Title	The weak isotropy of quadratic forms over field extensions
Authors(s)	O'Shea, James
Publication date	2014-02-28
Publication information	O’Shea, James. “The Weak Isotropy of Quadratic Forms over Field Extensions.” Springer, February 28, 2014. https://doi.org/10.1007/s00229-014-0671-0.
Publisher	Springer
Item record/more information	http://hdl.handle.net/10197/6343
Publisher's statement	The final publication is available at www.springerlink.com
Publisher's version (DOI)	10.1007/s00229-014-0671-0

THE WEAK ISOTROPY OF QUADRATIC FORMS OVER FIELD EXTENSIONS

JAMES O'SHEA

Abstract. The weak isotropy index (or equivalently, sublevel) of arbitrary quadratic forms is studied. Its relationship to the level of a form is investigated. The problem of determining the set of values of the weak isotropy index of a form as it ranges over field extensions is addressed, with both admissible and inadmissible numbers being determined. An analogous investigation with respect to the level of a form is also undertaken. A treatment of forms for which the above invariants coincide concludes this article, with some recently-raised questions being resolved.

1. Introduction

This article is primarily devoted to the study of the weak isotropy of quadratic forms through an analysis of the behaviour of a related invariant, known as the weak isotropy index. Given a (quadratic) form q, the *weak isotropy index of* q, denoted $wi(q)$, as introduced by Becher in [1], is the least number n such that the orthogonal sum of n copies of q is isotropic, that is, has a non-trivial representation of zero. If no such number n exists, then $wi(q)$ is defined to be infinite. For q an anisotropic form over a field F, the *q-sublevel of* F, defined by Berhuy, Grenier-Boley and Mahmoudi in [2], is the number $\underline{s}_q(F)$ such that $\underline{s}_q(F) = wi(q)-1$. We study the weak isotropy index in the second, third and fifth sections of this article, with the results contained therein being formulated in terms of the q-sublevel. Having established relationships between $\underline{s}_q(F)$ and a number of other invariants in Section 2, the third section addresses its behaviour with respect to field extensions K/F. In particular, we seek to determine the entries of the set $\{\underline{s}_q(K) \mid K/F,\, q \text{ anisotropic over } K\}$. Drawing upon known results regarding isotropy over function fields of forms, we establish criteria for the containment of numbers within this set, and determine entries without placing restrictions on the form q (see Propositions 3.1 and 3.7). Restricting to forms q of a specific type, we can identify further entries and, indeed, are able to obtain a complete determination of the set in certain cases. In particular, for any prescribed value of $\underline{s}_q(F)$ over an ordered field F, Corollary 3.9 establishes the existence of forms q such that $\{1, \ldots, \underline{s}_q(F)\} = \{\underline{s}_q(K) \mid K/F,\, q \text{ anisotropic over } K\}$. In the complementary direction, Theorem 3.12 determines intervals of numbers that do not belong to the set $\{\underline{s}_q(K) \mid K/F,\, q \text{ anisotropic over } K\}$. These intervals, which cannot be extended in either direction in general (and, indeed, never to the left), are described in terms of the first Witt index of multiples of q. Recent results regarding the values of the first Witt indices of such forms are referenced in Remark 3.15.

Letting R be a non-trivial ring, the *level of* R, denoted $s(R)$, is the least number n such that -1 is a sum of n squares in R if such a number n exists, and is infinite otherwise. Interest in this invariant first arose on account of the Artin-Schreier theorem, which states that a field F has an ordering if and only if $s(F) = \infty$, and

---

James O'Shea: Department of Mathematics and Statistics, National University of Ireland Maynooth, Ireland. e-mail: james.oshea@maths.nuim.ie

Mathematics Subject Classification: 11E04, 11E10, 11E25, 11E81, 12D15.

its behaviour with respect to various classes of rings continues to be a topic of study. In [2], Berhuy, Grenier-Boley and Mahmoudi introduced the concept of the level of a field with respect to a form q, or the q-level, as a generalisation of the level, and undertook a wide-ranging investigation of this invariant. For q a form over F, the *q-level of F*, denoted $s_q(F)$, is the least $n \in \mathbb{N}$ such that the orthogonal sum of n copies of q represents -1 over F, and is infinite if no such number n exists. Continuing on from the investigations undertaken in [2], we devote Section 4 to considerations of the behaviour of this invariant with respect to field extensions, seeking a determination of the set $\{s_q(K) \mid K/F,\, q \text{ anisotropic over } K\}$. In this regard, we establish analogues of our results with respect to the q-sublevel. In particular, for all $n \in \mathbb{N}$, we establish the existence of an n-dimensional form q over an ordered field that can attain any prescribed number as its q-level over a suitable extension (see Theorem 4.3).

In the final section of this article, we consider classes of forms q over F such that the aforementioned invariants coincide, that is, $\underline{s}_q(F) = s_q(F)$. By [2, Proposition 4.1], this equality holds whenever the elements represented by q form a group, whereby q is said to be a "group form". We note that this result is not a characterisation however, with Examples 5.1 and 5.2 serving to demonstrate that it is not possible to characterise those forms q over F such that $\underline{s}_q(F) = s_q(F)$, thereby answering [2, Question 6.1]. A related characterisation is established in Proposition 5.3, namely of those forms q over F such that $\underline{s}_q(K) = s_q(K)$ holds for all K/F where q_K is anisotropic. The classes of round and Pfister forms constitute two important classes of group forms. In [2, Section 4.3], the behaviour of Pfister forms with respect to quadratic extensions was studied, with [2, Lemma 4.8] and [2, Proposition 4.10] being established. Addressing [2, Remark 4.24], we show that [2, Lemma 4.8] holds for round forms (see Proposition 5.9), and hence that a weakened version of [2, Proposition 4.10] also applies to this class of forms (see Proposition 5.10). Restricting to Pfister forms, we formulate a sufficient condition for their sublevel (and thus, their level) to remain unchanged after passing to the function field of an arbitrary quadratic form (see Proposition 5.11), and establish that [2, Lemma 4.8] and a strengthened version of [2, Proposition 4.10] hold in this more general context (see Propositions 5.12 and 5.14).

Henceforth, we will let F denote a field of characteristic different from two (indeed, if $\operatorname{char}(F) = 2$ then every anisotropic quadratic form q over F satisfies $\underline{s}_q(F) = 1$). The term "form" will refer to a regular quadratic form. Every form over F can be diagonalised. Given $a_1, \ldots, a_n \in F^\times$ for $n \in \mathbb{N}$, we denote by $\langle a_1, \ldots, a_n \rangle$ the n-dimensional quadratic form $a_1X_1^2 + \ldots + a_nX_n^2$. If p and q are forms over F, we denote by $p \perp q$ their orthogonal sum and by $p \otimes q$ their tensor product. For $n \in \mathbb{N}$, we will denote the orthogonal sum of n copies of q by $n \times q$. We use aq to denote $\langle a \rangle \otimes q$ for $a \in F^\times$. We write $p \simeq q$ to indicate that p and q are isometric, and say that p and q are *similar* (over F) if $p \simeq aq$ for some $a \in F^\times$. For q a form over F and K/F a field extension, we will employ the notation q_K when viewing q as a form over K via the canonical embedding. A form p is a *subform of* q if $q \simeq p \perp r$ for some form r, in which case we will write $p \subset q$. A form q *represents* $a \in F^\times$ if there exists a vector v such that $q(v) = a$. We denote by $D_F(q)$ the set of values in $F^\times$ represented by q. A form over F is *isotropic* if it represents zero non-trivially, and *anisotropic* otherwise. A form q over F is *universal* if $D_F(q) = F^\times$. In particular, isotropic forms are universal [10, Theorem I.3.4]. Every form q has a decomposition $q \simeq q_{\text{an}} \perp i(q) \times \langle 1, -1 \rangle$ where the anisotropic form q_{an} and the integer $i(q)$ are uniquely determined. A form q is *hyperbolic* if q_{an} is trivial, whereby $i(q) = \frac{1}{2}\dim q$. Two anisotropic forms p and q over F are *isotropy equivalent* if for every field extension K/F we have that p_K is isotropic if and only if q_K is isotropic. The following basic fact (see [10, Exercise I.16]) will be employed frequently.

Lemma 1.1. *If* $\tau \subset \varphi$ *with* $\dim \tau \geqslant \dim \varphi - i(\varphi) + 1$, *then* τ *is isotropic.*

An *ordering of* F is a set $P \subset F^\times$ such that $P \cup -P = F^\times$ and $x + y, xy \in P$ for all $x, y \in P$. We will let X_F denote the space of orderings of F. If X_F is non-empty, we say that F is a *formally real field.* For $a \in F^\times$ a sum of squares in $F^\times$, denoted $a \in \sum {F^\times}^2$, the *length of* a, $\ell_F(a)$, is the least number of squares in $F^\times$ that sum to a (we set $\ell_F(a) = \infty$ if $a \notin \sum {F^\times}^2$). The *Pythagoras number of* F is $p(F) = \sup\{\ell_F(a) \mid a \in \sum {F^\times}^2\}$. Given a form q over F and an ordering $P \in X_F$, the *signature of* q *at* P, denoted $\mathrm{sgn}_P(q)$, is the number of coefficients in a diagonalisation of q that are in P minus the number that are not in P. A form q over F is *indefinite at* $P \in X_F$ if $|\mathrm{sgn}_P(q)| < \dim q$. For F a field without orderings, the *u-invariant of* F is $u(F) = \sup\{\dim q \mid q \text{ is an anisotropic form over } F\}$. For $n, m \in \mathbb{N}$, we will often invoke the following identity concerning the floor and ceiling functions: $\lceil \frac{n}{m} \rceil = \lfloor \frac{n-1}{m} \rfloor + 1$.

For $n \in \mathbb{N}$, an *n-fold Pfister form* over F is a form isometric to $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle$ for some $a_1, \ldots, a_n \in F^\times$ (the form $\langle 1 \rangle$ is the 0-fold Pfister form). Isotropic Pfister forms are hyperbolic [10, Theorem X.1.7]. A form τ over F is a *neighbour* of a Pfister form π if $\tau \subset a\pi$ for some $a \in F^\times$ and $\dim \tau > \frac{1}{2} \dim \pi$. An anisotropic form q is isotropy equivalent to a Pfister form π if and only if q is a neighbour of π [6, Proposition 2]. A form q over F is a *group form* if $D_F(q)$ is a subgroup of $F^\times$. A form q over F is *round* if $D_F(q) = \{a \in F^\times \mid aq \simeq q\}$, the group of similarity factors of q. Pfister forms are round (see [10, Theorem X.1.8]). Indeed, Witt's Round Form Theorem [10, Theorem X.1.14] states that the product of a Pfister form and a round form is round.

For a form q over F with $\dim q = n \geqslant 2$ and $q \not\simeq \langle 1, -1 \rangle$, the *function field* $F(q)$ *of* q is the quotient field of the integral domain $F[X_1, \ldots, X_n]/(q(X_1, \ldots, X_n))$ (this is the function field of the affine quadric $q(X) = 0$ over F). To avoid case distinctions, we set $F(q) = F$ if $\dim q \leqslant 1$ or $q \simeq \langle 1, -1 \rangle$. The integer $i(q_{F(q)})$ (which is positive for all forms q of dimension greater than one) is called the *first Witt index of* q, and is denoted by $i_1(q)$. For all forms p over F and all extensions K/F such that q_K is isotropic, we have that $i(p_{F(q)}) \leqslant i(p_K)$ (see [9, Proposition 3.1 and Theorem 3.3]). In particular, we have that $i_1(q) \leqslant i(q_K)$ for all extensions K/F such that q_K is isotropic. An anisotropic form q is said to have *maximal splitting* if $\dim q - i_1(q)$ is a power of two. The field $F(q)$ is a purely-transcendental extension of F if and only if q is isotropic over F (see [10, Theorem X.4.1]). On account of this fact, one can see that two anisotropic forms p and q over F are isotropy equivalent if and only if $p_{F(q)}$ and $q_{F(p)}$ are isotropic. The behaviour of orderings with respect to function field extensions is governed by the following result due to Elman, Lam and Wadsworth [4, Theorem 3.5] and, independently, Knebusch [5, Lemma 10].

Theorem 1.2. *Let* q *be a form over a formally real field* F *such that* $\dim q \geqslant 2$. *Then* $P \in X_F$ *extends to* $F(q)$ *if and only if* q *is indefinite at* P.

[6, Theorem 1] and [8, Theorem 4.1] represent important isotropy criteria for function fields of quadratic forms. We will regularly invoke these results throughout this article, and therefore recall them below.

Theorem 1.3. (*Hoffmann*) *Let* p *and* q *be forms over* F *such that* p *is anisotropic. If* $\dim p \leqslant 2^n < \dim q$ *for some integer* $n \geqslant 0$, *then* $p_{F(q)}$ *is anisotropic.*

Theorem 1.4. (*Karpenko, Merkurjev*) *Let* p *and* q *be anisotropic forms over* F *such that* $p_{F(q)}$ *is isotropic. Then*

(*i*) $\dim p - i_1(p) \geqslant \dim q - i_1(q)$;

(*ii*) $\dim p - i_1(p) = \dim q - i_1(q)$ *if and only if* $q_{F(p)}$ *is isotropic.*

2. Basic properties of the weak isotropy index

For q an anisotropic form over F, the q-sublevel of F, $\underline{s}_q(F)$, and the q-level of F, $s_q(F)$, are defined as follows:

$$\underline{s}_q(F) = \inf\{n \in \mathbb{N} \mid (n+1) \times q \text{ is isotropic over } F\}$$

and

$$s_q(F) = \inf\{n \in \mathbb{N} \mid \langle 1 \rangle \perp n \times q \text{ is isotropic over } F\}.$$

An important distinction between these concepts is the fact that the q-sublevel is invariant with respect to scaling, whereas the q-level is generally not. For example, if q is a form over a formally real field F such that $s_q(F) < \infty$ and $\operatorname{sgn}_P(q) = -\dim q$ for some $P \in X_F$, it follows that $s_{-q}(F) = \infty$.

Our opening result records some basic properties of the q-sublevel of a field, by establishing analogues of statements in [2, Lemma 3.1 and Proposition 3.3] concerning the q-level.

Proposition 2.1. *Let q be an anisotropic form over F.*

(*i*) $1 \leqslant \underline{s}_q(F) \leqslant s(F)$.

(*ii*) *If $q' \subset aq$ for some $a \in F^\times$, then $\underline{s}_q(F) \leqslant \underline{s}_{q'}(F)$.*

(*iii*) *If K/F is a field extension, then $\underline{s}_q(K) \leqslant \underline{s}_q(F)$.*

(*iv*) *If K/F is a field extension whose degree is odd, then $\underline{s}_q(K) = \underline{s}_q(F)$.*

(*v*) *If $q' \subset aq$ over F and K/F is a purely transcendental field extension, then*

$$\underline{s}_q(K) = \underline{s}_q(F) = \underline{s}_q(F(\!(x)\!)) = \underline{s}_{q\perp xq'}(F(\!(x)\!)).$$

(*vi*) *For every $n \in \mathbb{N}$, the number $\left\lfloor \frac{\underline{s}_q(F)}{n} \right\rfloor = \underline{s}_{n\times q}(F)$.*

(*vii*) *If $\underline{s}_q(F) < \infty$, then the number $p(F) - 1 \geqslant \underline{s}_q(F)$.*

(*viii*) *If F is not formally real, then $\underline{s}_q(F) \leqslant \left\lceil \frac{u(F)}{\dim q} \right\rceil \leqslant u(F)$.*

Proof. (*i*), (*ii*) and (*iii*) easily follow from the definition of the q-sublevel of a field, while (*iv*) can be proven by invoking Springer's Theorem [10, Theorem VII.2.7]. Statement (*v*) follows from invoking Springer's Theorem [10, Theorem VI.1.4] for complete discretely valuated fields

To prove (*vi*), we note that $\left(\left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil\right) n \times q$ is isotropic, whereby $\underline{s}_{n\times q}(F) \leqslant \left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil - 1$. Since $\left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil - 1 = \left\lfloor \frac{\underline{s}_q(F)}{n} + 1 \right\rfloor - 1 = \left\lfloor \frac{\underline{s}_q(F)}{n} \right\rfloor$, we have that $\left(\left\lceil \frac{\underline{s}_q(F)+1}{n} \right\rceil - 1\right) n \times q$ is anisotropic, establishing (*vi*).

To prove (*vii*), we may assume that $p(F) < \infty$. Since $(\underline{s}_q(F)+1) \times q$ is isotropic, we have that $p(F) \times q$ is isotropic. Hence, $\underline{s}_q(F) \leqslant p(F) - 1$. Statement (*viii*) follows from the fact that $\left(\left\lceil \frac{u(F)}{\dim q} \right\rceil + 1\right) \times q$ is isotropic. □

Remark 2.2. We remark that all of the bounds in Proposition 2.1 can be attained. As $\underline{s}_{\langle 1 \rangle}(F) = s(F)$, letting $q \simeq \langle 1 \rangle$ over $\mathbb{C}$, one realises the bounds in (*i*). Invoking (*v*), one sees that the upper bound in (*ii*) can be attained in the case where q' is a proper subform of q. The attainability of the upper bound in (*iii*) can be deduced from (*iv*) or (*v*). The upper bound in (*vii*) can be realised by letting $q \simeq \langle 1 \rangle$ over a field F of finite Pythagoras number satisfying $p(F) = s(F) + 1$ (see [12, Ch. 7, Proposition 1.5]). Finally, as in [2, Remark 3.4], letting $q \simeq \langle 1 \rangle$ over a field F such that $s(F) = u(F) = 2^m$ for some integer $m \geqslant 0$, one realises the upper bounds in (*viii*).

As was observed in [2, Lemma 3.1 (8)], if q is an anisotropic form over F such that $1 \in D_F(q)$, then $\underline{s}_q(F) \leqslant s_q(F)$, since $\langle 1\rangle \perp s_q(F) \times q \subset (s_q(F)+1) \times q$ in this case. Indeed, more generally, we have the following relation:

Proposition 2.3. *Let q be an anisotropic form over F and $a \in F^\times$. Then*

$$\underline{s}_q(F) = \inf\{s_{aq}(F) \mid a \in D_F(q)\}.$$

Proof. As $(n+1) \times q$ is isotropic over F if and only if there exists $a \in F^\times$ such that $a \in D_F(q)$ and $-a \in D_F(n \times q)$, we can conclude that

$$\underline{s}_q(F) = \inf\{n \in \mathbb{N} \mid -a \in D_F(n \times q) \text{ for some } a \in D_F(q)\}.$$

Hence, we have that $\underline{s}_q(F) = \inf\{n \in \mathbb{N} \mid -1 \in D_F(n \times aq) \text{ for some } a \in D_F(q)\}$. Thus, we can conclude that $\underline{s}_q(F) = \inf\{s_{aq}(F) \mid a \in D_F(q)\}$. ☐

The *q-length of $a \in F^\times$* is $\ell_q(a) := \inf\{n \in \mathbb{N} \mid n \times q \perp \langle -a\rangle \text{ is isotropic over } F\}$. The *Pythagoras q-number of F* is $p_q(F) := \sup\{\ell_q(a) \mid a \in F^\times \text{ with } \ell_q(a) < \infty\}$. For q an anisotropic form over F such that $\underline{s}_q(F) < \infty$, the form $(\underline{s}_q(F)+1) \times q$ is isotropic (and hence universal) over F, whereby we have that $s_q(F) \leqslant \underline{s}_q(F)+1$ (as in [2, Lemma 3.1 (7)]). Indeed, we have the following result.

Proposition 2.4. *For q an anisotropic form over F, the following are equivalent:*

- *(i)* $\underline{s}_q(F) < \infty$.
- *(ii)* $s_q(F) < \infty$ *and* $s_{-q}(F) < \infty$.
- *(iii)* $p_q(F) < \infty$ *and* $p_q(F) \times q$ *is universal.*

Proof. Assuming *(i)*, we have that $(\underline{s}_q(F)+1) \times q$ is isotropic, and thus universal, whereby $p_q(F) \times q$ is universal and $p_q(F) \leqslant \underline{s}_q(F)+1$, establishing *(iii)*.

Assuming *(iii)*, we have that $\{-1, 1\} \subset D_F(p_q(F) \times q)$, whereby $s_q(F) \leqslant p_q(F)$ and $s_{-q}(F) \leqslant p_q(F)$, establishing *(ii)*.

Assuming *(ii)*, we have that $(s_q(F) + s_{-q}(F)) \times q$ is isotropic, whereby $\underline{s}_q(F) \leqslant s_q(F) + s_{-q}(F) - 1$, establishing *(i)*. ☐

With respect to the above result, we note the existence of forms q over fields F such that $s_q(F) < \infty$ and $\underline{s}_q(F) = \infty$ (see Remark 4.4).

Proposition 2.5. *Let q be an anisotropic form over F such that $\underline{s}_q(F) < \infty$. Then $p_q(F) - 1 \leqslant \underline{s}_q(F) \leqslant p_q(F)$.*

Proof. As in the above proof, we have that $p_q(F)-1 \leqslant \underline{s}_q(F)$. Moreover, as $p_q(F) \times q$ is universal, we have that $(p_q(F)+1) \times q$ is isotropic, whereby $\underline{s}_q(F) \leqslant p_q(F)$. ☐

Remark 2.6. Letting $q \simeq \langle 1\rangle$ over a field F such that $\underline{s}_q(F) < \infty$, Proposition 2.5 states that $p(F) - 1 \leqslant s(F) \leqslant p(F)$. As in [12, Ch. 7, Proposition 1.5], there exist fields K and L over F satisfying $s(K) = p(K) - 1 < \infty$ and $s(L) = p(L) < \infty$.

3. Values of the weak isotropy index

In this section, we study the behaviour of the q-sublevel (or equivalently, the weak isotropy index) with respect to field extensions. In particular, for q an anisotropic form over F, we will study the set $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$. Clearly, $\underline{s}_q(F)$ always belongs to this set, with the remaining entries being less than $\underline{s}_q(F)$.

We begin by seeking to show that certain prescribed numbers belong to the above set. As motivated earlier, function fields of associated quadratic forms are the natural field extensions to consider in this regard, on account of the following fact:

A number $m \leqslant \underline{s}_q(F)$ is an element of the set $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ if and only if the form $m \times q$ is anisotropic over $F((m+1) \times q)$.

Invoking Theorem 1.3, if m is such that $m \dim q \leqslant 2^n < (m+1) \dim q$ for some $n \in \mathbb{N}_0$, then $m \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$. Our opening result determines those numbers to which this observation applies.

Proposition 3.1. *Let q be an anisotropic form over F. A number $m \leqslant \underline{s}_q(F)$ is such that $m \dim q \leqslant 2^n < (m+1) \dim q$ if and only if $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some $n \in \mathbb{N}_0$. In particular, $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ for all $m \leqslant \underline{s}_q(F)$.*

Proof. Let $m \leqslant \underline{s}_q(F)$ be such that $m \dim q \leqslant 2^n < (m+1) \dim q$ for some $n \in \mathbb{N}_0$. Since $m \dim q \leqslant 2^n$, it follows that $m \leqslant \frac{2^n}{\dim q}$, and thus $m \leqslant \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ as m is an integer. Moreover, as $2^n < (m+1) \dim q$, we have that $m > \frac{2^n}{\dim q} - 1$. Hence, we have that $m \geqslant \left\lfloor \frac{2^n}{\dim q} \right\rfloor$, and thus we can conclude that $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$.

Conversely, letting $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$, we clearly have that $\dim(m \times q) \leqslant 2^n$. Moreover, as $\left\lfloor \frac{2^n}{\dim q} \right\rfloor = \left\lceil \frac{2^n+1}{\dim q} \right\rceil - 1$, letting $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ gives us that $\dim((m+1) \times q) > 2^n$. The last statement now follows from applying Theorem 1.3. $\square$

Corollary 3.2. *Let q be an anisotropic form over F. Then we have that $1 \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.*

Proof. For $\dim q = 2^n - k$ for some $n, k \in \mathbb{N}_0$ such that $0 \leqslant k < 2^n$, Proposition 3.1 implies that $\left\lfloor \frac{2^n}{\dim q} \right\rfloor = 1 \in \{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$. $\square$

Corollary 3.3. *Let q be an anisotropic form over F of dimension 2^n for some $n \in \mathbb{N}_0$. Then $2^k \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ for all $k \in \mathbb{N}_0$ such that $2^k \leqslant \underline{s}_q(F)$.*

Proof. This follows immediately from Proposition 3.1. $\square$

With respect to certain forms q and numbers m, Proposition 3.1 enables us to determine whether or not m is in $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.

Proposition 3.4. *Let q be an anisotropic form over F such that $(m+1) \times q$ has maximal splitting for some number $m < \underline{s}_q(F)$. Then $m \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ if and only if $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some $n \in \mathbb{N}_0$.*

Proof. For $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some $n \in \mathbb{N}_0$, Proposition 3.1 implies that we have $m \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.

Conversely, let K/F be such that $\underline{s}_q(K) = m$. As $m \times q$ is anisotropic over K, Lemma 1.1 implies that $\dim(m \times q) \leqslant \dim((m+1) \times q) - i((m+1) \times q_K)$. We recall that $i((m+1) \times q_K) \geqslant i_1((m+1) \times q)$, as the form $(m+1) \times q$ is isotropic over K. Thus, we have that $\dim(m \times q) \leqslant \dim((m+1) \times q) - i_1((m+1) \times q)$. Since $(m+1) \times q$ has maximal splitting, $\dim((m+1) \times q) - i_1((m+1) \times q) = 2^n$ for some $n \in \mathbb{N}_0$. Hence, it follows that $\dim(m \times q) \leqslant 2^n < \dim((m+1) \times q)$. Invoking Proposition 3.1, we have that $m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$. □

Remark 3.5. If q is similar to the orthogonal sum of n copies of a Pfister form, for some $n \in \mathbb{N}$, then for all numbers m we have that $(m+1) \times q$ is a neighbour of a Pfister form similar to $2^k \times q$ for some $k \in \mathbb{N}$. Thus, the form $(m+1) \times q$ has maximal splitting for all m in this case, whereby Proposition 3.4 can be applied to give a complete determination of $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ (see Remark 5.6).

Theorem 1.4 provides another criterion for the admissibility of a prescribed number in the set $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.

Proposition 3.6. *For q an anisotropic form over F, let $m \in \mathbb{N}$ satisfy $m < \underline{s}_q(F)$. Then $m \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ if and only if $i_1((m+1) \times q) \neq \dim q + i_1(m \times q)$.*

Proof. Since $(m+1) \times q$ is isotropic over $F(m \times q)$, this follows from Theorem 1.4 (*ii*). □

As was the case with Theorem 1.3, Theorem 1.4 enables us to establish a general result regarding containment in $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ for forms q over F.

Proposition 3.7. *Let q be an anisotropic form over F. Let $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$ be such that $2^n l \leqslant \underline{s}_q(F)$, where l is odd. For every number $m = \left\lfloor 2^n l - \frac{i_1(2^n l \times q)}{\dim q} \right\rfloor$, we have that $m \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.*

Proof. For n and l as above, consider the number $m = \left\lfloor 2^n l - \frac{i_1(2^n l \times q)}{\dim q} \right\rfloor$. Since $m \leqslant 2^n l - 1$, we have that $(m+1) \times q \subset 2^n l \times q$, which is anisotropic over F. As $m + 1 > 2^n l - \frac{i_1(2^n l \times q)}{\dim q}$, we have that $\dim((m+1) \times q) > 2^n l \dim q - i_1(2^n l \times q)$, whereby $(m+1) \times q$ is isotropic over $F(2^n l \times q)$ by Lemma 1.1. Thus, Theorem 1.4 (*ii*) implies that $\dim((m+1) \times q) - i_1((m+1) \times q) = 2^n l \dim q - i_1(2^n l \times q)$. As $\dim(m \times q) \leqslant 2^n l \dim q - i_1(2^n l \times q)$, Theorem 1.4 (*i*) implies that $m \times q$ is anisotropic over $F((m+1) \times q)$. □

Although there exists an explicit determination of the possible values of the first Witt index of a given form in terms of its dimension (see [7]), pinpointing the exact value taken by this invariant remains problematic. Moreover, a determination of the precise value of $i_1(q)$ for a given form q does not, in general, enable one to determine $i_1((m+1) \times q)$ for $m \in \mathbb{N}$ (see Remark 3.15). Thus, for our purposes, the criteria provided by Proposition 3.6 and Proposition 3.7 are difficult to apply. In the case where the function field of the given form has an ordering however, the signature of the form with respect to this ordering imposes a natural bound on the first Witt index of the form and those of its multiples. Hence, for certain forms q over certain fields F, we can apply Proposition 3.6 to determine further entries of $\{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.

Theorem 3.8. *Let q be an anisotropic form over a formally real field F such that $|\mathrm{sgn}_P(q)| = \dim q - 2$ for some $P \in X_F$. Then, for $m' := \min\{\dim q - 1, \underline{s}_q(F) - 1\}$, we have that $\{1, \ldots, m'\} \cup \{\underline{s}_q(F)\} \subseteq \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$.*

Proof. As above, $\underline{s}_q(F) \in \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$.

Let m be a number such that $m \leqslant m'$. As $m \leqslant \underline{s}_q(F) - 1$, we have that $(m+1) \times q$ is anisotropic over F. Clearly, $|\mathrm{sgn}_P((m+1) \times q)| = (m+1)\dim q - 2(m+1)$, whereby Theorem 1.2 implies that P extends to $K = F((m+1) \times q)$. Since $((m+1) \times q)_K \simeq (((m+1) \times q)_K)_{\mathrm{an}} \perp i_1((m+1) \times q) \times \langle 1, -1\rangle_K$, a comparison of signatures with respect to P yields that $i_1((m+1) \times q) \leqslant m+1$. As $m+1 \leqslant \dim q$, we have that $i_1((m+1) \times q) < \dim q + i_1(m \times q)$, whereby Proposition 3.6 implies that $\underline{s}_q(K) = m$. □

If $\underline{s}_q(F) \leqslant \dim q$ for q as above, our determination of $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ is complete.

Corollary 3.9. *Let q be an anisotropic form over a formally real field F such that $|\mathrm{sgn}_P(q)| = \dim q - 2$ for some $P \in X_F$ and $\underline{s}_q(F) \leqslant \dim q$. Then we have that $\{1, \ldots, \underline{s}_q(F)\} = \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$.*

Proof. Since $\underline{s}_q(F) \leqslant \dim q$, the result follows from invoking Theorem 3.8. □

For the forms q treated in Theorem 3.8, the following example demonstrates that, in general, a complete determination of $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ remains outstanding in cases where $\underline{s}_q(F) > \dim q$.

Example 3.10. Let q be a 4-dimensional form over a formally real field F such that $|\mathrm{sgn}_P(q)| = 2$ for some $P \in X_F$ and $\underline{s}_q(F) \geqslant 5$ (for example, for F_0 a formally real field, one can let $F = F_0(X_1, X_2, X_3, X_4)$ and $q \simeq \langle X_1, X_2, X_3, X_4\rangle$, whereby we have that $\underline{s}_q(F) = \infty$ and $|\mathrm{sgn}_P(q)| = 2$ for $P \in X_F$ such that $\{X_1, -X_2, -X_3, -X_4\} \subset P$). By Theorem 3.8, we have that $\{1, 2, 3\} \cup \{\underline{s}_q(F)\} \subseteq \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$. For $K = F(5 \times q)$, it follows from Theorem 1.3 that $4 \times q$ is anisotropic over K, whereby $\underline{s}_q(K) = 4$. Hence, we can conclude that $\{1, 2, 3\} \cup \{\underline{s}_q(F)\} \subsetneq \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$.

For certain forms satisfying $\underline{s}_q(F) > \dim q$ however, Theorem 3.8 does provide a complete determination of $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$.

Example 3.11. Let q be a 3-dimensional form over a formally real field F such that q is indefinite at $P \in X_F$ and $\underline{s}_q(F) = 4$. This can be achieved, for example, by letting $F = F_0(X)$ and $q \simeq \langle 1, -a, X\rangle$ for F_0 a formally real field such that $p(F_0) \geqslant 5$ and $a \in {F_0}^{\times}$ such that $\ell_{F_0}(a) = 5$ (whereby $\langle 1\rangle \perp 4 \times \langle -a\rangle$ is anisotropic over F_0, implying that its associated Pfister form $4 \times \langle 1, -a\rangle$ is anisotropic over F_0, and thus that $4 \times q$ is anisotropic over F by [10, Exercise IX.1]). By Theorem 3.8, $\{1, 2, 4\} \subseteq \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$. Let K be any extension of F such that $4 \times q$ is isotropic over K. Since $4 \times q \simeq (4 \times \langle 1\rangle) \otimes q$, we have that $i((4 \times q)_K) \geqslant 4$ by [13, Theorem 2]. Hence, $3 \times q$ is isotropic over K by Lemma 1.1, whereby $\underline{s}_q(K) \leqslant 2$. Thus, $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\} = \{1, 2, 4\}$.

For an arbitrary form q over F, in order to establish that certain numbers less than $\underline{s}_q(F)$ do not belong to the set $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$, function fields of associated quadratic forms are once again the appropriate extensions to consider. As before, in order to show that a prescribed number $m < \underline{s}_q(F)$ is not an element of the set $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$, it suffices to show that $m \times q$ is isotropic over $F((m+1) \times q)$.

Theorem 3.12. *Let q be an anisotropic form over F. Let $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$ be such that $2^n l \leqslant \underline{s}_q(F)$, where l is odd. For all numbers $m \in \left(2^n l - \frac{i_1(2^n l \times q)}{\dim q}, 2^n l\right)$, we have that $m \notin \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$.*

Proof. Since $2^n l \leqslant \underline{s}_q(F)$, we have that $2^n l \times q$ is anisotropic over F. Moreover, as $m < 2^n l$, we have that $(m+1) \times q \subset 2^n l \times q$. Let K/F be such that $(m+1) \times q$ is isotropic over K, whereby $i((2^n l \times q)_K) \geqslant i_1(2^n l \times q)$. Since $m > 2^n l - \frac{i_1(2^n l \times q)}{\dim q}$, it follows that $m \times q \subset 2^n l \times q$ of codimension less than $i((2^n l \times q)_K)$. Thus, Lemma 1.1 implies that $m \times q$ is isotropic over K, whereby $\underline{s}_q(K) \leqslant m - 1$. □

Remark 3.13. In accordance with Proposition 3.7, every number $m = \left\lfloor 2^n l - \frac{i_1(2^n l \times q)}{\dim q} \right\rfloor$ belongs to the set $\{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$. Moreover, since every number is expressible in the form $2^n l$ for $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$ odd, Theorem 3.12 can be viewed as providing a complete description of those numbers $m < \underline{s}_q(F)$ such that $m \notin \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$.

The intervals in Theorem 3.12 are empty if $\dim q \geqslant i_1(2^n l \times q)$. Indeed, the following example establishes the existence of forms q over F such that $\dim q = i_1(2^n l \times q)$ and every number $m < \underline{s}_q(F)$ is attainable as $\underline{s}_q(K)$ for some extension K/F.

Example 3.14. Let $K(x)$ be a formally real field such that $s(K(x)) \geqslant 2^{2n}$ and consider the form $q \simeq (2^n - 1) \times \langle 1 \rangle \perp \langle x \rangle$ over $K(x)$. Invoking Springer's Theorem [10, Theorem VI.1.4] with respect to the x-adic valuation, the form $(2^n + 1) \times q$ is anisotropic over $K(x)$, as $s(K) \geqslant s(K(x)) \geqslant 2^{2n}$. Consider an ordering P of $K(x)$ such that $-x \in P$, whereby $sgn_P(q) = 2^n - 2$. Letting $F = K(x)((2^n + 1) \times q)$, the ordering P extends to F, by Theorem 1.2, and the form $2^n \times q$ is anisotropic over F, by Theorem 1.3, whereby $\underline{s}_q(F) = 2^n$. Hence, Corollary 3.9 implies that $\{1, \ldots, \underline{s}_q(F)\} = \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$, whereby we may conclude that $i_1(2^n \times q) \leqslant 2^n$ in light of Theorem 3.12. As $i_1(2^n \times q) \geqslant 2^n$ by [13, Theorem 2], we have that $\dim q = i_1(2^n \times q)$ as desired.

Remark 3.15. In accordance with Corollary 3.9, for prescribed $m \in \mathbb{N}$, there exist forms q over F of dimension m satisfying $\{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\} = \{1, \ldots, \underline{s}_q(F)\}$, provided that the inequality $m \geqslant \underline{s}_q(F)$ holds. This is not the case, in general, with respect to prescribed numbers $m < \underline{s}_q(F)$. In particular, for $\dim q = 2^n < \underline{s}_q(F)$ for some $n \in \mathbb{N}_0$, it is known that $i_1(2^n \times q) \geq 2^n$, whereby Theorem 3.12 implies that the non-empty interval $\left(2^n - \frac{i_1(2^n \times q)}{2^n}, 2^n\right)$ does not belong to the set $\{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$. The aforementioned bound on $i_1(2^n \times q)$ arises as a consequence of [13, Theorem 2], since, for π a Pfister form over F such that $\pi \otimes q$ is anisotropic, one has that $i_1(\pi \otimes q) \geqslant \dim \pi$. In fact, this bound can be refined to $i_1(\pi \otimes q) \geq \dim \pi(i_1(q))$ (details to appear in [11]). As a consequence, with respect to Theorem 3.12, we note that $i_1(2^n l \times q) \geq 2^n(i_1(l \times q))$. In particular, as $i_1(2^n \times q) \geq 2^n(i_1(q))$, Theorem 3.12 may be formulated in terms of the first Witt index of q (although this weakens the result in general). Moreover, for certain forms q, such as those with maximal splitting for example, we can establish that the equality $i_1(\pi \otimes q) = \dim \pi(i_1(q))$ holds, and therefore that $i_1(2^n l \times q) = 2^n(i_1(l \times q))$ for such forms, whereby Proposition 3.7 may be reformulated.

We finish this section with an example to illustrate the use of certain of the results established above, in addition to those alluded to in Remark 3.15.

Example 3.16. Let q be a 6-dimensional Pfister neighbour over F such that $\underline{s}_q(F) = 16$. Proposition 3.1 implies that $1, 2, 5, 10 \in \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$.

Indeed, since q has maximal splitting, it follows that $2^n \times q$ has maximal splitting for all $n \leqslant 4$ (see Remark 3.15), whereby Proposition 3.4 or Proposition 3.7 can be invoked to recover this observation. Applying Theorem 3.12, we have that $3, 6, 7, 11, 12, 13, 14, 15 \notin \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$, an observation that can be recovered via Proposition 3.4 in this instance. Our results do not enable us to determine whether or not $m \in \{\underline{s}_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ for $m = 4, 8$ or 9.

4. Values of the q-level

In analogy with the preceding section, for q an anisotropic form over F, we study the behaviour of the q-level with respect to field extensions, seeking to determine the entries of the set $\{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$. As with the q-sublevel, $s_q(F)$ always belongs to this set, with the remaining entries being less than $s_q(F)$.

In accordance with [2, Proposition 3.13 (2)], if a number $m \leqslant s_q(F)$ has the property that $1 + (m-1)\dim q \leqslant 2^n < 1 + m \dim q$ for some $n \in \mathbb{N}_0$, then it follows that $s_q(K) = m$ for $K = F(\langle 1 \rangle \perp m \times q)$. As with the q-sublevel, we can establish the values of m to which this criterion applies. In the statement of the following result, we combine this observation with other analogues of our results with respect to the q-sublevel.

Theorem 4.1. *Let $m \in \mathbb{N}$ such that $m < s_q(F)$ for q an anisotropic form over F.*

- *(i)* $1 + (m-1)\dim q \leqslant 2^n < 1 + m \dim q$ *for some* $n \in \mathbb{N}_0$ *if and only if* $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$. *In particular, every* $m = \left\lceil \frac{2^n}{\dim q} \right\rceil \in \{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.
- *(ii)* $\{1, 2\} \subseteq \{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$.
- *(iii)* *If* $\dim q = 2^n$ *for some* $n \in \mathbb{N}_0$, *then* $2^k \in \{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ *for all* $k \in \mathbb{N}_0$ *such that* $2^k \leqslant s_q(F)$.
- *(iv)* *If* $\langle 1 \rangle \perp m \times q$ *has maximal splitting, then* $m \in \{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ *if and only if* $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$ *for some* $n \in \mathbb{N}_0$.
- *(v)* $m \in \{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ *if and only if* $\dim q \neq i_1(\langle 1 \rangle \perp m \times q) - i_1(\langle 1 \rangle \perp (m-1) \times q)$.

Proof. *(i)* Suppose that $1+(m-1)\dim q \leqslant 2^n < 1+m\dim q$ for some $n \in \mathbb{N}_0$. Since $1+(m-1)\dim q \leqslant 2^n$, we have that $m \leqslant \frac{2^n+\dim q-1}{\dim q}$. Moreover, as $2^n < 1+m\dim q$, it follows that $m > \frac{2^n-1}{\dim q}$. Thus, since $m \in \mathbb{N}$, it follows that $m \geqslant \left\lfloor \frac{2^n-1}{\dim q} \right\rfloor + 1 = \left\lfloor \frac{2^n+\dim q-1}{\dim q} \right\rfloor$. Hence, we have that $m = \left\lfloor \frac{2^n+\dim q-1}{\dim q} \right\rfloor = \left\lceil \frac{2^n}{\dim q} \right\rceil$. Conversely, letting $m = \left\lceil \frac{2^n}{\dim q} \right\rceil$, we clearly have that $\dim(\langle 1 \rangle \perp m \times q) > 2^n$. Moreover, as $\left\lceil \frac{2^n}{\dim q} \right\rceil = \left\lfloor \frac{2^n-1}{\dim q} \right\rfloor + 1$, it follows that $\dim(\langle 1 \rangle \perp (m-1) \times q) \leqslant 2^n$.

Statements *(ii)* and *(iii)* are immediate corollaries of *(i)*. Arguing in an analogous manner to the proof of Proposition 3.4, one can also establish Statement *(iv)* as a corollary of *(i)*. Statement *(v)* follows from invoking Theorem 1.4 *(i)*. □

Remark 4.2. If q is the orthogonal sum of n copies of a Pfister form, for some $n \in \mathbb{N}$, then for all numbers m we have that $\langle 1 \rangle \perp m \times q$ is a neighbour of the Pfister form $2^k \times q$ for some $k \in \mathbb{N}$. Thus, the form $\langle 1 \rangle \perp m \times q$ has maximal splitting for all m in this case, whereby Theorem 4.1 *(iv)* can be applied to give a complete determination of $\{s_q(K) \mid K/F,\, q_K \text{ anisotropic}\}$ (see Remark 5.6). Unlike the situation with respect to the q-sublevel however, this observation does not apply to all forms that are similar to q.

As in the proof of Theorem 3.8, over formally real fields F we can invoke the signatures of q to bound the Witt indices of certain forms containing q. Applying this methodology to considerations of the q-level, we can establish, for all $n \in \mathbb{N}$, the existence of n-dimensional forms q over F that can attain any prescribed number less than their level over F as their level over a suitable extension.

Theorem 4.3. *Let F be a formally real field. Let q be a form over F such that $\operatorname{sgn}_P(q) = -\dim q$ for some $P \in X_F$, the space of orderings of F. Then $\{1, \dots, s_q(F)\} = \{s_q(K) \mid K/F,\ q_K$ anisotropic$\}$.*

Proof. Clearly $\{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\} \subseteq \{1, \dots, s_q(F)\}$, as $s_q(K) \leqslant s_q(F)$.

Let $n \in \mathbb{N}$ be such that $n < s_q(F)$, whereby $\langle 1\rangle \perp n \times q$ is anisotropic over F. Let $K = F(\langle 1\rangle \perp n \times q)$. Since $\langle 1\rangle \perp n \times q$ is indefinite with respect to P, Theorem 1.2 implies that P extends to K. Moreover, as $|\operatorname{sgn}_P(\langle 1\rangle \perp n \times q)| = n \dim q - 1$, we have that $i_1(\langle 1\rangle \perp n \times q) = 1$. Thus, Theorem 1.4 (i) implies that $\langle 1\rangle \perp (n-1) \times q$ is anisotropic over K, whereby $s_q(K) = n$. □

Remark 4.4. One can invoke the above proof to establish that, in general, the q-level does not impose an upper bound on the q-sublevel. Let q be a form over a formally real field F such that $\operatorname{sgn}_P(q) = -\dim q$ for some $P \in X_F$. As in the above proof, for $n \in \mathbb{N}$ such that $n < s_q(F)$, one has that $s_q(K) = n$ for $K = F(\langle 1\rangle \perp n \times q)$. As $\operatorname{sgn}_P(q) = -\dim q$ and P extends to K, it follows that $\underline{s}_q(K) = \infty$. Furthermore, letting $m \in \mathbb{N}$ be such that $1 + (n-1)\dim q \leqslant m \dim q \leqslant 2^r < (m+1)\dim q$ for some $r \in \mathbb{N}$, Theorem 1.3 implies that $s_q(L) = n$ and $\underline{s}_q(L) = m$ for $L = K((m+1) \times q)$.

Remark 4.5. In accordance with [2, Corollary 3.14], for all forms q over F of dimension 1 or 2 (respectively 3) such that $s_q(F) = \infty$, all numbers of the form 2^k (respectively $\frac{2^{2k}+2}{3}$ and $\frac{2^{2k+1}+1}{3}$) belong to $\{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$. We note that these values can be recovered by invoking Theorem 4.1 (i). Indeed, for all forms q over F of dimension $r \in \mathbb{N}$ such that $s_q(F) = \infty$, Theorem 4.1 (i) implies that all numbers $\left\lceil \frac{2^n}{\dim q} \right\rceil$ belong to $\{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$. Conversely, we can establish that the only numbers belonging to $\{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ for all forms q over F of dimension r such that $s_q(F) = \infty$ are those of the form $\left\lceil \frac{2^n}{\dim q} \right\rceil$. This observation follows from invoking Theorem 4.1 (iv) in conjunction with the following example, wherein for all $r, m \in \mathbb{N}$, the existence is established of forms q over F of dimension r such that $s_q(F) = \infty$ and $\langle 1\rangle \perp m \times q$ has maximal splitting.

Example 4.6. Assuming the existence of forms q over F such that $s_q(F) = \infty$, we can conclude that F is a formally real field. Thus, $q \simeq n \times \langle 1\rangle$ is an n-dimensional form over F such that $s_q(F) = \infty$ for all $n \in \mathbb{N}$. Moreover, for all $m \in \mathbb{N}$, the form $\langle 1\rangle \perp m \times q$ is a Pfister neighbour of $2^r \times \langle 1\rangle$, for $2^{r-1} \leqslant mn < 2^r$, whereby it has maximal splitting.

Whereas $\underline{s}_q(F) \leqslant s_q(F) + s_{-q}(F) - 1$ for q an anisotropic form over F, Remark 4.4 demonstrates that finiteness of $s_q(F)$ does not imply that of $\underline{s}_q(F)$. Thus, in general, we cannot hope to argue as in the proof of Theorem 3.12 (respectively, Proposition 3.7) to establish that certain numbers do not (respectively, do) belong to $\{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ (indeed, in accordance with Theorem 4.3, there exist anisotropic forms q that can take any prescribed number as their level over a suitable extension). For those forms q such that finiteness of their level implies finiteness of their sublevel, such as forms q with $1 \in D_F(q)$ for example, one can argue as in the proofs of Proposition 3.7 and Theorem 3.12 to establish analogous results.

5. Forms with equal sublevel and level

In [2, Question 6.1], it was asked whether it is possible to characterise those forms q such that $\underline{s}_q(F) = s_q(F)$. The following two examples serve to demonstrate that this is not possible in general.

Given a field F, we can establish the existence of an extension K/F such that $\underline{s}_q(K) = s_q(K)$ holds for all forms q over F of 2-power dimension:

Example 5.1. Consider a form q over F of dimension 2^k. For $n \in \mathbb{N}$ such that $\underline{s}_q(F) > 2^{n-k}$ and $s_q(F) > 2^{n-k}$, consider the field $K = F((2^{n-k}+1) \times q)$. By Proposition 3.1, we have that $\underline{s}_q(K) = 2^{n-k}$. Moreover, Theorem 1.3 implies that $s_q(K) \geqslant 2^{n-k}$. If $s_q(K) = 2^{n-k}$, then we are done. Otherwise, consider the field $L = K(\langle 1 \rangle \perp 2^{n-k} \times q)$. By Theorem 4.1 (i), we have that $s_q(L) = 2^{n-k}$. Moreover, Theorem 1.3 implies that $2^{n-k} \times q$ is anisotropic over L, whereby $\underline{s}_q(L) = 2^{n-k}$.

Over an ordered field, we can achieve the equality $\underline{s}_q(F) = s_q(F)$ without placing restrictions on the dimension of q:

Example 5.2. Over an ordered field F, let q be a form such that $\mathrm{sgn}_P q = -\dim q$ for some $P \in X_F$ and $s_q(F) > m = \left\lfloor \frac{2^n}{\dim q} \right\rfloor$ for some $n \in \mathbb{N}$. Theorem 1.2 implies that P extends to $K = F\left(\langle 1 \rangle \perp m \times q\right)$, whereby we have that $i_1\left(\langle 1 \rangle \perp m \times q\right) = 1$. Invoking Theorem 1.4, we have that $s_q(K) = m$. As $\mathrm{sgn}_P q = -\dim q$, it follows that $\underline{s}_q(K) = \infty$. Letting $L = K((m+1) \times q)$, Proposition 3.1 implies that $\underline{s}_q(L) = m$. As $\langle 1 \rangle \perp (m-1) \times q$ is anisotropic over L by Theorem 1.3, we furthermore have that $s_q(L) = m$.

In the spirit of [2, Question 6.1], we note that it is possible to characterise those forms q over F such that $\underline{s}_q(K) = s_q(K)$ holds for all K/F where q_K is anisotropic. Clearly, in order for this equality to hold, we require as a prerequisite that the sets $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ and $\{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ are equal, whereby it follows that $\underline{s}_q(F) = s_q(F)$ (these being maximal elements of the sets). We recall that two anisotropic forms p and q over F are isotropy equivalent if for every field extension K/F we have that p_K is isotropic if and only if q_K is isotropic, which is the case if and only if $p_{F(q)}$ and $q_{F(p)}$ are isotropic.

Proposition 5.3. *Let q be a form over F such that $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\} = \{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$. Then $\underline{s}_q(K) = s_q(K)$ holds for all such K/F if and only if the forms $\langle 1 \rangle \perp m \times q$ and $(m+1) \times q$ are isotropy equivalent for all $m \in \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$.*

Proof. For all $m \in \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\} = \{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$, we have that $\underline{s}_q(F((m+1) \times q)) = m = s_q(F(\langle 1 \rangle \perp m \times q))$. Assume that $\underline{s}_q(K) = s_q(K)$ for all such K/F. Thus, we have that $s_q(F((m+1) \times q)) = m = \underline{s}_q(F(\langle 1 \rangle \perp m \times q))$, whereby it follows that $\langle 1 \rangle \perp m \times q$ and $(m+1) \times q$ are isotropy equivalent for all $m \in \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$.

For $m \in \{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$, consider L/F such that $\underline{s}_q(L) = m$. Thus, by assumption, we have that $\langle 1 \rangle \perp m \times q$ is isotropic over L, whereby $s_q(L) \leqslant m$. Suppose, for the sake of contradiction, that $s_q(L) = n < m$. By assumption, we have that $\langle 1 \rangle \perp n \times q$ and $(n+1) \times q$ are isotropy equivalent, whereby it follows that $m \times q$ is isotropic over L, a contradiction. Hence, we may conclude that $s_q(L) = m$, whereby the result follows. $\square$

Corollary 5.4. *Let q be a form over F such that $\{\underline{s}_q(K) \mid K/F,\ q_K \text{ anisotropic}\} = \{s_q(K) \mid K/F,\ q_K \text{ anisotropic}\}$ and $1 \in D_F(q)$. Then $\underline{s}_q(K) = s_q(K)$ holds for* *all such K/F if and only if $i_1((m+1)\times q) = \dim q - 1 + i_1(\langle 1\rangle \perp m \times q)$ for all $m \in \{\underline{s}_q(K) \mid K/F,\ q_K$ anisotropic$\}$.*

Proof. As $1 \in D_F(q)$, we have that $\langle 1\rangle \perp m\times q \subset (m+1)\times q$, whereby $(m+1)\times q$ is isotropic over $F(\langle 1\rangle \perp m\times q)$. Hence, invoking Theorem 1.4 (*ii*), we have that the forms $\langle 1\rangle \perp m\times q$ and $(m+1)\times q$ are isotropy equivalent if and only if $i_1((m+1)\times q) = \dim q - 1 + i_1(\langle 1\rangle \perp m\times q)$, whereby the statement follows as a corollary of Proposition 5.3. □

In [2, Proposition 4.1], it was shown that $\underline{s}_q(F) = s_q(F)$ holds for all anisotropic group forms q over F. In the case where q is round, it is additionally known that $s_q(F)$ is a 2-power if finite (see [2, Proposition 4.3]). More specifically still, for q a Pfister form over F, [2, Theorem 4.4] states the following:

Theorem 5.5. *(Berhuy, Grenier-Boley, Mahmoudi) If q is an anisotropic Pfister form over F, then $\{s_q(K) \mid K/F,\ q_K$ anisotropic$\} = \{1,\dots,2^i,\dots,s_q(F)\}$.*

Remark 5.6. As before, in the case where q is an anisotropic Pfister form over F, the forms $\langle 1\rangle \perp m\times q$ and $(m+1)\times q$ have maximal splitting for all $m\in\mathbb{N}$, whereby Theorem 5.5 can be recovered by invoking Proposition 3.4 or Theorem 4.1 (*iv*) (indeed, in the case of the sublevel, the statement holds for all forms similar to q).

Thus, for q an anisotropic Pfister form over F, it follows that $\underline{s}_q(K) = s_q(K)$ holds for all K/F such that q_K is anisotropic. As demonstrated in [2, Remark 4.5], anisotropic round forms over F need not remain round over extensions of F (indeed, the property of remaining round over all extensions of F characterises the anisotropic Pfister forms over F). Thus, anisotropic round forms q over F need not satisfy $\underline{s}_q(K) = s_q(K)$ for all K/F such that q_K is anisotropic. Indeed, with respect to the round form $q \simeq \langle 1,1,1\rangle$ over $\mathbb{R}$, Theorem 4.1 (*i*) implies that $3 \in \{s_q(K) \mid K/\mathbb{R},\ q_K$ anisotropic$\}$, whereas Theorem 3.12 implies that $3 \notin \{\underline{s}_q(K) \mid K/\mathbb{R},\ q_K$ anisotropic$\}$. This same example also demonstrates that anisotropic group forms need not remain group forms over field extensions, since there exists an extension $K/\mathbb{R}$ such that $s_q(K)=3$, whereby it follows that $s_q(K) \neq \underline{s}_q(K)$ and hence that $q\simeq\langle 1,1,1\rangle$ is not a group form over K. As with round forms, the anisotropic group forms over F that remain group forms over all extensions of F are precisely the anisotropic Pfister forms over F. Given the above, it seems reasonable to ask whether Pfister forms q are the only forms over F satisfying $\underline{s}_q(K) = s_q(K)$ for all K/F such that q_K is anisotropic. The following example demonstrates that this is not the case.

Example 5.7. Let $q\simeq\langle 1,1,1,2\rangle$ over $\mathbb{Q}$. We note that q is not a Pfister form over $\mathbb{Q}$, since $\det q = 2 \notin \mathbb{Q}^2$. As q is positive definite, we have that $\underline{s}_q(\mathbb{Q}) = s_q(\mathbb{Q}) = \infty$. For all $n\in\mathbb{N}$, the Pfister form $2^n\times\langle 1\rangle$ represents 2 over $\mathbb{Q}$, whereby it follows that $2^n\times q \simeq 2^{n+2}\times\langle 1\rangle$. Hence, for all $m\in\mathbb{N}$, the form $(m+1)\times q$, being a Pfister neighbour of $2^n\times q$ for some $n\in\mathbb{N}$, has maximal splitting, whereby Proposition 3.4 implies that $\{\underline{s}_q(K) \mid K/\mathbb{Q},\ q_K$ anisotropic$\} = \{1,\dots,2^i,\dots,\infty\}$. As $1\in D_{\mathbb{Q}}(q)$, for all $m\in\mathbb{N}$ we have that $\langle 1\rangle\perp m\times q$ is a Pfister neighbour of $2^n\times q$ for some $n\in\mathbb{N}$, whereby $\langle 1\rangle\perp m\times q$ has maximal splitting for all $m\in\mathbb{N}$. Hence, Theorem 4.1 (*iv*) implies that $\{s_q(K) \mid K/\mathbb{Q},\ q_K$ anisotropic$\} = \{1,\dots,2^i,\dots,\infty\}$. As $1\in D_{\mathbb{Q}}(q)$, we have that $\langle 1\rangle\perp m\times q \subset (m+1)\times q$, whereby $(m+1)\times q$ is isotropic over $\mathbb{Q}(\langle 1\rangle\perp m\times q)$ for all $m\in\{\underline{s}_q(K) \mid K/\mathbb{Q},\ q_K$ anisotropic$\}$. Moreover, for all $m\in\{\underline{s}_q(K) \mid K/\mathbb{Q},\ q_K$ anisotropic$\}$, we have that $\langle 1\rangle\perp m\times q$ is isotropic over $\mathbb{Q}((m+1)\times q)$ by Lemma 1.1. Hence, Proposition 5.3 implies that $\underline{s}_q(K) = s_q(K)$ for all $K/\mathbb{Q}$ such that q_K is anisotropic.

For q a Pfister form over F, the behaviour of the q-level with respect to quadratic extensions of F was studied in [2], with [2, Lemma 4.8] and [2, Proposition 4.10] being established. These results are stated below, for ease of reference.

Proposition 5.8. *(Berhuy, Grenier-Boley, Mahmoudi) Let q be an anisotropic Pfister form over F and let $K = F(\sqrt{d})$ be a quadratic field extension of F.*

(i) [2, Lemma 4.8] *We have that $\ell_q(-d) \leqslant 2s_q(K)$.*

(ii) [2, Proposition 4.10] *If $\ell_q(-d) = n$, then we have that $s_q(K) = 2^r$ or 2^{r-1} where r is determined by $2^r \leqslant n < 2^{r+1}$.*

In [2, Remark 4.24], it was suggested that the above results are unlikely to hold for round forms, since round forms do not necessarily remain round over extensions. The following result demonstrates that Proposition 5.8 (i) does hold for round forms.

Proposition 5.9. *Let q be an anisotropic round form over F and let $K = F(\sqrt{d})$ be a quadratic field extension of F. Then we have that $s_q(K) \leqslant \ell_q(-d) \leqslant 2s_q(K)$.*

Proof. As q is a round form over F, we have that $1 \in D_F(q)$. Hence, we have that $-d \in D_F(\ell_q(-d) \times q)$, and thus that $-1 \in D_K(\ell_q(-d) \times q)$. Thus, $s_q(K) \leqslant \ell_q(-d)$.

If $s_q(K) = s_q(F)$, then $(s_q(K)+1) \times q$ is isotropic (and hence universal) over F, as $1 \in D_F(q)$. Thus, $\ell_q(-d) \leqslant s_q(K) + 1 \leqslant 2s_q(K)$.

If $s_q(K) < s_q(F)$, then $\langle 1 \rangle \perp s_q(K) \times q$ is anisotropic over F and isotropic over K. Thus, there exists $a \in F^\times$ such that $a\langle 1, -d \rangle \subset \langle 1 \rangle \perp s_q(K) \times q$ by [10, Theorem VII.3.1]. Letting $s_q(K) = 2^n + k$ for $n, k \in \mathbb{N}_0$ such that $0 \leqslant k < 2^n$, we have that $a\langle 1, -d \rangle \subset 2^{n+1} \times q$ since $1 \in D_F(q)$. By Witt's Round Form Theorem, $2^{n+1} \times q$ is a round form. Hence, since $a \in D_F(2^{n+1} \times q)$, we can conclude that $\langle 1, -d \rangle \subset a(2^{n+1} \times q) \simeq 2^{n+1} \times q$. Thus, $\ell_q(-d) \leqslant 2^{n+1} \leqslant 2s_q(K)$. □

As a consequence of the above, we can establish that a weakened version of Proposition 5.8 (ii) holds for round forms.

Proposition 5.10. *Let q be an anisotropic round form over F and let $K = F(\sqrt{d})$ be a quadratic field extension of F. If $\ell_q(-d) = n$, then we have that $2^{r-1} \leqslant s_q(K) < 2^{r+1}$ where r is determined by $2^r \leqslant n < 2^{r+1}$.*

Proof. This follows as an immediate corollary of Proposition 5.9. □

In light of the above, it is justified to seek to distinguish between the behaviour of the level of round and Pfister forms with respect to quadratic extensions. A fundamental distinction in this regard is that for q a Pfister form over F, we have that $s_q(F) \times q$ is a Pfister form, whereby $s_q(F) \times q$ becomes hyperbolic over those extensions K/F such that $s_q(K) < s_q(F)$. Bearing this distinction in mind, for q a Pfister form over F, we will henceforth consider the behaviour of the q-level with respect to arbitrary function fields of quadratic forms, as opposed to the special case of quadratic extensions. We begin by highlighting a sufficient condition for the level of a Pfister form to remain unchanged with respect to such extensions.

Proposition 5.11. *Let q be an anisotropic Pfister form over F and φ an anisotropic form over F. If $\dim \varphi > (s_q(F)) \dim q$, then $s_q(F(\varphi)) = s_q(F)$.*

Proof. For $\dim \varphi > (s_q(F)) \dim q$, we necessarily have that $s_q(F)$ is finite, whereby Theorem 5.5 implies that $s_q(F) = 2^k$ for some $k \in \mathbb{N}_0$. Hence, $s_q(F) \times q$ is an anisotropic Pfister form over F. Suppose that $s_q(F(\varphi)) < s_q(F)$, whereby $s_q(F) \times q$ becomes hyperbolic over $F(\varphi)$. Invoking [10, Theorem X.4.5], for every $a \in D_F(\varphi)$ we have that $a\varphi \subset s_q(F) \times q$, whereby it follows that $\dim \varphi \leqslant (s_q(F)) \dim q$. Thus, our statement follows by contraposition. □

Given the preceding result, for q a Pfister form over F, it is justified to restrict our study of the behaviour of the q-level with respect to those function field extensions $F(\varphi)$ where φ is an anisotropic form over F such that $\dim\varphi \leqslant (s_q(F))\dim q$. Our concluding results establish that Proposition 5.8 (*i*) and a strengthened version of Proposition 5.8 (*ii*) hold in this more general setting.

In analogy with the q-length of $a \in F^\times$, for q and φ forms over F, we define the *q-length of φ* to be $\ell_q(\varphi) = \min\{n \in \mathbb{N} \mid \varphi \subset n \times q\}$ if such numbers n exist, and set it to be infinite otherwise.

Proposition 5.12. *Let q be an anisotropic Pfister form over F and φ an anisotropic form over F such that* $\dim\varphi \leqslant (s_q(F))\dim q$. *Then* $s_q(F(\varphi)) \leqslant \ell_q(a\varphi) \leqslant 2s_q(F(\varphi))$ *for every* $a \in D_F(\varphi)$.

Proof. Since $a\varphi \subset \ell_q(a\varphi) \times q$, the form $\ell_q(a\varphi) \times q$ is isotropic over $F(\varphi)$, whereby $s_q(F(\varphi)) \leqslant \ell_q(a\varphi)$. To establish the remaining inequality, we may assume that $s_q(F(\varphi)) < \infty$.

If $s_q(F(\varphi)) = s_q(F)$, then the Pfister form $2s_q(F(\varphi)) \times q$ is hyperbolic over F. Thus, for $a \in D_F(\varphi)$, we have that $a\varphi \subset a\varphi \perp -a\varphi \subset 2s_q(F(\varphi)) \times q$ since $\dim\varphi \leqslant (s_q(F))\dim q$, establishing the result in this case.

If $s_q(F(\varphi)) < s_q(F)$, then we have that $2s_q(F(\varphi)) \leqslant s_q(F)$ by Theorem 5.5. As $2s_q(F(\varphi)) \times q$ is a Pfister form, it follows that it is anisotropic over F, as otherwise Lemma 1.1 would imply that $\langle 1\rangle \perp s_q(F(\varphi)) \times q$ is isotropic over F, a contradiction in this case. Since $\langle 1\rangle \perp s_q(F(\varphi)) \times q$ is isotropic over $F(\varphi)$, it follows that $2s_q(F(\varphi)) \times q$ becomes hyperbolic over $F(\varphi)$. Thus, invoking [10, Theorem X.4.5], we have that $a\varphi \subset 2s_q(F(\varphi)) \times q$ for every $a \in D_F(\varphi)$, establishing the result. □

The following example shows that the above bounds can be attained.

Example 5.13. If $a\varphi \subset q$, then clearly $s_q(F(\varphi)) = \ell_q(a\varphi) = 1$. Next, let F be a field of q-level at least two. If $a\varphi \simeq 2 \times q$, then $\ell_q(a\varphi) = 2$. Moreover, as the Pfister form $2 \times q$ is hyperbolic over $F(\varphi)$ in this case, we have that $\langle 1\rangle \perp q$ is isotropic over $F(\varphi)$ by Lemma 1.1. Thus, we have that $\ell_q(a\varphi) = 2s_q(F(\varphi))$ for $a\varphi \simeq 2 \times q$.

Proposition 5.14. *Let q be an anisotropic Pfister form over F and φ an anisotropic form over F such that* $\dim\varphi \leqslant (s_q(F))\dim q$. *If* $\ell_q(a\varphi) = n$ *for some* $a \in D_F(\varphi)$, *then we have that* $s_q(F(\varphi)) = 2^r$ *where r is determined by* $2^r < n \leqslant 2^{r+1}$.

Proof. We will first prove that $s_q(F(\varphi)) \leqslant 2^r$. If $s_q(F) \leqslant 2^r$, then this is clear. Hence, we may assume that $s_q(F) \geqslant 2^{r+1}$. In this case, the Pfister form $2^{r+1} \times q$ is anisotropic over F, as otherwise Lemma 1.1 would imply that $\langle 1\rangle \perp 2^r \times q$ is isotropic over F, a contradiction. Since $a\varphi \subset n \times q$ for some $a \in D_F(\varphi)$, we have that $2^{r+1} \times q$ is hyperbolic over $F(\varphi)$, and thus that $\langle 1\rangle \perp 2^r \times q$ is isotropic over $F(\varphi)$ by Lemma 1.1. Hence, we have that $s_q(F(\varphi)) \leqslant 2^r$.

Proposition 5.12 implies that $n \leqslant 2s_q(F(\varphi))$. As $s_q(F(\varphi))$ is necessarily a 2-power by Theorem 5.5, we can conclude that $2^{r+1} \leqslant 2s_q(F(\varphi))$. Hence, $s_q(F(\varphi)) = 2^r$. □

Acknowledgements. I gratefully acknowledge the support I received through an International Mobility Fellowship from the Irish Research Council, co-funded by Marie Curie Actions under FP7. I would like to thank Thomas Unger, Karim Becher, Ahmed Laghribi and Jean-Pierre Tignol for their helpful comments.

References

[1] K. J. Becher, Minimal weakly isotropic forms, *Mathematische Zeitschrift* **252**, no. 1, 91 − 102 (2006).

[2] G. Berhuy, N. Grenier-Boley, M. G. Mahmoudi, Sums of values represented by a quadratic form, *Manuscripta Mathematica* **140**, no. 3-4, 531 − 556 (2013).

[3] R. Elman, N. A. Karpenko, A. S. Merkurjev, The algebraic and geometric theory of quadratic forms, American Mathematical Society Colloquium Publications **56**, *American Mathematical Society* (2008).

[4] R. Elman, T. Y. Lam, A. R. Wadsworth, Orderings under field extensions, *Journal für die reine und angewandte Mathematik* **306**, 7 − 27 (1979).

[5] E. R. Gentile, D. B. Shapiro, Conservative quadratic forms, *Mathematische Zeitschrift* **163**, 15 − 23 (1978).

[6] D. W. Hoffmann, Isotropy of quadratic forms over the function field of a quadric, *Mathematische Zeitschrift* **220**, no. 3, 461 − 476 (1995).

[7] N. A. Karpenko, On the first Witt index of quadratic forms, *Inventiones mathematicae* **153**, 455 − 462 (2003).

[8] N. A. Karpenko, A. S. Merkurjev, Essential dimension of quadrics, *Inventiones mathematicae* **153**, 361 − 372 (2003).

[9] M. Knebusch, Generic Splitting of Quadratic Forms, I, *Proceedings of the London Mathematical Society* (3) **33**, 65 − 93 (1976).

[10] T. Y. Lam, Introduction to Quadratic Forms over Fields, *American Mathematical Society* (2005).

[11] J. O'Shea, Products of Pfister forms, in preparation.

[12] A. Pfister, Quadratic forms with applications to algebraic geometry and topology, *London Math. Soc. Lect. Notes* **217**, Cambridge University Press (1995).

[13] A. R. Wadsworth, D. B. Shapiro, On multiples of round and Pfister forms, *Mathematische Zeitschrift* **157**, 53 − 62 (1977).