra regularity assumption $d \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K}^m)$. Then X is also a mild solution in the sense that (25) holds true for all $t \geq 0$.

Proof of Theorem 4. Let $X_0 \in \mathcal{H}$, $d \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K}^m)$, $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$, and $T > 0$ be arbitrarily given. We denote by X the weak solution associated with (X_0, d, U) . As $\mathcal{C}^2([0, T]; \mathbb{K}^m)$ is dense in $\mathcal{C}^1([0, T]; \mathbb{K}^m)$ endowed with the usual norm $\|f\|_{\mathcal{C}^1([0, T]; \mathbb{K}^m)} = \|f\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)} + \|\dot{f}\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)}$, we can select, similarly to the proof of Theorem 3, approximating sequences $(X_{0,n})_n \in D(\mathcal{A})^{\mathbb{N}}$, $(d_n)_n \in \mathcal{C}^2([0, T]; \mathbb{K}^m)^{\mathbb{N}}$, and $(U_n)_n \in \mathcal{C}^1([0, T]; \mathcal{H})^{\mathbb{N}}$ such that $\mathcal{B}X_{0,n} = d_n(0)$, $X_{0,n} \xrightarrow{n \rightarrow +\infty} X_0$,

$$\|d_n - d\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)} \xrightarrow{n \rightarrow +\infty} 0,$$

$$\|\dot{d}_n - \dot{d}\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)} \xrightarrow{n \rightarrow +\infty} 0,$$

$$\|U_n - U\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \xrightarrow{n \rightarrow +\infty} 0.$$

We denote by X_n the unique classical solution of the abstract system (1) over $[0, T]$ associated with $(X_{0,n}, d_n, U_n)$. From (25), we obtain for all $t \in [0, T]$ and all $n \in \mathbb{N}$,

$$X_n(t) = S(t)(X_{0,n} - Bd_n(0)) + Bd_n(t) + \int_0^t S(t-\tau) \left\{ -B\dot{d}_n(\tau) + ABd_n(\tau) + U_n(\tau) \right\} d\tau. \quad (26)$$

Furthermore, from the proof of Theorem 3, we know that $\|X_n - X\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \xrightarrow{n \rightarrow +\infty} 0$. Thus, by letting $n \rightarrow +\infty$ in (26), we obtain that (25) holds true for all $t \in [0, T]$. As $T > 0$ has been arbitrarily chosen, this concludes the proof. \square

4.4.2 Semigroup property

It is well known that the classical/mild solutions of the abstract system (1) satisfy the semigroup property in the sense that if X is the classical/mild solution associated with (X_0, d, U) , then $X(\cdot + t_0)$ is the classical/mild solution associated with $(X(t_0), d(\cdot + t_0), U(\cdot + t_0))$ for any $t_0 > 0$. The following result shows that this semigroup property extends to the concept of weak solutions.

Theorem 5 Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system such that the assumptions of Theorem 3 hold true. Let

X be the weak solution associated with an initial condition $X_0 \in \mathcal{H}$, a boundary disturbance $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$, and a distributed disturbance $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$. Then, for any $t_0 > 0$, $X(\cdot + t_0)$ is the weak solution associated with $(X(t_0), d(\cdot + t_0), U(\cdot + t_0))$.

Proof of Theorem 5 Note first that, based on the developments preliminary to the introduction of Definition 5, we have for any classical solution X associated with (X_0, d, U) , any $T > 0$, and any $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ such that ⁶ $\mathcal{A}_0^* z \in \mathcal{C}^0([0, T]; \mathcal{H})$,

$$\begin{aligned} & \int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt \\ &= \langle X(T), z(T) \rangle_{\mathcal{H}} - \langle X_0, z(0) \rangle_{\mathcal{H}} \\ &+ \int_0^T \langle Bd(t), \mathcal{A}_0^* z(t) \rangle_{\mathcal{H}} dt \\ &- \int_0^T \langle ABd(t), z(t) \rangle_{\mathcal{H}} dt - \int_0^T \langle U(t), z(t) \rangle_{\mathcal{H}} dt. \end{aligned} \quad (27)$$

By resorting to the same density argument as the one employed in Step 4 of the proof of Theorem 3, this yields that any weak solution X associated with (X_0, d, U) also satisfies (27) for all $T > 0$ and all $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^* z \in \mathcal{C}^0([0, T]; \mathcal{H})$.

Now, let X be the weak solution associated with (X_0, d, U) . Let $t_0, T > 0$ and a test function $\hat{z} \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ over $[0, T]$ be arbitrarily given. We define the test function $\tilde{z} \in \mathcal{C}^0([0, t_0 + T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, t_0 + T]; \mathcal{H})$ over $[0, t_0 + T]$ as $\tilde{z} = \varphi(\cdot - t_0)|_{[0, t_0 + T]}$ where $\varphi \in \mathcal{C}^0(\mathbb{R}; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1(\mathbb{R}; \mathcal{H})$ is given for all $t \in [0, T]$ and all $k \in \mathbb{Z}$ by $\varphi(t + 2kT) = \hat{z}(t) - 2k\hat{z}(0)$ and $\varphi(t + (2k+1)T) = -\hat{z}(T-t) - 2k\hat{z}(0)$. In particular we have $\tilde{z}(t_0 + T) = \varphi(T) = \hat{z}(T) = 0$ and for all $t \in [0, T]$, $\tilde{z}(t + t_0) = \varphi(t) = \hat{z}(t)$. By using (27) once with $z = \tilde{z}|_{[0, t_0]}$ over $[0, t_0]$ and once with $z = \tilde{z}$ over $[0, t_0 + T]$, we obtain after a change of variable:

$$\begin{aligned} & \int_0^T \left\langle X(t + t_0), \mathcal{A}_0^* \hat{z}(t) + \frac{d\hat{z}}{dt}(t) \right\rangle_{\mathcal{H}} dt \\ &= - \langle X(t_0), \hat{z}(0) \rangle_{\mathcal{H}} + \int_0^T \langle Bd(t + t_0), \mathcal{A}_0^* \hat{z}(t) \rangle_{\mathcal{H}} dt \\ &- \int_0^T \langle ABd(t + t_0), \hat{z}(t) \rangle_{\mathcal{H}} dt \\ &- \int_0^T \langle U(t + t_0), \hat{z}(t) \rangle_{\mathcal{H}} dt. \end{aligned}$$

As T , \hat{z} and t_0 have been arbitrarily selected, it follows from Definition 5 that for all $t_0 > 0$, $X(\cdot + t_0)$ is the weak solution associated with $(X(t_0), d(\cdot + t_0), U(\cdot + t_0))$. \square

⁶ We do not impose here the condition $z(T) = 0$ as in the case of the test functions.

5 Application

Among the examples of applications, one can find 1D parabolic PDEs [19] and a flexible damped string [23]. In this section, we detail another example of structural vibrations, namely a damped Euler-Bernoulli beam.

We denote by $L^2(0, 1)$ and $H^m(0, 1)$ the set of square (Lebesgue) integrable functions over $(0, 1)$ and the usual Sobolev space of order m over $(0, 1)$, respectively. We also introduce $H_0^1(0, 1) \triangleq \{f \in H^1(0, 1) : f(0) = f(1) = 0\}$.

For a function $f : \mathbb{R}_+ \rightarrow L^2(0, 1)$, we denote, with a slight abuse of notation, $f(t, \xi) \triangleq [f(t)](\xi)$. When $f \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$, we denote $\frac{df}{dt}(t, \xi) \triangleq \left[\frac{df}{dt}(t) \right](\xi)$. Finally, when $f : \mathbb{R}_+ \rightarrow H^1(0, 1)$, we denote $f'(t, \xi) \triangleq [f(t)]'(\xi)$.

5.1 Damped Euler-Bernoulli beam

We consider a damped Euler-Bernoulli beam with point torque boundary conditions described by [8]:

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} - 2\alpha \frac{\partial^3 y}{\partial t \partial x^2} &= u, & \text{in } \mathbb{R}_+^* \times (0, 1) \\ y(t, 0) &= 0, & t \in \mathbb{R}_+^* \\ y(t, 1) &= 0, & t \in \mathbb{R}_+^* \\ \frac{\partial^2 y}{\partial x^2}(t, 0) &= d_1(t), & t \in \mathbb{R}_+^* \\ \frac{\partial^2 y}{\partial x^2}(t, 1) &= d_2(t), & t \in \mathbb{R}_+^* \\ y(0, x) &= y_0(x), & x \in (0, 1) \\ \frac{\partial y}{\partial t}(0, x) &= y_{t0}(x), & x \in (0, 1) \end{aligned}$$

where $\alpha \in \mathbb{R}_+^*$ is a constant parameter. The functions $u \in \mathcal{C}^0(\mathbb{R}_+; L^2(0, 1))$ and $d_1, d_2 \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K})$ are distributed and boundary perturbations, respectively. The functions $y_0 \in H^2(0, 1) \cap H_0^1(0, 1)$ and $y_{t0} \in L^2(0, 1)$ are the initial conditions.

Introducing the Hilbert space

$$\mathcal{H} = (H^2(0, 1) \cap H_0^1(0, 1)) \times L^2(0, 1)$$

with the inner product defined for all $(x_1, x_2), (\hat{x}_1, \hat{x}_2) \in \mathcal{H}$ by

$$\langle (x_1, x_2), (\hat{x}_1, \hat{x}_2) \rangle_{\mathcal{H}} = \int_0^1 x_1''(\xi) \overline{\hat{x}_1''(\xi)} + x_2(\xi) \overline{\hat{x}_2(\xi)} d\xi,$$

the distributed parameter system can be written as the abstract system (1) with $\mathcal{A}(x_1, x_2) = (x_2, -x_1'''' + 2\alpha x_1'')$

defined over the domain

$$D(\mathcal{A}) = (H^4(0, 1) \cap H_0^1(0, 1)) \times (H^2(0, 1) \cap H_0^1(0, 1)),$$

the boundary operator $\mathcal{B}(x_1, x_2) = (x_1''(0), x_1''(1))$ defined over the domain $D(\mathcal{B}) = D(\mathcal{A})$, the state vector $X(t) = (y(t, \cdot), y_t(t, \cdot)) \in \mathcal{H}$, the initial condition $X_0 = (y_0, y_{t0}) \in \mathcal{H}$, $d = (d_1, d_2) \in C^0(\mathbb{R}_+; \mathbb{K}^2)$, and $U = (0, u) \in C^0(\mathbb{R}_+; \mathcal{H})$.

The linear operator B defined such that for all $d = (d_1, d_2) \in \mathbb{K}^2$ and for all $x \in [0, 1]$,

$$(Bd)(x) = \left(\frac{d_2 - d_1}{6} x^3 + \frac{d_1}{2} x^2 - \frac{2d_1 + d_2}{6} x, 0 \right),$$

is a lifting operator associated with $(\mathcal{A}, \mathcal{B})$. Following [8, Exercise 2.23], it can be shown for $\alpha \in \mathbb{R}_+^* \setminus \{1\}$ that the disturbance-free operator \mathcal{A}_0 is a Riesz-spectral operator generating a C_0 -semigroup of contractions. Thus, $(\mathcal{A}, \mathcal{B})$ is a Riesz spectral boundary control system. Furthermore, the eigenvalues of \mathcal{A}_0 are given by $\{-n^2\pi^2(\alpha \pm i\sqrt{1-\alpha^2}), n \in \mathbb{N}^*\}$ when $\alpha \in (0, 1)$ while given by $\{-n^2\pi^2(\alpha \pm \sqrt{\alpha^2-1}), n \in \mathbb{N}^*\}$ when $\alpha > 1$. In both cases, the eigenvalue constraints (6) are satisfied with $\omega_0 = -\alpha\pi^2$ and $\zeta = \alpha^{-1}$ when $\alpha \in (0, 1)$ while $\omega_0 = -(\alpha - \sqrt{\alpha^2-1})\pi^2$ and $\zeta = 1$ when $\alpha > 1$.

5.2 Application of the main results

The adjoint operator \mathcal{A}_0^* is defined over the domain $D(\mathcal{A}_0^*) = D(\mathcal{A}_0)$ by

$$\mathcal{A}_0^*(x_1, x_2) = (-x_2, x_1'''' + 2\alpha x_2'').$$

Thus, for initial conditions $y_0 \in H^2(0, 1) \cap H_0^1(0, 1)$ and $y_{t0} \in L^2(0, 1)$, and disturbances $d = (d_1, d_2) \in C^0(\mathbb{R}_+; \mathbb{K}^2)$ and $u \in C^0(\mathbb{R}_+; L^2(0, 1))$, $X = (x_1, x_2) \in C^0(\mathbb{R}_+; \mathcal{H})$ is the weak solution of the abstract boundary control system (1) associated with $((y_0, y_{t0}), d, (0, u))$ if for all $T > 0$ and for all test function $z = (z_1, z_2) \in C^0([0, T]; D(\mathcal{A}_0^*)) \cap C^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^*z \in C^0([0, T]; \mathcal{H})$ and $z(T) = 0$, the following equality is satisfied:

$$\begin{aligned} & \int_0^T \int_0^1 x_1''(t, \xi) \overline{\left\{ -z_1'' + \left[\frac{dz_1}{dt} \right]'' \right\}}(t, \xi) d\xi dt \\ & + \int_0^T \int_0^1 x_2(t, \xi) \overline{\left\{ z_1'''' + 2\alpha z_2'' + \frac{dz_2}{dt} \right\}}(t, \xi) d\xi dt \\ & = - \int_0^1 y_0''(\xi) \overline{z_1''(0, \xi)} d\xi - \int_0^1 y_{t0}(\xi) \overline{z_2(0, \xi)} d\xi \\ & + \int_0^T d_1(t) \overline{z_2'(t, 0)} dt - \int_0^T d_2(t) \overline{z_2'(t, 1)} dt \\ & - \int_0^T \int_0^1 u(t, \xi) \overline{z_2(t, \xi)} d\xi dt. \end{aligned}$$

As the chosen lifting operator satisfies $\mathcal{A}\mathcal{B} = 0$, the constants C_i provided by both Theorems 1 and 2 are identical. Following [8, Exercise 2.23] and using the same approach that the one used in [23] for establishing the constants m_R and M_R related to the Riesz basis, we have:

- (1) Case $\alpha \in (0, 1)$: $\kappa_0 = \alpha\pi^2$, $\zeta = \alpha^{-1}$, $m_R = 1 - \alpha$, $M_R = 1 + \alpha$;
- (2) Case $\alpha > 1$: $\kappa_0 = (\alpha - \sqrt{\alpha^2-1})\pi^2$, $\zeta = 1$, $m_R = 1 - \alpha^{-1}$, $M_R = 1 + \alpha^{-1}$.

The boundary disturbance evolves into the two dimensional ($m = 2$) space $(\mathbb{K}^2, \|\cdot\|_2)$ endowed with the usual euclidean norm. By selecting $\mathcal{E} = \{e_1, e_2\}$ as the canonical basis of \mathbb{K}^2 , we obtain $c(\mathcal{E}) = 1$ and $\|Be_1\|_{\mathcal{H}} = \|Be_2\|_{\mathcal{H}} = 1/\sqrt{3}$. Thus, the constants of the ISS estimate are given by

$$C_0 = \sqrt{\frac{1+\alpha}{1-\alpha}}, \quad C_1 = \frac{2}{\alpha\sqrt{3}} \sqrt{\frac{1+\alpha}{1-\alpha}}, \quad C_2 = \frac{1}{\alpha\pi^2} \sqrt{\frac{1+\alpha}{1-\alpha}} \quad (28)$$

when $\alpha \in (0, 1)$, while

$$C_0 = \sqrt{\frac{\alpha+1}{\alpha-1}}, \quad C_1 = \frac{2}{\sqrt{3}} \sqrt{\frac{\alpha+1}{\alpha-1}}, \quad (29)$$

$$C_2 = \frac{1}{(\alpha - \sqrt{\alpha^2-1})\pi^2} \sqrt{\frac{\alpha+1}{\alpha-1}}$$

when $\alpha > 1$.

We observe that the ISS constants provided by Theorems 1 and 2 diverge to $+\infty$ when the quantity $\alpha \rightarrow 1$. This is due to the fact that the disturbance-free operator \mathcal{A}_0 loses its Riesz-spectral properties when $\alpha = 1$ with in particular $m_R \xrightarrow{\alpha \rightarrow 1} 0$. Thus, the obtained ISS constants become conservative when $\alpha \rightarrow 1$ due to the fact that the constants of the Riesz basis are such that $M_R/m_R \xrightarrow{\alpha \rightarrow 1} +\infty$. An approach for reducing such a conservatism is discussed in the next subsection.

It is interesting to evaluate the impact of an increased damping term α . Constants C_0 and C_1 are decreasing functions of $\alpha > 1$ and we have $C_0 \xrightarrow{\alpha \rightarrow +\infty} 1$ and $C_1 \xrightarrow{\alpha \rightarrow +\infty} 2/\sqrt{3}$. We also observe that $C_2 \xrightarrow{\alpha \rightarrow +\infty} +\infty$; this divergent behavior is induced by the fact that Theorems 1 and 2 deal with distributed perturbations evolving in the full space \mathcal{H} . However, the distributed disturbance evolves in the subspace $\{0\} \times L^2(0, 1)$. To provide a tighter constant C_2 , we directly estimate (11) with the sparse structure $U = (0, u)$. This yields the following version of the ISS constant:

$$C_2 = \sqrt{M_R \sum_{\substack{n \in \mathbb{N}^* \\ \epsilon \in \{-1, +1\}}} \frac{\|\psi_{n, \epsilon}^2\|_{L^2(0, 1)}^2}{|\operatorname{Re} \lambda_{n, \epsilon}|^2}}$$

where $\psi_{n,\varepsilon}^2$ denotes the second component of $\psi_{n,\varepsilon}$. For $\alpha > 1$, the eigenvalues of the disturbance-free operator \mathcal{A}_0 are given by $\lambda_{n,\varepsilon} = -n^2\pi^2(\alpha + \varepsilon\sqrt{\alpha^2 - 1}) \in \mathbb{R}_-^*$, $n \in \mathbb{N}^*$ and $\varepsilon \in \{-1, +1\}$. The corresponding eigenvectors are given by

$$\phi_{n,\varepsilon} = \frac{1}{n^2\pi^2\sqrt{\alpha(\alpha + \varepsilon\sqrt{\alpha^2 - 1})}} \begin{pmatrix} \sin(n\pi\cdot) \\ \lambda_{n,\varepsilon} \sin(n\pi\cdot) \end{pmatrix}.$$

The associated biorthogonal sequence is given by

$$\psi_{n,\varepsilon} = \frac{2\sqrt{\alpha(\alpha + \varepsilon\sqrt{\alpha^2 - 1})}}{n^2\pi^2(1 - (\alpha + \varepsilon\sqrt{\alpha^2 - 1})^2)} \begin{pmatrix} \sin(n\pi\cdot) \\ -\lambda_{n,\varepsilon} \sin(n\pi\cdot) \end{pmatrix}.$$

Now, straightforward computations show that⁷

$$C_2 = \frac{1}{3\sqrt{10}} \sqrt{\frac{\alpha}{\alpha - 1}} \xrightarrow{\alpha \rightarrow +\infty} \frac{1}{3\sqrt{10}}.$$

5.3 Improvement in the neighborhood of $\alpha = 1$ by adaptation of the spectral decomposition

We propose to adapt the spectral decomposition (9) in order to reduce the conservatism observed in the neighborhood of $\alpha = 1$ for the constants C_0 and C_1 . To do so, we introduce $\psi_n = \frac{1}{n^2\pi^2} (\sin(n\pi\cdot), n^2\pi^2 \sin(n\pi\cdot)) \in D(\mathcal{A}_0^*)$, $n \in \mathbb{N}^*$, which are the eigenvectors of \mathcal{A}_0^* in the case $\alpha = 1$ associated with the eigenvalues $\mu_n = -n^2\pi^2$. As $\{\psi_n, n \in \mathbb{N}\}$ is not maximal in \mathcal{H} , we introduce the vectors $\psi_n^d = \frac{1}{n^2\pi^2} (\sin(n\pi\cdot), -n^2\pi^2 \sin(n\pi\cdot)) \in D(\mathcal{A}_0^*)$. Straightforward computations show that $\{\psi_n, \psi_n^d, n \in \mathbb{N}^*\}$ is a Hilbert basis of \mathcal{H} and, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} \mathcal{A}_0^* \psi_n &= -\alpha n^2 \pi^2 \psi_n + (\alpha - 1) n^2 \pi^2 \psi_n^d, \\ \mathcal{A}_0^* \psi_n^d &= (\alpha + 1) n^2 \pi^2 \psi_n - \alpha n^2 \pi^2 \psi_n^d. \end{aligned}$$

Then, introducing $c_n(t) \triangleq \langle X(t), \psi_n \rangle_{\mathcal{H}}$ and $c_n^d(t) \triangleq \langle X(t), \psi_n^d \rangle_{\mathcal{H}}$, we have $\|X(t)\|_{\mathcal{H}}^2 = \sum_{n \geq 1} |c_n(t)|^2 + |c_n^d(t)|^2$.

Furthermore, by using the same approach as the one employed to derive (9), the following decomposition holds true for any classical solution X associated with an initial condition X_0 and a boundary disturbance d ,

$$\begin{bmatrix} \dot{c}_n(t) \\ \dot{c}_n^d(t) \end{bmatrix} = n^2 \pi^2 M \begin{bmatrix} c_n(t) \\ c_n^d(t) \end{bmatrix} - n^2 \pi^2 M \begin{bmatrix} \langle D(t), \psi_n \rangle_{\mathcal{H}} \\ \langle D(t), \psi_n^d \rangle_{\mathcal{H}} \end{bmatrix}$$

⁷ We used $\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.

with

$$M \triangleq \begin{bmatrix} -\alpha & \alpha - 1 \\ \alpha + 1 & -\alpha \end{bmatrix}$$

and where $D(t)$ is defined by (21). Then, for all $t \geq 0$, we have

$$\begin{bmatrix} c_n(t) \\ c_n^d(t) \end{bmatrix} = e^{n^2 \pi^2 M t} \begin{bmatrix} c_n(0) \\ c_n^d(0) \end{bmatrix} - n^2 \pi^2 M \int_0^t e^{n^2 \pi^2 M (t-\tau)} \begin{bmatrix} \langle D(\tau), \psi_n \rangle_{\mathcal{H}} \\ \langle D(\tau), \psi_n^d \rangle_{\mathcal{H}} \end{bmatrix} d\tau.$$

In the case $\alpha > 1$, we have

$$e^{n^2 \pi^2 M t} = e^{-\alpha n^2 \pi^2 t} \begin{bmatrix} \cosh(\theta_n t) & \sqrt{\frac{\alpha - 1}{\alpha + 1}} \sinh(\theta_n t) \\ \sqrt{\frac{\alpha + 1}{\alpha - 1}} \sinh(\theta_n t) & \cosh(\theta_n t) \end{bmatrix}$$

where $\theta_n \triangleq n^2 \pi^2 \sqrt{\alpha^2 - 1}$. Then, with a similar approach as the one employed in the proof of Theorem 1 and using the inequalities $\cosh(x) \leq e^x$, $\sinh(x) \leq e^x/2$, $\sinh(x) \leq x \cosh(x)$, and $x e^{-\gamma x} \leq e^{-1}/\gamma$ that hold for all $x \geq 0$ and $\gamma > 0$, we obtain for any $\varepsilon \in (0, 1)$ the following versions of the ISS constants:

$$\begin{aligned} \kappa_0 &= (1 - \varepsilon)(\alpha - \sqrt{\alpha^2 - 1})\pi^2, \\ C_0 &= 2 \max \left(1, \frac{e^{-1}(\alpha + 1)}{\varepsilon(\alpha - \sqrt{\alpha^2 - 1})} \right), \\ C_1 &= \frac{4(\alpha + 1)(2\alpha - \sqrt{\alpha^2 - 1})}{\sqrt{3}(\alpha - \sqrt{\alpha^2 - 1})^2}. \end{aligned} \quad (30)$$

Similar computations show that the above constants are also valid in the case $\alpha = 1$ while the case $\alpha \in (0, 1)$ yields the following versions of the constants:

$$\kappa_0 = (1 - \varepsilon)\alpha\pi^2,$$

$$C_0 = 2 \max \left(1, \frac{e^{-1}(\alpha + 1)}{\varepsilon\alpha} \right), \quad C_1 = \frac{8(\alpha + 1)}{\sqrt{3}\alpha}. \quad (31)$$

Comparing to (28-29), the above ISS constants (30-31) are less conservative in a small neighborhood of $\alpha = 1$ because finite in $\alpha = 1$. However, outside such a neighborhood, they become more conservative. For instance, the above version of C_1 given by (30-31), which is such that $C_1 \underset{\alpha \rightarrow +\infty}{\sim} 16\alpha^4/\sqrt{3}$, is only less conservative than the version (28-29) over the (approximate) range $\alpha \in (0.967, 1.020)$. The evolution of the constant C_1 obtained by taking the minimum of the versions (28-29) and (30-31) is depicted in Fig. 1.

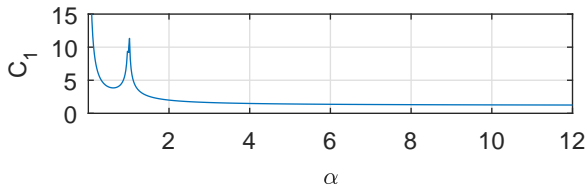


Fig. 1. ISS constant C_1 for the Euler-Bernoulli beam

6 Conclusion

This paper discussed the establishment of Input-to-State Stability (ISS) estimates for a class of Riesz-spectral boundary control system with respect to both boundary and distributed perturbations. First, a spectral decomposition depending only on the boundary disturbance (but not on its time derivative) was obtained by projecting the system trajectories over an adequate Riesz basis. This was used to derive an ISS estimate with respect to classical solutions. Then, in order to relax the regularity assumptions required for assessing the existence of classical solutions, a concept of weak solution that applies to a large class of boundary control systems (which is not limited to Riesz-spectral ones) has been introduced under a variational formulation. Assuming that an ISS estimate holds true with respect to classical solutions, various properties of the weak solutions were derived, including their existence and uniqueness, as well as their ISS property.

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A Proof of Lemma 3

By linearity, we must show that if $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ satisfies

$$\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt = 0 \quad (\text{A.1})$$

for all $T > 0$ and for all $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^* z \in \mathcal{C}^0([0, T]; \mathcal{H})$ and $z(T) = 0$, then $X = 0$. Denoting by S the C_0 -semigroup generated by \mathcal{A}_0 , then S^* is the C_0 -semigroup generated by \mathcal{A}_0^* (see, e.g., [8, Thm 2.2.6]). Let $z_0 \in D(\mathcal{A}_0^*)$

and $\alpha > 0$ be arbitrarily given. For any given $T > 0$, we consider the function $z_{z_0, \alpha, T}$ defined for any $t \in [0, T]$ by $z_{z_0, \alpha, T}(t) = S^*(\alpha t)z_0 - S^*(\alpha T)z_0$. As $z_0 \in D(\mathcal{A}_0^*)$, we obtain that for any $t \geq 0$, $S^*(\alpha t)z_0 \in D(\mathcal{A}_0^*)$, $\mathcal{A}_0^* S^*(\alpha t)z_0 = S^*(\alpha t)\mathcal{A}_0^* z_0$, and

$$\frac{dz_{z_0, \alpha, T}}{dt}(t) = \alpha \mathcal{A}_0^* S^*(\alpha t)z_0 = \alpha S^*(\alpha t)\mathcal{A}_0^* z_0.$$

Thus, $z_{z_0, \alpha, T}$ is a test function over $[0, T]$ and (A.1) shows:

$$\begin{aligned} (\alpha + 1) \int_0^T \langle X(t), S^*(\alpha t)\mathcal{A}_0^* z_0 \rangle_{\mathcal{H}} dt \\ = \int_0^T \langle X(t), S^*(\alpha T)\mathcal{A}_0^* z_0 \rangle_{\mathcal{H}} dt. \end{aligned}$$

Using the definition of the adjoint operator, the fact that $S(\alpha T) \in \mathcal{L}(\mathcal{H})$, and the properties of the Bochner integral, the equation above is equivalent to

$$\left\langle (\alpha + 1) \int_0^T S(\alpha t)X(t) dt - S(\alpha T) \int_0^T X(t) dt, \mathcal{A}_0^* z_0 \right\rangle_{\mathcal{H}} = 0.$$

Because \mathcal{H} is a Hilbert space with \mathcal{A}_0 closed and densely defined, we have that $\ker(\mathcal{A}_0)^\perp = \overline{\mathbb{R}(\mathcal{A}_0^*)}$ (see, e.g., [4, Chap. 2, Rem. 17]). Since \mathcal{A}_0 is assumed to be injective, this yields that $\overline{\mathbb{R}(\mathcal{A}_0^*)} = \mathcal{H}$. Consequently, we obtain that the following equality holds true for all $\alpha > 0$ and $T \geq 0$

$$(\alpha + 1) \int_0^T S(\alpha t)X(t) dt = S(\alpha T) \int_0^T X(t) dt. \quad (\text{A.2})$$

This implies that, for any $h > 0$,

$$\begin{aligned} \frac{\alpha + 1}{h} \int_T^{T+h} S(\alpha t)X(t) dt \\ = S(\alpha(T+h)) \left\{ \frac{1}{h} \int_T^{T+h} X(t) dt \right\} \\ + \frac{S(\alpha h) - I_{\mathcal{H}}}{\alpha h} \left\{ \alpha S(\alpha T) \int_0^T X(t) dt \right\}. \end{aligned}$$

As $t \rightarrow X(t)$ and $t \rightarrow S(\alpha t)X(t)$ are continuous over \mathbb{R}_+ , we obtain by the continuity property of the C_0 -semigroups that

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{S(\alpha h) - I_{\mathcal{H}}}{\alpha h} \left\{ S(\alpha T) \int_0^T X(t) dt \right\} \\ = \frac{\alpha + 1}{\alpha} S(\alpha T)X(T) - \frac{1}{\alpha} S(\alpha T)X(T) \\ = S(\alpha T)X(T). \end{aligned}$$

Thus we have $S(\alpha T) \int_0^T X(t) dt \in D(\mathcal{A}_0)$ and

$$\mathcal{A}_0 \left\{ S(\alpha T) \int_0^T X(t) dt \right\} = S(\alpha T)X(T).$$

From (A.2), we deduce that $\int_0^T S(\alpha t)X(t) dt \in D(\mathcal{A}_0)$ and

$$(\alpha + 1)\mathcal{A}_0 \int_0^T S(\alpha t)X(t) dt = S(\alpha T)X(T)$$

for all $\alpha > 0$ and $T \geq 0$. Introducing $y_\alpha(T) = \int_0^T S(\alpha t)X(t) dt$ and noting that $t \rightarrow S(\alpha t)X(t)$ is continuous over \mathbb{R}_+ , we obtain that y_α satisfies over \mathbb{R}_+ the differential equation $\frac{dy_\alpha}{dT} = (\alpha + 1)\mathcal{A}_0 y_\alpha$ with the initial condition $y_\alpha(0) = 0$. As \mathcal{A}_0 generates the C_0 -semigroup S and $\alpha + 1 > 0$, we deduce that $(\alpha + 1)\mathcal{A}_0$ generates the C_0 -semigroup $S((\alpha + 1)\cdot)$. Thus we have $y_\alpha = S((\alpha + 1)\cdot)y_\alpha(0) = 0$. By taking the time derivative of y_α , we deduce that $S(\alpha t)X(t) = 0$ for all $\alpha > 0$ and $t \geq 0$. From the continuity property of the C_0 -semigroups, we obtain by letting $\alpha \rightarrow 0^+$ that $X(t) = 0$ for all $t \geq 0$. \square

B Proof of Theorem 3

To establish the uniqueness part, we only need to show that \mathcal{A}_0 is injective. In that case, the conclusion will follow from the application of Lemma 3. Let $x_0 \in \ker(\mathcal{A}_0)$ be arbitrarily given. Introducing $X(t) = x_0$ for all $t \geq 0$, $X_0 = x_0$, $d = 0$, and $U = 0$, one has for all $t \geq 0$, $\frac{dX}{dt}(t) = 0 = \mathcal{A}_0 x_0 = \mathcal{A}X(t) + U(t)$, $\mathcal{B}X(t) = \mathcal{B}x_0 = 0 = d(t)$, and $X(0) = x_0$. Thus X is the classical solution associated with $(X_0, d, U) = (x_0, 0, 0)$. The ISS estimate (24) gives $\|x_0\|_{\mathcal{H}} \leq \beta(\|x_0\|_{\mathcal{H}}, t) \xrightarrow{t \rightarrow +\infty} 0$. This yields $x_0 = 0$, ensuring the injectivity of \mathcal{A}_0 .

To show the existence part, let an initial condition $X_0 \in \mathcal{H}$, and disturbances $d \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ be arbitrarily given. We also consider an arbitrarily given lifting operator B associated with $(\mathcal{A}, \mathcal{B})$.

Step 1: Construction of a weak solution candidate $X \in \mathcal{C}^0([0, T]; \mathcal{H})$ by density arguments.

Let $T > 0$ be arbitrarily given. As $\mathcal{C}^2([0, T]; \mathbb{K}^m)$ and $\mathcal{C}^1([0, T]; \mathcal{H})$ are dense in $\mathcal{C}^0([0, T]; \mathbb{K}^m)$ and $\mathcal{C}^0([0, T]; \mathcal{H})$, respectively, there exist sequences $(d_n)_n \in \mathcal{C}^2([0, T]; \mathbb{K}^m)^{\mathbb{N}}$ and $(U_n)_n \in \mathcal{C}^1([0, T]; \mathcal{H})^{\mathbb{N}}$ (we can actually use here approximating sequences of smooth functions, i.e., of class \mathcal{C}^∞) such that

$$\|d_n - d\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)} \xrightarrow{n \rightarrow +\infty} 0,$$

$$\|U_n - U\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \xrightarrow{n \rightarrow +\infty} 0.$$

Now, as $\overline{D(\mathcal{A}_0)} = \mathcal{H}$, there exists $(\tilde{X}_{0,n})_n \in (D(\mathcal{A}_0))^{\mathbb{N}}$ such that $\tilde{X}_{0,n} \xrightarrow{n \rightarrow +\infty} X_0 - Bd(0)$. Introducing $X_{0,n} \triangleq \tilde{X}_{0,n} + Bd_n(0) \in D(\mathcal{A})$, the bounded nature of B gives $X_{0,n} \xrightarrow{n \rightarrow +\infty} X_0$. Recalling that $D(\mathcal{A}_0) \subset \ker(\mathcal{B})$ and $\mathcal{B}\mathcal{B} = I_{\mathbb{K}}$, we get $\mathcal{B}X_{0,n} = \mathcal{B}\tilde{X}_{0,n} + \mathcal{B}Bd_n(0) = d_n(0)$.

For any $n \in \mathbb{N}$, let $X_n \in \mathcal{C}^0([0, T]; D(\mathcal{A})) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ be the classical solution of the abstract system (1) over $[0, T]$ associated with $(X_{0,n}, d_n, U_n)$. For any $n, m \in \mathbb{N}$, by linearity, $X_n - X_m$ is the unique classical solution of the abstract system (1) over $[0, T]$ associated with $(X_{0,n} - X_{0,m}, d_n - d_m, U_n - U_m)$. Thus the ISS estimate for classical solutions (24) yields for all $n, m \in \mathbb{N}$,

$$\begin{aligned} \|X_n - X_m\|_{\mathcal{C}^0([0, T]; \mathcal{H})} &\leq \beta(\|X_{0,n} - X_{0,m}\|_{\mathcal{H}}, 0) \\ &\quad + \gamma_1(\|d_n - d_m\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)}) \\ &\quad + \gamma_2(\|U_n - U_m\|_{\mathcal{C}^0([0, T]; \mathcal{H})}). \end{aligned}$$

Since $(X_{0,n})_n$, $(d_n)_n$, and $(U_n)_n$ are Cauchy sequences, so is $(X_n)_n$. As $\mathcal{C}^0([0, T]; \mathcal{H})$ is a Banach space, there exists $X \in \mathcal{C}^0([0, T]; \mathcal{H})$ such that $\|X_n - X\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \xrightarrow{n \rightarrow +\infty} 0$. Writing the ISS estimate (24) for each classical solution X_n and letting $n \rightarrow +\infty$ shows that (24) holds true for X for all $t \in [0, T]$.

Step 2: The obtained $X \in \mathcal{C}^0([0, T]; \mathcal{H})$ is independent of the chosen approximating sequences of X_0 , d , and U .

For a given $T > 0$, we show that the construction of Step 1 provides a $X \in \mathcal{C}^0([0, T]; \mathcal{H})$ that uniquely depends on (X_0, d, u) in the sense that it is independent of the employed approximation sequences. Assume that, following the construction of Step 1, $(X_{1,0,n})_n \in D(\mathcal{A})^{\mathbb{N}}$, $(d_{1,n})_n \in \mathcal{C}^2([0, T]; \mathbb{K}^m)^{\mathbb{N}}$, $(U_{1,n})_n \in \mathcal{C}^1([0, T]; \mathcal{H})^{\mathbb{N}}$ and $(X_{2,0,n})_n \in D(\mathcal{A})^{\mathbb{N}}$, $(d_{2,n})_n \in \mathcal{C}^2([0, T]; \mathbb{K}^m)^{\mathbb{N}}$, $(U_{2,n})_n \in \mathcal{C}^1([0, T]; \mathcal{H})^{\mathbb{N}}$ converge to X_0 , $d|_{[0, T]}$, $U|_{[0, T]}$, respectively. For any $n \in \mathbb{N}$ and $i \in \{1, 2\}$, let $X_{i,n}$ be the unique classical solution associated with $(X_{i,0,n}, d_{i,n}, U_{i,n})$ over $[0, T]$. We know from Step 1 that $X_{i,n}$ converges to some $X_i \in \mathcal{C}^0([0, T]; \mathcal{H})$ when $n \rightarrow +\infty$. By linearity $X_{1,n} - X_{2,n}$ is the unique classical solution associated with $(X_{1,0,n} - X_{2,0,n}, d_{1,n} - d_{2,n}, U_{1,n} - U_{2,n})$. Thus the ISS estimate for classical solutions (24) yields for all $n \in \mathbb{N}$,

$$\begin{aligned} \|X_{1,n} - X_{2,n}\|_{\mathcal{C}^0([0, T]; \mathcal{H})} &\leq \beta(\|X_{1,0,n} - X_{2,0,n}\|_{\mathcal{H}}, 0) \\ &\quad + \gamma_1(\|d_{1,n} - d_{2,n}\|_{\mathcal{C}^0([0, T]; \mathbb{K}^m)}) \\ &\quad + \gamma_2(\|U_{1,n} - U_{2,n}\|_{\mathcal{C}^0([0, T]; \mathcal{H})}). \end{aligned}$$

Letting $n \rightarrow +\infty$, it gives $\|X_1 - X_2\|_{\mathcal{C}^0([0, T]; \mathcal{H})} = 0$, i.e., $X_1 = X_2$.

Step 3: Definition of a weak solution candidate $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$.

Let $0 < T_1 < T_2$ be arbitrarily given and let $X_1 \in \mathcal{C}^0([0, T_1]; \mathcal{H})$ and $X_2 \in \mathcal{C}^0([0, T_2]; \mathcal{H})$ as provided by the construction of Step 1. It is easy to see that, by restricting the approximation sequences of $d|_{[0, T_2]}$ and $U|_{[0, T_2]}$ from $[0, T_2]$ to $[0, T_1]$ and by resorting to the uniqueness result of Step 2, that $X_1 = X_2|_{[0, T_1]}$. Therefore, we can define $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ such that for any $T > 0$, $X|_{[0, T]} \in \mathcal{C}^0([0, T]; \mathcal{H})$ is the result of the construction of Step 1. As (24) holds true for all $t \in [0, T]$ and for all $T > 0$ with functions $\beta, \gamma_1, \gamma_2$ that are independent of T , then (24) holds true for the built function $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ for all $t \geq 0$.

Step 4: The obtained candidate $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ is the unique weak solution associated with (X_0, d, U) .

Let $T > 0$ be arbitrarily given. Let $(X_{0,n})_n \in D(\mathcal{A})^{\mathbb{N}}$, $(d_n)_n \in \mathcal{C}^2([0, T]; \mathbb{K}^m)^{\mathbb{N}}$, and $(U_n)_n \in \mathcal{C}^1([0, T]; \mathcal{H})^{\mathbb{N}}$ be approximating sequences, compliant with the procedure of Step 1, converging to X_0 , $d|_{[0, T]}$, and $U|_{[0, T]}$, respectively. Thus, the corresponding sequence of classical solutions $(X_n)_n$ converges to $X|_{[0, T]}$. Based on Corollary 1, X_n is also a weak solution for all $n \in \mathbb{N}$. Thus, we have for all $z \in \mathcal{C}^0([0, T]; D(\mathcal{A}_0^*)) \cap \mathcal{C}^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^*z \in \mathcal{C}^0([0, T]; \mathcal{H})$ and $z(T) = 0$,

$$\begin{aligned} & \int_0^T \left\langle X_n(t), \mathcal{A}_0^*z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt \\ &= -\langle X_{0,n}, z(0) \rangle_{\mathcal{H}} + \int_0^T \langle Bd_n(t), \mathcal{A}_0^*z(t) \rangle_{\mathcal{H}} dt \quad (\text{B.1}) \\ & \quad - \int_0^T \langle ABd_n(t), z(t) \rangle_{\mathcal{H}} dt - \int_0^T \langle U_n(t), z(t) \rangle_{\mathcal{H}} dt. \end{aligned}$$

From $X_{0,n} \xrightarrow{n \rightarrow +\infty} X_0$, we have $\langle X_{0,n}, z(0) \rangle_{\mathcal{H}} \xrightarrow{n \rightarrow +\infty} \langle X_0, z(0) \rangle_{\mathcal{H}}$. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \left| \int_0^T \left\langle X_n(t) - X(t), \mathcal{A}_0^*z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt \right| \\ & \leq T \left\| \mathcal{A}_0^*z + \frac{dz}{dt} \right\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \|X_n - X\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \\ & \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Applying a similar procedure to the three integral terms on the right hand side of (B.1), and recalling that operators B and AB are bounded, one can show their convergence when $n \rightarrow +\infty$. Thus, letting $n \rightarrow +\infty$ in (B.1), we obtain that X satisfies (22) for all $T > 0$ and all test function z over $[0, T]$. Thus, X is the unique weak solution associated with (X_0, d, U) . \square