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# GENERALIZED FISHBURN NUMBERS AND TORUS KNOTS

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ABSTRACT. Andrews and Sellers recently initiated the study of arithmetic properties of Fishburn numbers. In this paper we prove prime power congruences for generalized Fishburn numbers. These numbers are the coefficients in the  $1 - q$  expansion of the Kontsevich-Zagier series  $\mathcal{F}_t(q)$  for the torus knots  $T(3, 2^t)$ ,  $t \geq 2$ . The proof uses a strong divisibility result of Ahlgren, Kim and Lovejoy and a new “strange identity” for  $\mathcal{F}_t(q)$ .

## 1. INTRODUCTION

The Fishburn numbers  $\xi(n)$  are the coefficients in the formal power series expansion

$$F(1 - q) =: \sum_{n \geq 0} \xi(n)q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \dots \quad (1.1)$$

where  $F(q) := \sum_{n \geq 0} (q)_n$  is the Kontsevich-Zagier “strange” series [31] and

$$(a_1, a_2, \dots, a_j)_n = (a_1, a_2, \dots, a_j; q)_n := \prod_{k=1}^n (1 - a_1 q^{k-1})(1 - a_2 q^{k-1}) \dots (1 - a_j q^{k-1})$$

is the standard  $q$ -hypergeometric notation, valid for  $n \in \mathbb{N} \cup \{\infty\}$ . Here, the moniker “strange” is used as  $F(q)$  does not converge on any open subset of  $\mathbb{C}$ , but is well-defined when  $q$  is a root of unity (where it is finite) and when  $q$  is replaced by  $1 - q$  as in (1.1). The Fishburn numbers are of interest for their numerous combinatorial variants (see A022493 in [26]), asymptotics [27, 31] and arithmetic properties [1, 3, 7, 10, 28]. In their marvelous paper, Andrews and Sellers [3] proved congruences for  $\xi(n)$  modulo primes which were then extended to prime powers [1, 28]. For example, we have

$$\xi(5^r m - 1) \equiv \xi(5^r m - 2) \equiv 0 \pmod{5^r}, \quad (1.2)$$

$$\xi(7^r m - 1) \equiv 0 \pmod{7^r} \quad (1.3)$$

and

$$\xi(11^r m - 1) \equiv \xi(11^r m - 2) \equiv \xi(11^r m - 3) \equiv 0 \pmod{11^r} \quad (1.4)$$

for all natural numbers  $r$  and  $m$ . Our interest in this paper lies in the knot theoretic interpretation of  $F(q)$  as it leads to a natural generalization of the coefficients  $\xi(n)$ .

Let  $K$  be a knot and  $J_N(K; q)$  be the usual colored Jones polynomial, normalized to be 1 for the unknot. As a knot invariant, the colored Jones polynomial plays the lead role in many open

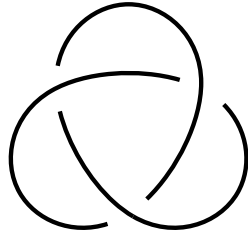
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problems in quantum topology. The sequence  $\{J_N(K; q)\}_{N \in \mathbb{N}}$  appears to encode many subtle geometric and topological properties of the knot  $K$  at a remarkably deep level. For example, the Volume Conjecture [17, 23, 24] relates the value at  $\zeta_N = e^{2\pi i/N}$  of the  $N$ th colored Jones polynomial (or, equivalently, the  $N$ th Kashaev invariant) of a knot to its hyperbolic volume; the Strong Slope Conjecture [6, 16] posits that the maximal and minimal degrees of  $J_N(K; q)$  in  $q$  contain information about essential surfaces in knot exteriors; the AJ Conjecture [5] connects the recurrence relation for  $J_N(K; q)$  to the  $A$ -polynomial of the knot [4], a plane curve describing the  $SL(2, \mathbb{C})$  character variety of the knot complement. Explicit formulas for  $J_N(K; q)$  in terms of  $q$ -hypergeometric series are also of importance and have been proven for various families of knots [11, 13, 14, 19, 20, 21, 22, 29, 30]. For example, if  $T(3, 2) = T(2, 3)$  is the right-handed trefoil knot as in Figure 1, then [11, 19]

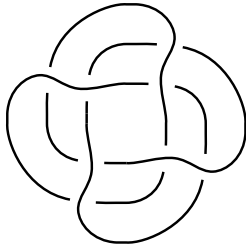
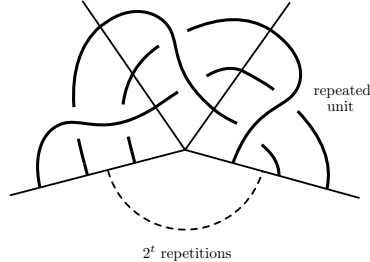
$$J_N(T(3, 2); q) = q^{1-N} \sum_{n \geq 0} q^{-nN} (q^{1-N})_n. \quad (1.5)$$

FIGURE 1.  $T(3, 2)$ 

Observe that the Kontsevich-Zagier series  $F(q)$  matches the colored Jones polynomial for  $T(3, 2)$  at roots of unity, that is, for  $q = \zeta_N$  we have

$$\zeta_N F(\zeta_N) = J_N(T(3, 2); \zeta_N). \quad (1.6)$$

Consider the family of torus knots  $T(3, 2^t)$  for  $t \geq 2$  as in Figures 2 and 3.

FIGURE 2.  $T(3, 4)$ FIGURE 3.  $T(3, 2^t)$ 

In this case, a  $q$ -hypergeometric expression for the colored Jones polynomial has been computed, namely (see page 41, Théorème 3.2 in [18])

$$\begin{aligned}
J_N(T(3, 2^t); q) &= (-1)^{h''(t)} q^{2^t-1-h'(t)-N} \sum_{n \geq 0} (q^{1-N})_n q^{-Nnm(t)} \\
&\times \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ j_\ell \equiv 1 \pmod{m(t)}}} (-q^{-N})^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\
&\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}
\end{aligned} \tag{1.7}$$

where

$$h''(t) = \begin{cases} \frac{2^t-1}{3} & \text{if } t \text{ is even,} \\ \frac{2^t-2}{3} & \text{if } t \text{ is odd,} \end{cases} \quad h'(t) = \begin{cases} \frac{2^t-4}{3} & \text{if } t \text{ is even,} \\ \frac{2^t-5}{3} & \text{if } t \text{ is odd,} \end{cases} \quad a(t) = \begin{cases} \frac{2^{t-1}+1}{3} & \text{if } t \text{ is even,} \\ \frac{2^t+1}{3} & \text{if } t \text{ is odd,} \end{cases}$$

$m(t) = 2^{t-1}$ ,  $I(*)$  is the characteristic function and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the  $q$ -binomial coefficient. We note that the  $t = 2$  case of (1.7) recovers equation (16) in [13]. We now define the *Kontsevich-Zagier series for torus knots*  $T(3, 2^t)$  as<sup>1</sup>

$$\begin{aligned}
\mathcal{F}_t(q) &= (-1)^{h''(t)} q^{-h'(t)} \sum_{n \geq 0} (q)_n \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ j_\ell \equiv 1 \pmod{m(t)}}} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\
&\times \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}
\end{aligned} \tag{1.8}$$

and notice that, similar to (1.6), we have

$$\zeta_N^{2^t-1} \mathcal{F}_t(\zeta_N) = J_N(T(3, 2^t); \zeta_N).$$

As with the original Kontsevich-Zagier series,  $\mathcal{F}_t(q)$  does not converge on any open subset of  $\mathbb{C}$ , but is well-defined when  $q$  is a root of unity (where it is finite) and when  $q$  is replaced by  $1 - q$ . Thus, we may write

$$\mathcal{F}_t(1 - q) =: \sum_{n \geq 0} \xi_t(n) q^n.$$

For example,

$$\mathcal{F}_2(1 - q) = 1 + 3q + 11q^2 + 50q^3 + 280q^4 + 1890q^5 + \dots$$

and

$$\mathcal{F}_3(1 - q) = 1 + 7q + 49q^2 + 420q^3 + 4515q^4 + 59367q^5 + \dots$$

<sup>1</sup>For  $t = 1$ , one may define the sum over the  $j_\ell$  to be 1 in (1.7) and (1.8) to recover (1.5) and  $F(q)$ .

The purpose of this paper is to illustrate how a recent result of Ahlgren, Kim and Lovejoy [2] combined with a new “strange identity” for  $\mathcal{F}_t(q)$  allow one to prove prime power congruences akin to (1.2)–(1.4) for the *generalized Fishburn numbers*  $\xi_t(n)$ . For natural numbers  $s$  and  $t \geq 2$  and the periodic function

$$\chi_t(n) = \chi_{3 \cdot 2^{t+1}}(n) := \begin{cases} 1 & \text{if } n \equiv 2^{t+1} - 3, 3 + 2^{t+2} \pmod{3 \cdot 2^{t+1}}, \\ -1 & \text{if } n \equiv 2^{t+1} + 3, 2^{t+2} - 3 \pmod{3 \cdot 2^{t+1}}, \\ 0 & \text{otherwise,} \end{cases} \quad (1.9)$$

we define the set

$$S_{t,\chi_t}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}} \pmod{s} \text{ where } \chi_t(n) \neq 0 \right\}.$$

Our main result is now the following.

**Theorem 1.1.** *If  $p \geq 5$  is a prime and  $j \in \{1, 2, \dots, p-1 - \max S_{t,\chi_t}(p)\}$ , then*

$$\xi_t(p^r m - j) \equiv 0 \pmod{p^r}$$

for all natural numbers  $r$ ,  $m$  and  $t \geq 2$ .

One can check that  $S_{2,\chi_2}(5) = \{0, 2, 3\}$ ,  $S_{2,\chi_2}(17) = \{0, 2, 3, 4, 7, 8, 9, 11, 14\}$ ,  $S_{3,\chi_3}(7) = \{0, 2, 3, 4\}$  and  $S_{3,\chi_3}(13) = \{0, 2, 5, 6, 7, 8, 11\}$ . Thus, by Theorem 1.1, we have

$$\begin{aligned} \xi_2(5^r m - 1) &\equiv 0 \pmod{5^r}, \\ \xi_2(17^r m - 1) &\equiv \xi_2(17^r m - 2) \equiv 0 \pmod{17^r}, \\ \xi_3(7^r m - 1) &\equiv \xi_3(7^r m - 2) \equiv 0 \pmod{7^r} \end{aligned}$$

and

$$\xi_3(13^r m - 1) \equiv 0 \pmod{13^r}$$

for all natural numbers  $r$  and  $m$ .

The paper is organized as follows. In Section 2, we recall the main result from [2] and then record some preliminaries, including the new “strange identity” (see Theorem 2.4). Theorem 2.4 is of independent interest for at least three reasons. First, there seem to be few such explicit instances in which the underlying identity has been proven (for example, see equation (70) in [14] and Theorem 2 in [31]). Second, this result implies (via Theorem 2.1) a strong divisibility property for the coefficients of the partial sums of the dissection of  $\mathcal{F}_t(q)$ . This is used in turn to prove Theorem 1.1. Third, it is a key component in determining the quantum modularity for  $\mathcal{F}_t(q)$  [9]. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we discuss some possibilities for future work.

## 2. PRELIMINARIES

Our first step is to recall the setup from [2]. Let  $\mathcal{F}$  be a function of the form

$$\mathcal{F}(q) = \sum_{n \geq 0} (q)_n f_n(q) \quad (2.1)$$

where  $f_n(q) \in \mathbb{Z}[q]$ . For positive integers  $s$  and  $N$ , consider the partial sum

$$\mathcal{F}(q; N) := \sum_{n=0}^N (q)_n f_n(q)$$

and its  $s$ -dissection

$$\mathcal{F}(q; N) = \sum_{i=0}^{s-1} q^i A_{\mathcal{F},s}(N, i, q^s)$$

where  $A_{\mathcal{F},s}(N, i, q) \in \mathbb{Z}[q]$ . Now, consider the partial theta functions

$$P_{a,b,\chi}^{(\nu)}(q) := \sum_{n \geq 0} n^\nu \chi(n) q^{\frac{n^2-a}{b}} \quad (2.2)$$

where  $\nu \in \{0, 1\}$ ,  $a \geq 0$ ,  $b > 0$  are integers and  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  is a function satisfying the following two conditions:

$$\chi(n) \neq 0 \text{ only if } \frac{n^2 - a}{b} \in \mathbb{Z} \quad (2.3)$$

and for each root of unity  $\zeta$ ,

$$\text{the function } n \rightarrow \zeta^{\frac{n^2-a}{b}} \chi(n) \text{ is periodic and has mean value zero.} \quad (2.4)$$

Finally, define the set  $S_{a,b,\chi}(s)$  by

$$S_{a,b,\chi}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - a}{b} \pmod{s} \text{ where } \chi(n) \neq 0 \right\}.$$

We can now state the main result in [2].

**Theorem 2.1.** *Suppose that  $\mathcal{F}$  and  $P_{a,b,\chi}^{(\nu)}$  are functions as in (2.1) and (2.2) and for each root of unity  $\zeta$ , we have the asymptotic expansion*

$$\mathcal{F}(\zeta e^{-t}) \sim P_{a,b,\chi}^{(\nu)}(\zeta e^{-t})$$

as  $t \rightarrow 0^+$ . Suppose that  $s$  and  $N$  are positive integers and  $i \notin S_{a,b,\chi}(s)$ . Then

$$(q)_{\lambda(N,s)} \mid A_{\mathcal{F},s}(N, i, q)$$

where  $\lambda(N, s) := \lfloor \frac{N+1}{s} \rfloor$ .

We now turn to our situation and record the following result. Throughout, we assume that  $t \geq 2$ .

**Proposition 2.2.** *The periodic function  $\chi_t$  (as defined by (1.9)) satisfies (2.3) and (2.4).*

*Proof.* A straightforward calculation using (1.9) confirms (2.3) with  $a = (2^{t+1} - 3)^2$  and  $b = 3 \cdot 2^{t+2}$ . Suppose  $\zeta$  is a root of unity of order  $M$  and define

$$\psi_t(n) = \zeta^{\frac{n^2 - (2^{t+1} - 3)^2}{3 \cdot 2^{t+2}}} \chi_t.$$

Note that  $\psi_t$  has period  $M(3 \cdot 2^{t+1})$ . We now claim that

$$\sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) = 0. \quad (2.5)$$

Suppose  $M$  is even. By (1.9), we obtain

$$\chi_t(n + 3 \cdot 2^t M) = \chi_t(n) \quad (2.6)$$

for all  $n$ . We also have

$$\zeta^{\frac{(n+3 \cdot 2^t M)^2 - (2^{t+1}-3)^2}{3 \cdot 2^{t+2}}} = \zeta^{\frac{M(n+3 \cdot 2^{t-1} M^2)}{2}} \zeta^{\frac{n^2 - (2^{t+1}-3)^2}{3 \cdot 2^{t+2}}}. \quad (2.7)$$

As  $\psi_t$  is supported on odd integers, we can assume that  $n$  is odd. Then  $M(n + 3 \cdot 2^{t-1} M^2)$  is an odd multiple of  $M$  and so (2.7) equals  $-\zeta^{\frac{n^2 - (2^{t+1}-3)^2}{3 \cdot 2^{t+2}}}$ . This fact and (2.6) imply

$$\sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) = \sum_{n=1}^{M(3 \cdot 2^t)} (\psi_t(n) + \psi_t(n + 3 \cdot 2^t M)) = 0.$$

Now, suppose  $M$  is odd. We break up (2.5) into four sums, one for each congruence class modulo  $3 \cdot 2^{t+1}$  in (1.9). Specifically,

$$\begin{aligned} \sum_{n=1}^{M(3 \cdot 2^{t+1})} \psi_t(n) &= \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+2}+3)m + 2^t + 3} - \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+2}-3)m + 2^t - 1} \\ &\quad + \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+1}-3)m} - \sum_{m=0}^{M-1} \zeta^{3 \cdot 2^t m^2 + (2^{t+1}+3)m + 2}. \end{aligned} \quad (2.8)$$

Let  $i_{t,M}$  be the unique solution to  $2^{t i} \equiv 1 \pmod{M}$ . Performing the shift  $m \rightarrow m + i_{t,M}$  (respectively,  $m \rightarrow m - i_{t,M}$ ) to the second (respectively, fourth) sum in (2.8) followed by a routine simplification implies (2.5). The result now follows.  $\square$

For  $\chi_t$  as in (1.9), consider the series

$$H_t(x) = H_t(x, q) = \sum_{n \geq 0} \chi_t(n) q^{\frac{n^2 - (2^{t+1}-3)^2}{3 \cdot 2^{t+2}}} x^{\frac{n - (2^{t+1}-3)}{2}}.$$

**Proposition 2.3.** *Let  $h(t) = 2^t - 2$ . We have*

$$\begin{aligned} H_t(x) &= (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} \sum_{n \geq 0} (x)_{n+1} x^{nm(t)} \\ &\quad \times \sum_{\substack{3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \\ \pmod{m(t)}}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\ &\quad \times \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[ \begin{matrix} n + I(\ell \leq k) \\ j_\ell \end{matrix} \right]. \end{aligned} \quad (2.9)$$

*Proof.* Let  $f_t(x)$  denote the right-hand side of (2.9). Then  $f_t(x)$  satisfies the difference equation (see (3.3.11) in [18])

$$f_t(x) = 1 - q^2 x^3 - q^{2t-1} x^{2t} + q^{3+2t} x^{3+2t} + q^{5 \cdot 2^t - 3} x^{3 \cdot 2^t} f_t(q^2 x). \quad (2.10)$$

One can directly verify that  $H_t(x)$  also satisfies (2.10) using (1.9). The result now easily follows.  $\square$

Recall that the Kontsevich-Zagier series  $F(q)$  satisfies the “strange identity”

$$F(q) \text{ “ = ” } - \frac{1}{2} \sum_{n \geq 1} n \left( \frac{12}{n} \right) q^{\frac{n^2-1}{24}} \quad (2.11)$$

where “ = ” means that the two sides agree to all orders at every root of unity (for further details, see Sections 2 and 5 in [31]) and  $\left( \frac{12}{*} \right)$  is the quadratic character of conductor 12. We now prove the following new strange identity for the Kontsevich-Zagier series for torus knots  $T(3, 2^t)^2$ .

**Theorem 2.4.** *For each root of unity  $\zeta$ , we have the asymptotic expansion*

$$\mathfrak{F}_t(\zeta e^{-u}) \sim -\frac{1}{2} P_{(2^{t+1}-3)^2, 3 \cdot 2^{t+2}, \chi_t}^{(1)}(\zeta e^{-u}) \quad (2.12)$$

as  $u \rightarrow 0^+$ .

*Proof of Theorem 2.4.* For ease of notation, let  $'$  denote the condition

$$3 \sum_{\ell=1}^{m(t)-1} j_\ell \equiv 1 \pmod{m(t)}$$

occurring in the second sum in (1.7) on the  $j_\ell$ ,

$$v = v(j_1, \dots, j_{m(t)-1}) := \frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}$$

and  $\bar{c}$  the reduction of an integer  $c$  modulo  $m(t)$ . The following identity<sup>3</sup> implies (2.12) upon setting  $q = \zeta e^{-u}$ , then letting  $u \rightarrow 0^+$ :

$$\begin{aligned} & \frac{1}{2} \sum_{n \geq 0} n \chi_t(n) q^{\frac{n^2 - (2^{t+1}-3)^2}{3 \cdot 2^{t+2}}} - \frac{2^{t+1} - 3}{2} (q^{2^t-1}, q^{2^t+1}, q^{2^{t+1}}; q^{2^{t+1}})_\infty (q^2, q^{2^{t+2}-2}; q^{2^{t+2}})_\infty \\ &= (-1)^{h''(t)+1} q^{-h'(t)} \sum_{n \geq 0} [(q)_n - (q)_\infty] \\ & \quad \times \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \left[ \begin{matrix} n + I(\ell \leq k) \\ j_\ell \end{matrix} \right] \end{aligned}$$

<sup>2</sup>Using the convention in footnote 1, we can take  $t = 1$  in (1.9) and (2.12) to recover (2.11).

<sup>3</sup>The  $t = 2$  case of (2.13) gives an alternative (and corrected) version of Proposition 5 in [15].

$$+ (-1)^{h''(t)+1} q^{-h'(t)}(q)_\infty \left( \sum_{i=1}^{\infty} \frac{q^i}{1-q^i} \right) \sum_{n \geq 0} b_{n,t}(q) + (-1)^{h''(t)} q^{-h'(t)}(q)_\infty \sum_{n \geq 0} (n-h(t)) b_{n,t}(q) \quad (2.13)$$

where

$$b_{n,t}(q) = a_{n,t}(q) - a_{n-1,t}(q) \quad (2.14)$$

and

$$a_{n,t}(q) = \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \prod_{\ell=1}^{m(t)-1} \left[ \frac{n - \sum_{\ell=1}^{m(t)-1} j_\ell - \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}}{m(t)} + I(\ell \leq \overline{(n - \sum_{\ell=1}^{m(t)-1} j_\ell)}) \right]. \quad (2.15)$$

To prove (2.13), we begin by rewriting (2.9) as

$$\begin{aligned} H_t(x) &= (-1)^{h''(t)} q^{-h'(t)} (1-x) \sum_{n \geq 0} \left[ (qx)_n - (qx)_\infty \right] x^{nm(t)} \\ &\quad \times \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{-h(t) + \sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[ n + I(\ell \leq k) \right] \\ &\quad + (qx)_\infty (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} (1-x) M_t(x, q) \end{aligned} \quad (2.16)$$

where

$$M_t(x, q) := \sum_{n \geq 0} x^{nm(t)} \sum'_{j_1, \dots, j_{m(t)-1}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \left[ n + I(\ell \leq k) \right]. \quad (2.17)$$

We now claim that

$$(1-x) M_t(x, q) = \sum_{n \geq 0} b_{n,t}(q) x^n \quad (2.18)$$

where  $b_{n,t}(q)$  is given by (2.14) and (2.15). To see (2.18), we first write

$$\sum_{\ell=1}^{m(t)-1} j_\ell = m(t) \left[ \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right] + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}. \quad (2.19)$$

We now have that  $M_t(x, q)$  equals

$$\begin{aligned} &\sum'_{j_1, \dots, j_{m(t)-1}} \sum_{k=0}^{m(t)-1} \sum_{n \geq 0} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{m(t) \left( n + \left[ \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right] \right) + k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell}} q^v \prod_{\ell=1}^{m(t)-1} \left[ n + I(\ell \leq k) \right] \\ &\quad \text{(using (2.17) and (2.19))} \end{aligned}$$

$$\begin{aligned}
&= \sum'_{j_1, \dots, j_{m(t)-1}} \sum_{k=0}^{m(t)-1} \sum_{n \geq 0} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^{m(t)n+k+\overline{\sum_{\ell=1}^{m(t)-1} j_\ell}} q^v \prod_{\ell=1}^{m(t)-1} \left[ n - \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor + I(\ell \leq k) \right] \\
&\quad \left( \text{letting } n \rightarrow n - \left\lfloor \frac{\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \right\rfloor \right) \\
&= \sum_{k=0}^{m(t)-1} \sum'_{j_1, \dots, j_{m(t)-1}} \sum_{n \equiv k + \overline{\sum_{\ell=1}^{m(t)-1} j_\ell} \pmod{m(t)}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} x^n q^v \prod_{\ell=1}^{m(t)-1} \left[ \frac{n-k-\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} + I(\ell \leq k) \right] \\
&\quad \left( \text{letting } n \rightarrow \frac{n-k-\sum_{\ell=1}^{m(t)-1} j_\ell}{m(t)} \text{ and using (2.19)} \right) \\
&= \sum_{n \geq 0} \sum'_{j_1, \dots, j_{m(t)-1}} x^n (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \prod_{\ell=1}^{m(t)-1} \left[ \frac{n-\sum_{\ell=1}^{m(t)-1} j_\ell - \overline{(n-\sum_{\ell=1}^{m(t)-1} j_\ell)}}{m(t)} + I(\ell \leq \overline{n-\sum_{\ell=1}^{m(t)-1} j_\ell}) \right] \\
&\quad \left( \text{combining the outer and innermost sum and replacing } k \text{ with } n - \sum_{\ell=1}^{m(t)-1} j_\ell \right) \\
&= \sum_{n \geq 0} a_{n,t}(q) x^n.
\end{aligned}$$

Thus, we obtain (2.18) via (2.14). We now substitute (2.18) into (2.16), then take the derivative of both sides of (2.16) with respect to  $x$  and set  $x = 1$ . This yields the right-hand side of (2.13). To obtain the left-hand side of (2.13), we additionally break up the resulting second sum into four sums, one for each congruence class modulo  $3 \cdot 2^{t+1}$  in (1.9). Namely,

$$\begin{aligned}
\sum_{n \geq 0} \chi_t(n) q^{\frac{n^2 - (2^{t+1}-3)n}{3 \cdot 2^{t+2}}} &= \sum_{k \geq 0} \left( q^{3 \cdot 2^t k^2 + (2^{t+1}-3)k} - q^{3 \cdot 2^t k^2 + (2^{t+2}-3)k + 2^t - 1} \right) \\
&\quad + \sum_{k \geq 0} \left( q^{3 \cdot 2^t k^2 + (2^{t+2}+3)k + 2^t + 3} - q^{3 \cdot 2^t k^2 + (2^{t+1}+3)k + 2} \right). \tag{2.20}
\end{aligned}$$

Let  $k \rightarrow k - 1$ , then replace  $k$  with  $-k$  in the third and fourth sums of (2.20) and combine with the first and second sums in (2.20) to obtain

$$\sum_{k \in \mathbb{Z}} \left( q^{3 \cdot 2^t k^2 + (2^{t+1}-3)k} - q^{3 \cdot 2^t k^2 + (2^{t+2}-3)k + 2^t - 1} \right). \tag{2.21}$$

We now apply Watson's quintuple product identity

$$\sum_{k \in \mathbb{Z}} q^{\frac{k(3k-1)}{2}} x^{3k} (1 - xq^k) = (q, x, qx^{-1})_\infty (qx^2, qx^{-2}; q^2)_\infty$$

with  $q \rightarrow q^{2^{t+1}}$  and  $x = q^{2^t - 1}$  to (2.21). This proves the result.  $\square$

**Remark 2.5.** If we take  $x = 1$  in (2.16), apply Watson's quintuple product identity as above and use the fact that the sum in (2.18) telescopes, then we have

$$\begin{aligned} (q)_\infty (-1)^{h''(t)} q^{-h'(t)} \sum'_{j_1, \dots, j_{m(t)-1}} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} \frac{q^v}{(q)_{j_1} \cdots (q)_{j_{m(t)-1}}} \\ = (q^{2^t-1}, q^{2^t+1}, q^{2^{t+1}}, q^{2^{t+1}}; q^{2^{t+1}})_\infty (q^2, q^{2^{t+2}-2}, q^{2^{t+2}})_\infty. \end{aligned} \quad (2.22)$$

This recovers Slater's identity (see (86) in [25])

$$(q)_\infty \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(q)_{2n+1}} = (q^3, q^5, q^8; q^8)_\infty (q^2, q^{14}; q^{16})_\infty$$

upon taking  $t = 2$  in (2.22) and simplifying. It would be interesting to give an alternative proof of (2.22), possibly using Bailey pairs or a combinatorial interpretation.

### 3. PROOF OF THEOREM 1.1

*Proof of Theorem 1.1.* Let  $p$  be a prime  $\geq 5$  and  $n \geq r$  be an integer. Consider the truncation  $\mathcal{F}_t(q; pn - 1)$  of (1.8) and its  $p$ -dissection

$$\mathcal{F}_t(q; pn - 1) = \sum_{i=0}^{p-1} q^i A_{\mathcal{F}_t, p}(pn - 1, i, q^p).$$

We have

$$\mathcal{F}_t(1 - q; pn - 1) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(pn - 1, i, (1 - q)^p) + \sum_{i \notin S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(pn - 1, i, (1 - q)^p).$$

By Theorem 2.1, Proposition 2.2 and Theorem 2.4, we can write

$$\mathcal{F}_t(1 - q; pn - 1) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(pn - 1, i, (1 - q)^p) + (1 - (1 - q)^p)^n \sum_{i \notin S_{t, \chi_t}(p)} (1 - q)^i g_i(q)$$

for  $g_i(q) \in \mathbb{Z}[q]$ . By Lemma 3.3 in [28],

$$(1 - (1 - q)^p)^n \equiv O(q^{pn - (p-1)(r-1)}) \pmod{p^r}$$

and so

$$\mathcal{F}_t(1 - q; pn - 1) = \sum_{i \in S_{t, \chi_t}(p)} (1 - q)^i A_{\mathcal{F}_t, p}(pn - 1, i, (1 - q)^p) + O(q^{pn - (p-1)(r-1)}) \pmod{p^r}.$$

Choosing  $n$  large enough, it suffices to show that the coefficient of  $q^{p^r m - j}$  in  $(1 - q)^i A_{\mathcal{F}_t, p}(pn - 1, i, (1 - q)^p)$  vanishes modulo  $p^r$  for all  $i \in S_{t, \chi_t}(p)$ . Since  $A_{\mathcal{F}_t, p}(pn - 1, i, (1 - q)^p) \in \mathbb{Z}[q]$ , it suffices to show that

$$\binom{i + lp}{p^r m - j} \equiv 0 \pmod{p^r} \quad (3.1)$$

for natural numbers  $l$  and  $j \in \{1, 2, \dots, p-1 - \max S_{t, \chi_t}(p)\}$ . The condition on  $j$  implies that  $j < p-i$  and so (3.1) follows from Lemma 3.4 in [28]. This proves the result.  $\square$

#### 4. FURTHER COMMENTS

There are several avenues for further study. First, one could investigate combinatorial descriptions and asymptotic properties for the numbers  $\xi_t(n)$ . Second, the colored Jones polynomial of a knot  $K$  satisfies a *cyclotomic expansion* of the form

$$J_N(K; q) = \sum_{n \geq 0} (q^{1+N})_n (q^{1-N})_n C_n(K; q),$$

where the *cyclotomic coefficients*  $C_n(K; q)$  are Laurent polynomials independent of  $N$  [12]. It is highly desirable to find the cyclotomic expansion for the torus knots  $T(3, 2^t)$ ,  $t \geq 2$ . For  $t = 1$ , the cyclotomic expansion has been found by Masbaum [22]. However, it is unclear if his techniques are sufficient when  $t \geq 2$ . They require finding a link whose components are unknots, from which  $T(3, 2)$  can be recovered by introducing twists into a single region of one of the components. Employing this process for  $t \geq 2$  appears to require extending it to allow for multiple twist regions. Third, in relation to Theorem 2.4, a strange identity for the Kontsevich-Zagier series associated to the torus knots  $T(2, 2t+1)$  for  $t \geq 1$  has also been computed (see (15) in [14]). Do strange-type identities exist for similar  $q$ -series associated to satellite or hyperbolic knots? Fourth, as the Kontsevich-Zagier series  $F(q)$  is a foundational example of a quantum modular form [32], it is natural to wonder if the same is true for  $\mathcal{F}_t(q)$ . This is the subject of the paper [9]. Finally, we are pleased to report that the techniques in this paper have recently been extended in [8] to prove prime power congruences for additional families of generalized Fishburn numbers.

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