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# REAL BARGMANN SPACES, FISCHER DECOMPOSITIONS AND SETS OF UNIQUENESS FOR POLYHARMONIC FUNCTIONS

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ABSTRACT. In this paper a positive answer is given to the following question of W.K. Hayman: if a polyharmonic entire function of order  $k$  vanishes on  $k$  distinct ellipsoids in the euclidean space  $\mathbb{R}^n$  then it vanishes everywhere. Moreover a characterization of ellipsoids is given in terms of an extension property of solutions of entire data functions for the Dirichlet problem answering a question of D. Khavinson and H.S. Shapiro. These results are consequences from a more general result in the context of direct sum decompositions (Fischer decompositions) of polynomials or functions in the algebra  $A(B_R)$  of all real-analytic functions defined on the ball  $B_R$  of radius  $R$  and center 0 whose Taylor series of homogeneous polynomials converges compactly in  $B_R$ . The main result states that for a given elliptic polynomial  $P$  of degree  $2k$  and sufficiently large radius  $R > 0$  the following decomposition holds: for each function  $f \in A(B_R)$  there exist unique  $q, r \in A(B_R)$  such that  $f = Pq + r$  and  $\Delta^k r = 0$ . Another application of this result is the existence of polynomial solutions of the polyharmonic equation  $\Delta^k u = 0$  for polynomial data on certain classes of algebraic hypersurfaces.

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## 1. INTRODUCTION

Recall that a complex-valued function  $f$  defined on a domain  $G$  in the euclidean space  $\mathbb{R}^n$  is *polyharmonic of order  $k$*  if  $f$  is  $2k$ -times continuously differentiable and

$$\Delta^k f(x) = 0 \text{ for all } x \in G$$

where  $\Delta^k$  is the  $k$ -th iterate of the Laplace operator  $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ . For  $k = 1$  this class of functions are just the *harmonic functions*, while for  $k = 2$  the term *biharmonic function* is used which is important in elasticity theory. Fundamental work about polyharmonic functions is due to *E. Almansi* [2], *M. Nicolesco* (see e.g. [52]) and *N. Aronszajn* [4], and still this is an area of active research, see e.g. [29], [30], [41], [42], [47], [60]. Polyharmonic functions are also important in applied mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis, see e.g. [8], [37], [43], [44], [45], [46] and [48].

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In [35] *W.K. Hayman and B. Korenblum* proved that a polyharmonic function  $f : G \rightarrow \mathbb{C}$  of order  $k$  vanishing on  $k$  distinct spheres  $\{x \in \mathbb{R}^n : |x - x_m| = r_m\}$  for  $m = 1, \dots, k$  is identical zero provided that the balls  $\{x \in \mathbb{R}^n : |x - x_m| \leq r_m\}$  are contained in  $G$  for  $m = 1, \dots, k$ ; moreover they have proved that it is even possible to replace one of the spheres by the boundary of a domain whose closure is contained in  $G$ . They raised the question whether spheres can be replaced by more general hypersurfaces like ellipsoids in  $\mathbb{R}^n$ , see also [36]. In this paper we shall present a positive answer to this question for a large class of algebraic hypersurfaces (including the case of ellipsoids) for polyharmonic functions defined on  $\mathbb{R}^n$  for arbitrary dimension  $n$ , generalizing previous results of M. Balk and M. Mazalov in [9] and [10] for dimension  $n = 2$ . However, our method of proof goes far beyond the question of uniqueness and they allow us to prove a conjecture in [39] saying that ellipsoids are exactly the compact algebraic hypersurfaces in  $\mathbb{R}^n$  for which the solutions of the Dirichlet problem for entire data functions extends to entire harmonic functions. Moreover we shall give an analogue of this result for the polyharmonic equation  $\Delta^k f = 0$ .

Our methods of proof are related to two influential papers: the first one is due to *M. BreLOT and G. Choquet* [14] which provides important basic results about harmonic and polyharmonic polynomials. It is shown there that a non-trivial homogeneous *non-negative* polynomial  $P$  of  $n$  variables is *not* a harmonic divisor. Recall that a polynomial  $P$  is a *harmonic divisor* if there exists a non-zero polynomial  $q$  such that  $P \cdot q$  is harmonic, so

$$\Delta(Pq) = 0.$$

We refer to [1] and [3] for a characterization of all *quadratic* harmonic divisors and the significance of this notion. The result of BreLOT and Choquet is an immediate consequence of the fact that a homogeneous harmonic polynomial is orthogonal over the unit sphere to all polynomials of lower degree. We shall prove the following generalization: A homogeneous polynomial  $P$  of degree  $m$  is a solution of the equation  $\Delta^k u = 0$  if and only if  $P$  is orthogonal to all polynomials of degree less than  $m - 2k + 2$ . It follows that a non-trivial homogeneous *non-negative* polynomial of degree greater than  $2k - 2$  is *not* a  $k$ -harmonic divisor, so there is not a non-zero polynomial  $q$  such that

$$\Delta^k(Pq) = 0.$$

The second source is the fundamental work [50] of *D.J. Newman and H.S. Shapiro* about direct sum decompositions of entire functions which have been proven to be useful in a variety of mathematical problems like holomorphic solutions of the Dirichlet and Goursat problem, see [56] and [39], to mixed Cauchy problems, see [19] and [20], and for the discussion of weak maximum principles, see [21]. In [24] we will discuss applications of our method to the mixed Cauchy problem for data on *singular conics*, and in [25] to the Goursat problem for the Helmholtz operator.

In order to motivate direct sum decompositions we recall the theorem of Almansi saying that for every polyharmonic function  $f : G \rightarrow \mathbb{C}$  of order  $k$  on a star domain  $G$  there

exist harmonic functions  $h_0, \dots, h_k$  on  $G$  such that

$$(1) \quad f(x) = |x|^{2k} h_k(x) + |x|^{2(k-1)} h_{k-1}(x) + \dots + h_0(x),$$

where  $|x|^2 := x_1^2 + \dots + x_n^2$  is the euclidean distance of  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , see p. 4 in [4], and p. 122 in [4] for a further essential extension. This result, proved by Almansi in 1899 in [2], generalizes the *Gauß decomposition* of a polynomial  $f$ , which is basic in the theory of harmonic functions, see [5], [43] or [61]. The question which we address to is whether the special role of the function  $|x|^2$  in (1) can be taken over by a general polynomial  $P(x)$  where one allows the coefficient functions  $h_j, j = 1, \dots, k$  to be solutions of an accompanying linear differential operator  $Q(D)$  with constant coefficients. It is convenient to adopt a notion introduced by H.S. Shapiro in [56]: Suppose that  $E$  is a vector space of infinitely differentiable functions  $f : G \rightarrow \mathbb{C}$  defined on an open subset  $G \subset \mathbb{R}^n$  which is a module over  $\mathbb{C}[x_1, \dots, x_n]$ , the space of all polynomials in  $n$  variables with complex coefficients. A polynomial  $P$  and a differential operator  $Q(D)$  forms a *Fischer pair for the space  $E$* , shortly we say that  $(P, Q(D))$  is a Fischer pair, if for each  $f \in E$  there exist *unique* elements  $q \in E$  and  $r \in E$  such that

$$(2) \quad f = P \cdot q + r \text{ and } Q(D)r = 0.$$

Here  $Q(D)$  is the linear differential operator obtained from the polynomial  $Q$  by replacing the variables  $x_j$  by the partial derivatives  $\partial/\partial x_j$  for  $j = 1, \dots, n$ . It follows from the Gauß decomposition that  $(|x|^2, \Delta)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$ . This classical result was generalized by E. Fischer<sup>1</sup> in 1917 (see [27] or [17]):  $(P, P^*(D))$  is a Fischer pair for any *homogeneous* polynomial  $P$ , where  $P^*$  denotes the polynomial obtained from  $P$  by conjugating its coefficients. Important ingredient of the proof is the *Fischer inner product*  $[\cdot, \cdot]_F$  on  $\mathbb{C}[x_1, \dots, x_n]$  defined by

$$[P, Q]_F := [Q(D)P](0) = \sum_{\alpha \in \mathbb{N}_0^n} \alpha! c_\alpha \overline{d_\alpha}$$

where we use the standard notation for multi-indices  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ , and  $\alpha! = \alpha_1! \dots \alpha_n!$  and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ , and  $P$  and  $Q$  are given by  $P(x) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha$  and  $Q(x) = \sum_{\alpha \in \mathbb{N}_0^n} d_\alpha x^\alpha$  (where the sums are assumed to be finite). In [50] and [56] the corresponding Hilbert space norm  $\sqrt{[P, P]_F}$  is called the *Fischer norm*, while in [12] and [63] the term *Bombieri norm* is used. In passing, let us note that the Fischer inner product is also used in the recent work of C. de Boor, A. Ron and T. Sauer in the context of polynomial interpolation problems and algorithms for H-bases, see [16] and [55] and the references cited there.

In 1966 D.J. Newman and H.S. Shapiro proved a Fischer type decomposition (see [50] for the precise statement) for the *Bargmann space*  $\mathcal{F}_n$  (also called *Fock* or *Fischer space*)

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<sup>1</sup>See [32], p. 217 and p. 228, for some biographical comments and remarks on the Riesz-Fischer theorem for Hilbert spaces.

defined as the space of all entire functions  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  which satisfy

$$(3) \quad \|f\|_{\mathcal{F}_n}^2 := \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dz < \infty$$

where  $dz$  is Lebesgue measure on  $\mathbb{R}^{2n}$ . Since  $\|P\|_{\mathcal{F}_n} = \sqrt{[P, P]_F}$  for any  $P \in \mathbb{C}[x_1, \dots, x_n]$ , and polynomials are dense in  $\mathcal{F}_n$  the Bargmann space  $\mathcal{F}_n$  is the completion of  $\mathbb{C}[x_1, \dots, x_n]$  under the Fischer norm, see [11]. In [56] H.S. Shapiro has generalized Fischer's result:  $(P, P^*(D))$  is a Fischer pair for the algebra  $\mathcal{E}_n$  of all entire functions in  $\mathbb{C}^n$  for any homogeneous polynomial  $P$ . Furthermore A. Meril and D. Struppa showed in [49] that  $(x_j^k, Q(D))$  and  $(Q, \frac{\partial^k}{\partial x_j^k})$  are Fischer pairs for  $\mathcal{E}_n$  for polynomials  $Q$  having a non-zero coefficient at  $x_j^k$ .

A new type of a Fischer pair has been recently recognized in [6]:  $(P, \Delta)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  whenever  $P$  is a *nonhyperbolic quadratic polynomial*, so  $P$  is of the form

$$\sum_{j=1}^n b_j^2 x_j^2 + \sum_{j=1}^n c_j x_j + d,$$

where at least one  $b_j \neq 0$ . More generally, let us call a polynomial  $P$  *nonhyperbolic* if its principal part is non-negative on  $\mathbb{R}^n$ . Recall that the *principal part* (also called the *leading term*, see [16]) of a polynomial  $P$  is the non-zero polynomial  $P_k$  such that  $P = \sum_{j=0}^k P_j$  with homogeneous polynomials  $P_j$  of degree  $j$  for  $j = 0, \dots, k$ . Further a polynomial  $P$  is called *elliptic* if there exists  $C > 0$  such that its principal part  $P_k$  satisfies  $CP_k(x) \geq |x|^k$  for all  $x \in \mathbb{R}^n$ .

We will present the following far reaching generalization of Theorem 2.8 in [6]: *if the polynomial  $P$  of degree  $2k$  is nonhyperbolic then  $(P, \Delta^k)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$ .* For the proof of this result we introduce the space  $\mathcal{RF}_n$  of all measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that

$$(4) \quad \|f\|^2 := \int_{\mathbb{R}^n} |f(x)|^2 e^{-|x|^2} dx < \infty,$$

which we call the *real Bargmann space* and which is the basic technical tool in our investigations. Let us emphasize that  $\mathcal{RF}_n$  differs from the Bargmann space  $\mathcal{F}_n$  in many respects; however it is very suitable to introduce real methods like non-negativity for the concept of Fischer pairs. One basic observation is the following fact: a homogeneous polynomial  $f$  of degree  $\geq 2(k-1)$  is polyharmonic of order  $k$  if and only if it is orthogonal to all polynomials  $g$  of degree  $< \deg f - 2(k-1)$  with respect to the scalar product of the real Bargmann space. This result leads rather quickly to a solution of the question of W.K. Hayman for polyharmonic *polynomials*.

The next important step is the passage from the polynomial case to suitable spaces of real-analytic functions. It turns out that the following space is very useful: Let  $B_R := \{x \in \mathbb{R}^n : |x| < R\}$  be the open ball in  $\mathbb{R}^n$  with center 0 and radius  $0 < R \leq \infty$ , and let

$A(B_R)$  be the set of all infinitely differentiable functions  $f : B_R \rightarrow \mathbb{C}$  such that for any compact subset  $K \subset B_R$  the homogeneous Taylor series  $\sum_{m=0}^{\infty} f_m(x)$  converges absolutely and uniformly to  $f$  on  $K$ , where  $f_m$  are the homogeneous polynomials of degree  $m$  defined by the Taylor series of  $f$ , i.e. that

$$f_m(x) = \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0) x^\alpha \text{ for } m \in \mathbb{N}_0.$$

It is known that  $A(B_R)$  is isomorphic to the set of all holomorphic functions on the harmonicity hull of  $B_R$ , see Section 8.

Using our results for the polynomial case we can show that  $(P_{2k}, \Delta^k)$  is a Fischer pair for  $A(B_R)$  whenever  $P_{2k}$  is a *homogeneous elliptic* polynomial of degree  $2k$ . The question under which conditions  $(P, \Delta^k)$  is a Fischer pair for  $A(B_R)$  for *non-homogeneous* polynomials is much more involved and subtle; moreover the answer has applications to the Dirichlet problem and boundary value problems of higher order PDE's as we shall see below. Our main result says that for a given elliptic polynomial  $P$  the pair  $(P, \Delta^k)$  is a Fischer pair for  $A(B_R)$  whenever the radius  $R$  is large enough; an explicit bound is given depending on  $P$  and the ellipticity constant  $C$ . For  $R = \infty$  this implies that  $(P, \Delta^k)$  is a Fischer pair for the algebra  $\mathcal{E}_n$  of all entire functions for any elliptic polynomial.

Our results enable us to solve (at least partially, i.e. for a class of algebraic hypersurfaces) the following questions of D. Khavinson and H.S. Shapiro (see [39], p. 460): Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  such that  $\partial\Omega$  is a real-analytic hypersurface.

(I) Suppose that for every entire function  $f$  the solution of Dirichlet's problem for  $\Omega$  with data  $f|_{\partial\Omega}$  is harmonically extendible to  $\mathbb{R}^n$ . Must  $\Omega$  be an ellipsoid?

(II) Suppose that for every polynomial  $f$  the solution of Dirichlet's problem for  $\Omega$  with data  $f|_{\partial\Omega}$  is a polynomial. Must  $\Omega$  be an ellipsoid?

Problem (II) has been discussed by several authors: Based on ideas in [39] a positive solution of question (II) was given in [15] for the case  $n = 2$  such that the boundary  $\partial\Omega$  is equal to an algebraic set, i.e. the zero set of a polynomial  $\psi$  (which satisfies some natural conditions). We shall extend this result to arbitrary dimension.

It seems that problem (I) has not been solved even for the simplest domains in  $\mathbb{R}^2$ . We give the following solution of question (I): suppose that  $\psi = \psi_1 \dots \psi_r$  is elliptic where  $\psi_1, \dots, \psi_r \in \mathbb{R}[x_1, \dots, x_n]$  are irreducible and no  $\psi_j$  is a scalar multiple of some  $\psi_k$  for  $k \neq j$ . Assume that there exist open sets  $U_j, j = 1, \dots, r$  such that  $\psi_j$  changes sign on  $U_j$  and such that

$$U_j \cap \{x \in \mathbb{R}^n : \psi_j(x) = 0\} \subset \partial\Omega$$

for  $j = 1, \dots, r$ . If  $\deg \psi > 2$  then (I) is not satisfied. In other words, (I) implies that  $\psi$  has degree 2, and using the boundedness of the region, it follows that  $\psi$  defines an ellipsoid.

Let us now give a short outline of the paper. In Section 2 we show that nonhyperbolic polynomials  $P$  of degree  $2k$  induces Fischer pairs  $(P, \Delta^k)$  for  $\mathbb{C}[x_1, \dots, x_n]$ . Section 3 contains a proof of Hayman's conjecture for the polynomial case. In Section 4 it is shown

that  $(P_{2k}, \Delta^k)$  is a Fischer pair for  $A(B_R)$  whenever  $P_{2k}$  is homogeneous of degree  $2k$  and elliptic. In Section 5 we discuss uniqueness of the Fischer decomposition in the general case. In Section 6 we prove the main result about non-homogeneous Fischer pairs  $(P, \Delta^k)$  for  $A(B_R)$  and large radii  $R$ . In Section 7 we deduce an Almansi type theorem from our main result. In Section 8 it is shown how these results carry over to the space  $\mathcal{E}_n$  of all entire functions.

In Section 9 we present the main result, the solution of Hayman's conjecture for polyharmonic functions on  $\mathbb{R}^n$ . Section 10 contains the above-mentioned characterization of ellipsoids via condition (I) or (II) and some analogues for the polyharmonic equation.

In the appendix 11 we collect some basic results about Fischer pairs. Although most of the results are known and relatively simple to prove (the original source is [27]), we have included some proofs in order to hold the paper self-contained.

Finally let us introduce some notation. The natural numbers are denoted by  $\mathbb{N}$  and we put  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . A real-valued function  $f$  changes sign on a set  $U$  if there exists  $x, y \in U$  with  $f(x) < 0 < f(y)$ . If  $R$  is a ring we denote by  $R[x_1, \dots, x_n]$  the ring of polynomials in  $n$  variables with coefficients in  $R$ . By  $\deg P$  we denote the degree of a polynomial  $P$ . A polynomial  $P$  is called *homogeneous of degree  $m$*  if  $P(rx) = r^m P(x)$  for all  $r > 0$  and  $x \in \mathbb{R}^n$ . In order to emphasize in formulae that a polynomial  $P$  is homogeneous of degree  $m$  we write often  $P_m$  instead of  $P$ . Frequently we use spherical coordinates  $x = r\theta$  where  $r > 0$  and  $\theta$  is in the unit sphere  $\mathbb{S}^{n-1}$  defined by

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Moreover  $\omega_{n-1} := \int_{\mathbb{S}^{n-1}} 1 d\theta$  denotes the area of  $\mathbb{S}^{n-1}$ .

The author is indebted to *Prof. Ognyan Kouanchev* for stimulating discussions and valuable comments on the subject.

## 2. FISCHER PAIRS FOR $\mathbb{C}[x_1, \dots, x_n]$

Define a scalar product on  $\mathbb{C}[x_1, \dots, x_n]$  by setting for  $f, g \in \mathbb{C}[x_1, \dots, x_n]$

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) \overline{g(x)} e^{-|x|^2} dx.$$

A simple argument, using partial integration, shows that for  $j = 1, \dots, n$

$$\left\langle \frac{\partial}{\partial x_j} f, g \right\rangle = - \left\langle f, \frac{\partial}{\partial x_j} g \right\rangle + 2 \langle x_j \cdot f, g \rangle.$$

Then a straightforward computation gives

$$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle + 2 \sum_{j=1}^n \left\langle x_j \frac{\partial}{\partial x_j} f, g \right\rangle - \left\langle f, x_j \frac{\partial}{\partial x_j} g \right\rangle.$$

Assuming now that  $f, g$  are homogeneous one obtains the following conclusion:

**Proposition 1.** *For homogeneous polynomials  $f$  and  $g$  of degree  $m_f$  and  $m_g$  resp. the following identity holds:*

$$(5) \quad \langle \Delta f, g \rangle = \langle f, \Delta g \rangle + 2(m_f - m_g) \langle f, g \rangle.$$

**Theorem 2.** *Suppose that  $f$  is a homogeneous polynomial, and let  $k \in \mathbb{N}_0$  with  $2(k-1) \leq \deg f$ . Then  $\Delta^k f = 0$  if and only if  $\langle f, g \rangle = 0$  for all polynomials  $g$  with  $2(k-1) + \deg g < \deg f$ .*

*Proof.* Let us prove the necessity part. Clearly we can assume that  $g$  is a homogeneous polynomial. Let  $m_g = \deg g$  and  $m_f = \deg f$ . We prove the claim by induction over  $k$ . If  $k = 0$  then the assumption  $\Delta^k f = f = 0$  clearly implies  $\langle f, g \rangle = 0$  for any polynomial  $g$ . Suppose the statement holds for  $k$  and assume that  $\Delta^{k+1} f = 0$  for a polynomial  $f$  with  $\deg f \geq 2k$ . Then  $\Delta^k f_1 = 0$  with  $f_1 := \Delta f$  and  $\deg f_1 \geq 2(k-1)$ . By induction hypothesis we know that  $0 = \langle f_1, g \rangle = \langle \Delta f, g \rangle$  for all homogeneous polynomials  $g$  with  $\deg g < \deg f_1 - 2(k-1) = \deg f - 2k$ . Then (5) shows that

$$(6) \quad 0 = \langle f, \Delta g \rangle + 2(m_f - m_g) \langle f, g \rangle$$

for all homogeneous polynomials  $g$  with  $m_g < m_f - 2k$ . Hence it suffices to show that (6) implies  $0 = \langle f, g \rangle$  for all homogeneous polynomials  $g$  with  $m_g \leq m_f - 2k$ .

We prove the latter statement by a second induction over a new variable  $s \in \mathbb{N}$ : let  $k$  be fixed, we claim that  $0 = \langle f, g \rangle$  for all homogeneous polynomials  $g$  with  $m_g < m_f - 2k$  and  $\Delta^s g = 0$ . If  $s = 1$ , this means that  $g$  is a harmonic polynomial with  $m_g < m_f - 2k$ . Then equation (6) implies  $0 = \langle f, g \rangle$ . Assume that the statement holds for  $s$ , and let  $g_1$  be a homogeneous polynomial such that  $m_{g_1} < m_f - 2k$  and  $\Delta^{s+1} g_1 = 0$ . By the Gauß decomposition,  $g_1 = |x|^2 g + h$ , where  $h$  is a harmonic homogeneous polynomial of degree  $\leq \deg g_1$  and  $g$  is homogeneous. Then (6) implies

$$(7) \quad 0 = \langle f, \Delta(|x|^2 g) \rangle + 2(m_f - m_g - 2) \langle f, |x|^2 g \rangle.$$

We know that  $\Delta^s \Delta(|x|^2 g) = \Delta^{s+1} g_1 = 0$ . Thus by induction hypothesis  $\langle f, \Delta(|x|^2 g) \rangle = 0$ . Hence (7) yields  $0 = (m_f - m_g - 2) \langle f, |x|^2 g \rangle$ . Since  $m_g + 2 = m_{g_1} < m_f$  we conclude  $\langle f, |x|^2 g \rangle = 0$ . Since we already know that  $\langle f, h \rangle = 0$  we arrive at  $\langle f, g \rangle = 0$  and the induction is completed.

Now let us prove the sufficiency part (which is not needed later) by induction over  $k$ . If  $k = 0$  we may take  $g = f$  and obtain that  $\langle f, f \rangle = 0$  which implies  $f = 0$ . Assume the statement holds for  $k$ , and let  $f$  be a polynomial of degree  $\geq 2k$  such that (\*)  $\langle f, g \rangle = 0$  for all polynomials  $g$  of degree  $< \deg f - 2k$ . Define  $F := \Delta f$ . Then for any polynomial  $g$  with degree  $< \deg F - 2(k-1) = \deg f - 2k$  we have by (5) and our assumption (\*) (twice applied)

$$\langle F, g \rangle = \langle \Delta f, g \rangle = \langle f, \Delta g \rangle + 2(m_f - m_g) \langle f, g \rangle = 0.$$

By induction hypothesis we conclude that  $\Delta^k F = 0$ , hence  $\Delta^{k+1} f = 0$ . The proof is complete.  $\square$

**Theorem 3.** *Let  $P$  be a nonhyperbolic polynomial of degree  $2k$ . Then  $(P, \Delta^k)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$ . Moreover,*

$$(8) \quad \Delta^k(Pq) \neq 0 \text{ and } \Delta^k(P_{2k}q) \neq 0$$

for any polynomial  $q \neq 0$  where  $P_{2k}$  is the principal part of  $P$ .

*Proof.* Suppose that  $\Delta^k(P_{2k}q) = 0$  for a polynomial  $q \neq 0$ . By Theorem 2,  $\langle P_{2k}q, g \rangle = 0$  for all polynomials  $g$  of degree  $< \deg(Pq) - 2(k-1) = \deg q + 2$ . So we can take  $g := q$  and obtain that

$$0 = \langle P_{2k}q, q \rangle = \int_{\mathbb{R}^n} |q(x)|^2 P_{2k}(x) e^{-|x|^2} dx.$$

Since  $P_{2k} \geq 0$  we conclude that  $q = 0$ , a contradiction. Hence  $\Delta^k(P_{2k}q) \neq 0$  for all  $q \neq 0$ . By Theorem 37 in the appendix,  $(P_{2k}, |x|^{2k})$  is a Fischer pair. Finally Theorem 38 shows that  $(P, \Delta^k)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  and that  $\Delta^k(Pq) \neq 0$ .  $\square$

I am very indebted to H.S. Shapiro [57] for providing and permitting me to communicate the following example. It shows that the operator  $\Delta^k$  in Theorem 3 cannot be replaced by a general homogeneous *elliptic* differential operator.

**Example 4.** *There exist elliptic homogeneous polynomials  $P, Q$  of the same degree such that  $(P, Q(D))$  is not a Fischer pair: Let  $d = 2$  and  $P(x, y) = x^4 + y^4 + 12x^2y^2$  and  $Q(x, y) = x^4 + y^4 - x^2y^2$ . It is easy to see that  $P, Q$  are homogeneous elliptic polynomials and that  $\langle P, Q \rangle_F = 0$ . It follows that*

$$0 = \langle P, Q \rangle_F = \langle 1, P^*(D)(Q) \rangle = P^*(D)(Q)(0) = P^*(D)(Q \cdot 1).$$

Hence  $(Q, P^*(D))$  is not a Fischer pair.

### 3. HAYMAN'S CONJECTURE FOR POLYNOMIALS

Let us recall the assumptions in Hayman's conjecture: a polyharmonic function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  of order  $k$  vanishes on  $k$  different ellipsoids  $S_1, \dots, S_k$ . Hence we can find irreducible polynomials  $\psi_1, \dots, \psi_k \in \mathbb{R}[x_1, \dots, x_n]$  of degree 2 such that

$$S_j := \{x \in \mathbb{R}^n : \psi_j(x) = 0\} \text{ for } j = 1, \dots, k,$$

and it is clear that each  $\psi_j$ ,  $j = 1, \dots, k$ , changes sign on  $\mathbb{R}^n$ . Since  $S_1, \dots, S_k$  are pairwise different it is also clear that  $\psi_j$  is not a scalar multiple of  $\psi_l$  for  $k \neq l$ . We want to prove that  $f$  is identical zero.

These remarks motivate the following terminology:

**Definition 5.** *Let  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  be a polynomial and write  $\psi = \psi_1 \dots \psi_r$  as a product of irreducible polynomials in  $\mathbb{R}[x_1, \dots, x_n]$ . Then  $\psi$  is called square-free if no  $\psi_j$  is a multiple of some  $\psi_l$  for  $l \neq j$ . In other words:  $\psi$  does not have multiple prime factors. The polynomial  $\psi$  is called non-degenerate if each irreducible factor  $\psi_j$  changes sign on  $\mathbb{R}^n$  for  $j = 1, \dots, r$ .*

Let us emphasize that the zero set  $Z_\psi := \{x \in \mathbb{R}^n : \psi(x) = 0\}$  of a polynomial  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  is in general *not* connected, even if  $\psi$  is irreducible, cf. the examples on p. 60 in [13]. Theorem 4.5.1 in [13] shows that an irreducible polynomial  $\psi_j$  changes sign on  $\mathbb{R}^n$  if and only if the zero set

$$Z_{\psi_j} := \{x \in \mathbb{R}^n : \psi_j(x) = 0\}$$

has dimension  $n - 1$ , and it is also equivalent to the fact that  $\psi_j$  has a nonsingular zero. Hence the notion of a non-degenerate square-free polynomial incorporates the fact that we are dealing with a finite union of pairwise different hypersurfaces. The following is a simple extension of Theorem 4.5.1 in [13]:

**Theorem 6.** *Let  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  be square-free and assume that each irreducible factor  $\psi_j, j = 1, \dots, r$ , changes sign on given open sets  $U_j$  for  $j = 1, \dots, r$ . Suppose that  $f \in \mathbb{R}[x_1, \dots, x_n]$  vanishes on  $Z_{\psi_j} \cap U_j$  for  $j = 1, \dots, r$ . Then there exists  $q \in \mathbb{R}[x_1, \dots, x_n]$  such that  $f = q\psi_1 \dots \psi_r$ .*

Now we can give an affirmative answer to the conjecture of W. Hayman for the class of all polyharmonic *polynomials*. Let us recall that a nonhyperbolic polynomial has even degree.

**Theorem 7.** *Let  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  be square-free, nonhyperbolic and assume that each irreducible factor  $\psi_j, j = 1, \dots, r$ , changes sign on given open sets  $U_j$  for  $j = 1, \dots, r$ . Suppose that  $f \in \mathbb{R}[x_1, \dots, x_n]$  vanishes on  $Z_{\psi_j} \cap U_j$  for  $j = 1, \dots, r$ . If  $\Delta^k f = 0$  with  $k = \frac{1}{2} \deg \psi$  then  $f = 0$ .*

*Proof.* Suppose that  $f \neq 0$ . By Theorem 6 we can write  $f = q\psi$  for some polynomial  $q \neq 0$ . Theorem 3 shows that  $\Delta^k(f) = \Delta^k(\psi q) \neq 0$ . This contradiction shows that  $f = 0$ .  $\square$

As mentioned in the introduction, a positive solution to Hayman's conjecture for the case  $n = 2$  has been given in [9] and [10]. However, for the polynomial case our result is stronger since in [9] it is assumed that  $\psi$  is elliptic on  $\mathbb{R}^2$ . So the following example is not covered by their results:

**Example 8.** *Let  $\psi_1, \dots, \psi_k$  be nonhyperbolic, sign-changing irreducible polynomials in  $n$  variables of degree 2 (e.g. paraboloids). If the polynomial  $f$  vanishes on the pairwise different sets  $Z_{\psi_j} = \{x \in \mathbb{R}^n : \psi_j(x) = 0\}$  for  $j = 1, \dots, k$ , and  $\Delta^k f = 0$  then  $f$  is identically zero.*

The result in Example 8 was proved for  $k = 1$  in [6], p. 643; it generalizes a result in [28] saying that a harmonic polynomial  $p(x, y)$  in two variables  $x, y$  which vanishes on the parabola  $x = y^2$  must be identically zero. Moreover, the authors in [28] have shown that there exists a non-zero harmonic function  $f$  on the space  $\mathbb{R}^2$  vanishing on the parabola  $x = y^2$  (while this is not possible for the curves  $x = y^m$  for  $m > 2$ ). This shows that Theorem 7 does not hold for harmonic *functions*  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  (for  $k = 1$ ). We shall show that an analog of Theorem 7 for polyharmonic functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is valid if we assume

that  $\psi$  is elliptic. However, this needs some more efforts to be carried out in the next sections.

#### 4. HOMOGENEOUS FISCHER PAIRS FOR $A(B_R)$

In this section we want to prove that  $(P, \Delta^k)$  is a Fischer pair for the algebra  $A(B_R)$  whenever  $P$  is a *homogeneous* elliptic polynomial of degree  $2k$ . At first we need estimates between the norm  $\|f_m\|$  defined in (4) of a homogeneous polynomial  $f_m$  of degree  $m$  and the maximum norm on the sphere. Let us define

$$I_m := \int_0^\infty e^{-r^2} r^m dr \text{ for } m \in \mathbb{N}_0.$$

This integral can be computed explicitly: for the even case (see p. 265 in [54]) we have

$$(9) \quad I_{2m} = \frac{\sqrt{\pi} (2m)!}{2 m!} 2^{-2m} = \frac{\sqrt{\pi} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)}{2 \cdot 2^m} \leq m!$$

while in the odd case a simple substitution argument gives

$$(10) \quad I_{2m+1} = \int_0^\infty e^{-r^2} r^{2m+1} dr = \frac{1}{2} \int_0^\infty e^{-x} x^m dx = \frac{1}{2} m!.$$

We need some elementary estimates, and for convenience of the reader, we include the proof.

**Proposition 9.** *For  $n \geq 1$  define  $n^* := \frac{n}{2} - 1 \geq 0$  for even  $n$ , and  $n^* := (n-1)/2 \geq 0$  for odd  $n$ . Then for any  $m \geq 0$*

$$(11) \quad \frac{1}{4} (m + n^* - 1)! \leq I_{2m+n-1} \leq (m + n^*)!$$

where we define  $(-1)! := 2$ . For  $m, k \in \mathbb{N}_0$  with  $m + n^* \geq k$  the following estimates

$$(12) \quad \frac{1}{2} \leq \frac{(m + n^* - 1)!}{(m + n^* - k - 1)!} \leq \frac{I_{2m+n-1}}{I_{2m-2k+n-1}} \leq \frac{(m + n^*)!}{(m + n^* - k)!}$$

hold.

*Proof.* If  $n$  is even then  $2m + n - 1 = 2l + 1$  for some  $l \in \mathbb{N}_0$ , so  $m + n^* = l$ . Then  $I_{2m+n-1} = \frac{1}{2} (m + n^*)!$  for each  $m \geq 0$  by (10). Clearly  $I_{2m+n-1} \geq \frac{1}{2} (m + n^* - 1)!$  for  $m + n^* \geq 1$ . For  $m + n^* = 0$  we have  $I_{2m+n-1} = \frac{1}{2}$ , and with the convention  $(-1)! := 2$ , one obtains that (11) also holds in this case. The inequality (12) follows from the fact that  $\frac{I_{2m+n-1}}{I_{2m-2k+n-1}} = \frac{(m+n^*)!}{(m+n^*-k)!}$ .

If  $n$  is odd then  $2m + n - 1 = 2l$ , so  $m + n^* = l$ . Then  $I_{2m+n-1} \leq (m + n^*)!$  by (9). For the lower estimate in (11) note that for  $m + n^* \geq 1$

$$I_{2m+n-1} = \frac{\sqrt{\pi} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2(m + n^*) - 1)}{2 \cdot 2^{m+n^*}} \geq \frac{\sqrt{\pi}}{4} (m + n^* - 1)!.$$

If  $m + n^* = 0$  then  $I_{2m+n-1} = \frac{\sqrt{\pi}}{2} \geq \frac{1}{4}(-1)!$ . The upper estimate in (12) follows from the identity

$$\frac{I_{2m+n-1}}{I_{2m-2k+n-1}} = \frac{1}{2^k} (2m + 2n^* - 2k + 1) (2m + 2n^* - 2k + 3) \dots (2m + 2n^* - 1).$$

If  $m + n^* - k \geq 1$  one obtains the lower estimate as well. If  $m + n^* - k = 0$  one has to be careful with the first factor, but the lower estimate is valid by interpreting  $(m + n^* - k - 1)! = (-1)! = 2$ .  $\square$

The next estimate is straightforward:

**Proposition 10.** *Let  $f_m$  be a homogeneous polynomial of degree  $m$ . Then*

$$(13) \quad \|f_m\| \leq \sqrt{\omega_{n-1}} \sqrt{I_{2m+n-1}} \max_{\theta \in \mathbb{S}^{n-1}} |f_m(\theta)|.$$

*Proof.* Let us take spherical coordinates  $x = r\theta$  where  $r > 0$  and  $\theta \in \mathbb{S}^{n-1}$ . By homogeneity of  $f_m$  we have  $f_m(r\theta) = r^m f(\theta)$  for all  $r > 0$  and  $\theta \in \mathbb{S}^{n-1}$ . Thus

$$(14) \quad \langle f_m, f_m \rangle = \int_{\mathbb{R}^n} |f_m(x)|^2 e^{-|x|^2} dx = I_{2m+n-1} \int_{\mathbb{S}^{n-1}} |f_m(\theta)|^2 d\theta.$$

A standard estimate for the last integral gives now (13).  $\square$

**Theorem 11.** *Let  $n \in \mathbb{N}$  be fixed. Then for all homogeneous polynomials  $f_m$  of degree  $m \in \mathbb{N}_0$  and for all  $\theta \in \mathbb{S}^{n-1}$*

$$(15) \quad |f_m(\theta)| \leq \sqrt{2\omega_{n-1}} \frac{(1+m)^{(n-1)/2}}{\sqrt{I_{2m+n-1}}} \|f_m\|.$$

*Proof.* In this proof we use several results from the theory of spherical harmonics which can be found in [5]: By the Gauß decomposition we can write  $f = h_m + h_{m-2}|x|^2 + \dots + h_{m-2m_2}|x|^{2m_2}$  with harmonic homogeneous polynomials  $h_{m-2j}$  of degree  $m-2j$ , and  $m_2 := \lfloor \frac{m}{2} \rfloor$  where  $\lfloor x \rfloor$  denotes the largest integer  $\leq x$ . Then

$$(16) \quad f(\theta) = h_m(\theta) + h_{m-2}(\theta) + \dots + h_{m-2m_2}(\theta),$$

and  $|f(\theta)|^2 = \sum_{j,l=0}^{m_2} h_{m-2j}(\theta) \overline{h_{m-2l}(\theta)}$ . Since  $h_{m-2j}$  and  $h_{m-2l}$  are harmonic and homogeneous polynomials of degree  $m-2j$  and  $m-2l$  the orthogonality relations imply

$$(17) \quad \int_{\mathbb{S}^{n-1}} |f(\theta)|^2 d\theta = \sum_{j=0}^{m_2} \int_{\mathbb{S}^{n-1}} |h_{m-2j}(\theta)|^2 d\theta.$$

Let  $Z_m(\eta, \theta)$  be a zonal harmonic of degree  $m$  with pole at  $\theta \in \mathbb{S}^{n-1}$ , so for each harmonic polynomial  $h$  of degree  $m$  the following identity

$$(18) \quad h(\theta) = \int_{\mathbb{S}^{n-1}} h(\eta) \overline{Z_m(\eta, \theta)} d\eta$$

holds. Then  $f(\theta) = \sum_{j=0}^{m_2} \int_{\mathbb{S}^{n-1}} h_{m-2j}(\eta) \overline{Z_{m-2j}(\eta, \theta)} d\eta$  by (16) and (18). The Cauchy-Schwarz inequality, applied to each summand, yields

$$|f(\theta)| \leq \sum_{j=0}^{m_2} \left( \int_{\mathbb{S}^{n-1}} |h_{m-2j}(\eta)|^2 d\eta \right)^{\frac{1}{2}} \sqrt{a_{m-2j}}.$$

where  $\int_{\mathbb{S}^{n-1}} |Z_{m-2j}(\eta, \theta)|^2 d\eta = a_{m-2j}$  for  $j = 0, \dots, m_2$ , and  $a_m$  is the dimension of the space of all harmonic polynomials of degree  $m$ . The Cauchy-Schwarz inequality for  $\mathbb{R}^n$  yields

$$(19) \quad |f(\theta)| \leq \left( \sum_{j=0}^{m_2} \int_{\mathbb{S}^{n-1}} |h_{m-2j}(\eta)|^2 d\eta \right)^{\frac{1}{2}} \cdot \left( \sum_{j=0}^{m_2} a_{m-2j} \right)^{\frac{1}{2}}.$$

Since  $\sum_{j=0}^{m_2} a_{m-2j} \leq (m+1) a_m$ , (17) and (19) lead to

$$|f(\theta)| \leq \sqrt{(1+m) a_m} \left( \int_{\mathbb{S}^{n-1}} |f(\eta)|^2 d\eta \right)^{\frac{1}{2}}.$$

Now we use the estimate  $a_m \leq 2(m+1)^{n-2}$ , so with (14) we arrive at (15). The former estimate can be derived in the following way: For  $n > 2$  it is proved in [5] that

$$a_m = \frac{(n+m-3)!}{(n-2)!m!} (n+2m-2) = \frac{n-2+2m}{n-2} \prod_{l=1}^{n-3} \frac{m+l}{l}.$$

For  $l > 1$  and  $m > 1$  we have  $m(l-1) \geq 2(l-1) \geq l$ , so  $l+m \leq m \cdot l$ . Since  $n-2+2m \leq 2(m+1)(n-2)$  for  $n > 2$  the result easily follows for  $l > 1$  and  $m > 1$ . Clearly this estimate is also valid for  $m=0$  and  $m=1$ . In case of dimension  $n=2$  it is well known that  $a_m = 2$  for all  $m \geq 1$ .  $\square$

The following result is the analogue of Lemma 1 in [56].

**Proposition 12.** *Suppose that  $f_m$  are homogeneous polynomials of degree  $m$  for  $m \in \mathbb{N}_0$ . Then  $\sum_{m=0}^{\infty} f_m$  converges compactly in  $B_R$  if and only if*

$$(20) \quad \limsup_{m \rightarrow \infty} \left( \max_{\theta \in \mathbb{S}^{n-1}} |f_m(\theta)| \right)^{1/m} \leq R^{-1}$$

*if and only if*

$$(21) \quad \limsup_{m \rightarrow \infty} \left( \frac{\|f_m\|}{\sqrt{m!}} \right)^{1/m} \leq R^{-1}.$$

*Proof.* Suppose that  $\sum_{m=0}^{\infty} f_m$  converges compactly in  $B_R$ . Let  $\rho < R$ . Then there exists  $M > 0$  such that  $\sum_{m=0}^{\infty} |f_m(x)| \leq M$  for all  $|x| \leq \rho$ , in particular

$$\rho^m |f_m(\theta)| \leq M$$

for all  $\theta \in \mathbb{S}^{n-1}$ . This implies  $\limsup_{m \rightarrow \infty} (\max_{\theta \in \mathbb{S}^{n-1}} |f_m(\theta)|)^{1/m} \leq \rho^{-1}$ . Since this holds for any  $\rho < R$  we arrive at (20).

We show now that (20) implies (21). By (13) and the inequality  $I_{2m+n-1} \leq (m+n^*)! \leq m!(m+n^*)^{n^*}$  (see (11)) we obtain

$$\left( \frac{\|f_m\|}{\sqrt{m!}} \right)^{1/m} \leq \omega_{n-1}^{1/2m} (m+n^*)^{n^*/2m} \max |f_m(\theta)|^{1/m}.$$

By taking the limit superior  $m \mapsto \infty$  and using assumption (20) one obtains (21).

Suppose that (21) holds. Let  $\rho < R$  and take  $\delta > 0$  so small such that  $\frac{\rho}{R-\delta} < 1$ . Since  $R^{-1} < (R-\delta)^{-1}$ , there exists  $m_0$  such that  $\|f_m\|/\sqrt{m!} \leq (R-\delta)^{-m}$  for all  $m \geq m_0$ . Since  $I_{2m+n-1} \geq \frac{1}{4}(m-1)!$  we obtain from (15)

$$\begin{aligned} |f_m(\theta)| &\leq \frac{\sqrt{2\omega_{n-1}}(1+m)^{(n-1)/2}}{\sqrt{I_{2m+n-1}}} \|f_m\| \leq 2 \frac{\sqrt{2\omega_{n-1}}(1+m)^{n/2}}{\sqrt{m!}} \|f_m\| \\ &\leq 2\sqrt{2\omega_{n-1}}(1+m)^{n/2} (R-\delta)^{-m}. \end{aligned}$$

Let  $x = r\theta$  with  $r \leq \rho$ . Then

$$\sum_{m=0}^{\infty} r^m |f_m(\theta)| \leq 2\sqrt{2\omega_{n-1}} \sum_{m=0}^{\infty} (1+m)^{n/2} \left( \frac{\rho}{R-\delta} \right)^m$$

clearly converges uniformly for  $r \leq \rho$ , so  $\sum_{m=0}^{\infty} f_m(x)$  converges uniformly for  $|x| \leq \rho$ .  $\square$

The following is the main result of this section:

**Theorem 13.** *Let  $0 < R \leq \infty$  and  $P_{2k}$  be a homogeneous polynomial of degree  $2k$  such that  $CP_{2k}(x) \geq |x|^{2k}$  for all  $x \in \mathbb{R}^n$  for some constant  $C > 0$ . Then  $(P_{2k}, \Delta^k)$  is a Fischer pair for  $A(B_R)$ .*

*Proof.* For  $f \in A(B_R)$  write  $f = \sum_{m=0}^{\infty} f_m$  where  $f_m$  are homogeneous polynomials of degree  $m$ . Assume that  $m \geq 2k$ . By Theorem 3  $(P_{2k}, \Delta^k)$  is Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  hence we can write

$$(22) \quad f_m = P_{2k} \cdot T(f_m) + r_m$$

where  $T(f_m)$  is homogeneous of degree  $m-2k$ , and  $r_m$  is homogeneous of degree  $m$ , see Theorem 37 in the appendix. Since  $\Delta^k r_m = 0$ , Theorem 2 yields  $\langle r_m, T(f_m) \rangle = 0$  (note that  $\deg T(f_m) < \deg r_m - 2(k-1)$ ). Then

$$(23) \quad C \langle f_m, T(f_m) \rangle = C \langle P_{2k} T(f_m), T(f_m) \rangle \geq \left\langle |x|^{2k} T(f_m), T(f_m) \right\rangle,$$

where we have used for the last inequality our assumption  $CP_{2k}(x) \geq |x|^{2k}$ . Further we know that (cf. formula (14))

$$(24) \quad \frac{I_{2m-4k+n-1}}{I_{2m-2k+n-1}} \left\langle |x|^{2k} T(f_m), T(f_m) \right\rangle = \langle T(f_m), T(f_m) \rangle.$$

Now (24) and (23) and the Cauchy-Schwarz inequality imply

$$(25) \quad \|T(f_m)\|^2 \leq C \frac{I_{2m-4k+n-1}}{I_{2m-2k+n-1}} \|f_m\| \cdot \|T(f_m)\|.$$

Thus we have proved the fundamental inequality

$$(26) \quad \|T(f_m)\| \leq C \frac{I_{2m-4k+n-1}}{I_{2m-2k+n-1}} \|f_m\|$$

which will be used later frequently. Since  $I_{2m-4k+n-1} \leq 2I_{2m-2k+n-1}$  by (12) one arrives at the weaker estimate

$$(27) \quad \|T(f_m)\| \leq 2C \|f_m\|.$$

Note that  $m! \leq (m-2k)!m^{2k}$  for  $m \geq 2k$ . It follows that

$$\left( \frac{\|T(f_m)\|}{\sqrt{(m-2k)!}} \right)^{1/(m-2k)} \leq (2C)^{1/(m-2k)} m^{2k/(m-2k)} \left( \frac{\|f_m\|}{\sqrt{m!}} \right)^{1/(m-2k)}.$$

It is not difficult to see that the limit superior of the right hand side is less than or equal to  $R^{-1}$  since  $f \in A(B_R)$ . By Proposition 12,  $q := \sum_{m=2k}^{\infty} T(f_m)$  converges on compact subsets of  $B_R$ , so  $q$  is in  $A(B_R)$ . It follows from (22) that  $r := \sum_{m=0}^{\infty} r_m$  converges compactly in  $B_R$  and that  $f = P_{2k}q + r$ . Since the locally uniform limit of polyharmonic functions is polyharmonic, we infer  $\Delta^k r = 0$ . The proof is complete.  $\square$

In order to conclude that  $(P, Q(D))$  is a Fischer pair for  $A(B_R)$  it is not sufficient that  $(P, Q(D))$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$ . A counterexample is constructed in [56], p. 532, for the space  $A(\mathbb{R}^2)$  with the *homogeneous* polynomials  $Q(x, y) = x^2 + y^2$  and  $P_t(x, y) = y(tx - y)$  for suitable  $t \in \mathbb{R}$ .

## 5. UNIQUENESS OF DECOMPOSITIONS

It follows from Theorem 3 that  $(P_{2k} - d, \Delta^k)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  for all real numbers  $d$  if  $P_{2k}$  is non-negative. At first it might be a temptation to conjecture that  $(P_{2k} - d, \Delta^k)$  is a Fischer pair for  $A(B_R)$  for any  $R > 0$  and any real number  $d$ . However the following simple example shows that this far from being true: If  $d \geq R^{2k}$  then  $P := |x|^{2k} - d$  is invertible in  $A(B_R)$  and we can write for any  $f \in A(B_R)$  the following trivial and useless decomposition

$$f = u + Pq \text{ with } q := (f - u)P^{-1} \in A(B_R)$$

where  $u \in A(B_R)$  is an arbitrary harmonic function. Thus uniqueness fails, a sharp contrast to the polynomial case where existence and uniqueness are related, see Theorem 37. On the other hand, Theorem 19 in this section will show us that  $(|x|^{2k} - d, \Delta^k)$  is a Fischer pair for  $A(B_R)$  whenever  $d < R^{2k}$ .

It is also very instructive to consider the one-dimensional case: then  $\Delta^k$  is just the operator  $\frac{d^{2k}}{dx^{2k}}$  and a function  $\varphi$  is in  $A(B_R)$  if and only if its power series converges in  $\{x \in \mathbb{R} : |x| < R\}$ .

**Theorem 14.** *Let  $n = 1$  and let  $P$  be a polynomial of exact degree  $2k$  in one variable, and let  $R > 0$ . Then  $(P, \Delta^k)$  is a Fischer pair for  $A(B_R)$  if and only if all zeros of the polynomial  $P$  in the complex plane are contained in the disk  $\{z \in \mathbb{C} : |z| < R\}$ .*

*Proof.* Let  $z_1, \dots, z_{2k}$  be the zeros of  $P$ . Suppose that there exists a zero of  $P$ , say  $z_1$ , which is not in  $\{z \in \mathbb{C} : |z| < R\}$  and that  $(P, \Delta^k)$  is a Fischer pair for  $A(B_R)$ . Then  $u(x) := (x - z_2) \dots (x - z_{2k})$  has the property that  $d^{2k}u/dx^{2k} = 0$ , so we can write  $u = 0P + u$ . On the other hand, the decomposition  $u = P\frac{1}{x-z_1} + 0$  holds since  $1/(x - z_1)$  is in  $A(B_R)$ . This contradicts to the uniqueness of the Fischer decomposition. For the converse, note that the existence of the Fischer decomposition follows from standard interpolation theory. For uniqueness, suppose that  $f = Pq_1 + u_1 = Pq_2 + u_2$ . Then  $P\varphi = u$  with  $\varphi = q_1 - q_2$  and  $u := u_2 - u_1$  and  $d^{2k}u/dx^{2k} = 0$ . It follows that  $u(z_j) = 0$  for  $j = 1, \dots, 2k$ . Since  $u$  has degree  $2k - 1$  this yields  $u = 0$ , hence  $\varphi = 0$ .  $\square$

Let us turn now to the question of uniqueness of the Fischer decomposition. The next theorem shows that a solution  $\varphi \in A(B_R)$  of the equation  $P\varphi = u$  for a function  $u$  defined in neighborhood of 0 satisfying  $\Delta^k u = 0$ , must be zero if the radius  $R$  is sufficiently large. This yields the uniqueness of Fischer decompositions in the space  $A(B_R)$  for large radii since  $f = Pq_1 + u_1 = Pq_2 + u_2$  implies that  $P\varphi = u$  with  $\varphi = q_1 - q_2$  and  $u := u_2 - u_1$  and  $\Delta^k u = 0$ .

It is instructive to consider in Theorem 15 the case  $n = 1$  again: Then  $P$  is a univariate polynomial of degree  $2k$  and it is assumed that  $\varphi$  is of the form

$$\varphi(x) = \frac{u(x)}{P(x)}$$

where  $u$  is a polynomial of degree  $< 2k$ . The conclusion of the theorem is that the convergence radius of  $\varphi$  is bounded by a number given in terms of the coefficients of the polynomial  $P$ . Note that the assumption  $\deg u < \deg P = 2k$  is essential in order to guarantee that  $\varphi \neq 0$  has a singularity.

**Theorem 15.** *Let  $P$  be polynomial of degree  $2k$ , let  $P = P_{2k} + \dots + P_0$  be its homogeneous decomposition, and assume that  $CP_{2k}(x) \geq |x|^{2k}$  for all  $x \in \mathbb{R}^n$ . Let  $E_P := \{s \in \{0, \dots, 2k - 1\} : P_s \neq 0\}$  and  $l_P$  the cardinality of  $E_P$ ; further  $\alpha$  denotes the smallest and  $\beta$  the largest element in  $E_P$ . Define*

$$D := \max_{s=0, \dots, 2k-1} \max_{\theta \in \mathbb{S}^{n-1}} |P_s(\theta)|.$$

*If  $\varphi \in A(B_R)$ ,  $\varphi \neq 0$ , is a solution of the equation  $P\varphi = u$  in a neighborhood of zero for some real analytic function  $u$  with  $\Delta^k u = 0$  then*

$$R^\gamma \leq l_P CD \text{ for some } \gamma \text{ with } 2k - \beta \leq \gamma \leq 2k - \alpha.$$

*Proof.* Using the technique provided in (14) we obtain for any homogeneous polynomial  $\varphi_{m+j}$  of degree  $m+j$

$$\|P_{2k-j}\varphi_{m+j}\|^2 \leq \frac{I_{4k+2m+n-1}}{I_{2m+2j+n-1}} \max_{\theta \in \mathbb{S}^{n-1}} |P_{2k-j}(\theta)|^2 \|\varphi_{m+j}\|^2.$$

Further by (12) for  $j = 1, \dots, 2k$

$$\frac{I_{4k+2m+n-1}}{I_{2m+2j+n-1}} \leq \frac{(m+n^*+2k)!}{(m+j+n^*)!} \leq (m+n^*+2k)^{2k-j}.$$

Hence we have the estimate

$$(28) \quad \|P_{2k-j}\varphi_{m+j}\| \leq D_{2k-j} (m+n^*+2k)^{k-\frac{1}{2}j} \|\varphi_{m+j}\|.$$

The equation  $P\varphi = u$ , valid in a neighborhood of 0, implies that  $P_{2k}\varphi_m + \dots + P_0\varphi_{m+2k} = u_{m+2k}$  where we have written  $\varphi = \sum_{m=0}^{\infty} \varphi_m$  and  $u = \sum_{m=0}^{\infty} u_m$ , and  $\varphi_m$  and  $u_m$  are homogeneous polynomials of degree  $m$ . Since  $\varphi \neq 0$  there exists a natural number  $m$  with  $\|\varphi_m\| \neq 0$ . Note that  $\Delta^k u = 0$  implies that  $\Delta^k u_{m+2k} = 0$ . By Theorem 2 we conclude that  $\langle u_{m+2k}, \varphi_m \rangle = 0$ , and therefore  $\langle P_{2k}\varphi_m + \dots + P_0\varphi_{m+2k}, \varphi_m \rangle = 0$ . It follows that  $|\langle P_{2k}\varphi_m, \varphi_m \rangle| \leq \sum_{j=1}^{2k} |\langle P_{2k-j}\varphi_{m+j}, \varphi_m \rangle|$ . The Cauchy-Schwarz inequality and division by  $\|\varphi_m\|$  implies

$$\frac{|\langle P_{2k}\varphi_m, \varphi_m \rangle|}{\|\varphi_m\|} \leq \sum_{j=1}^{2k} \|P_{2k-j}\varphi_{m+j}\|.$$

On the other hand,

$$C \langle P_{2k}\varphi_m, \varphi_m \rangle \geq \langle |x|^{2k} \varphi_m, \varphi_m \rangle = \frac{I_{2k+2m+n-1}}{I_{2m+n-1}} \|\varphi_m\|^2.$$

Hence the last two equations imply that

$$(29) \quad C^{-1} \frac{I_{2k+2m+n-1}}{I_{2m+n-1}} \|\varphi_m\| \leq \sum_{j=1}^{2k} \|P_{2k-j}\varphi_{m+j}\|.$$

By (12), for  $m \geq 1$

$$(30) \quad \frac{I_{2k+2m+n-1}}{I_{2m+n-1}} \geq \frac{(k+m+n^*-1)!}{(m+n^*-1)!} \geq (m+n^*)^k \geq m^k.$$

With (29) and (30) and (28) we arrive at the inequality

$$(31) \quad C^{-1} m^k \|\varphi_m\| \leq \sum_{j=1}^{2k} (m+n^*+2k)^{k-\frac{1}{2}j} D_{2k-j} \|\varphi_{m+j}\|.$$

Note that the last sum has exactly  $l_P$  summands. Further we can estimate  $D_{2k-j} \leq D$  for the coefficients with  $D_{2k-j} \neq 0$ . For an inequality of the type  $a \leq b_1 + \dots + b_l$  with positive entries it is trivial to conclude that there must exist an index  $s \in \{1, \dots, l\}$  such

that  $a \leq lb_s$ . Thus for given  $m$  there exists  $m_1 > m$  and  $m_1 \leq m + 2k$  (writing  $m_1 = m + j$  with  $0 < j \leq 2k$ ) such that

$$C^{-1}m^k \|\varphi_m\| \leq Dl_P (m + n^* + 2k)^{k - \frac{1}{2}(m_1 - m)} \|\varphi_{m_1}\|.$$

Repeating this argument for  $\varphi_{m_1}$  (note that  $\|\varphi_{m_1}\| \neq 0$ ), one obtains  $m_2 > m_1$  and  $m_2 \leq m_1 + 2k$  such that

$$C^{-1}m_1^k \|\varphi_{m_1}\| \leq Dl_P (m_1 + n^* + 2k)^{k - \frac{1}{2}(m_2 - m_1)} \|\varphi_{m_2}\|.$$

Let us define  $m_0 := m$ . By induction we obtain a sequence  $m_j$  for  $j = 1, 2, \dots$  such that

$$(32) \quad C^{-j}m_0^k \dots m_{j-1}^k \|\varphi_m\| \leq D^j l_P^j \|\varphi_{m_j}\| \prod_{r=1}^j (m_{r-1} + n^* + 2k)^{k - \frac{1}{2}(m_r - m_{r-1})}.$$

Note that  $m_{j-1} < m_j$ , so  $m_j \rightarrow \infty$  for  $j \rightarrow \infty$ . It is easy to see from the construction of  $m_j$  and (31) that  $m_j \leq m_{j-1} + 2k - \alpha$  and  $m_j \geq m_{j-1} + 2k - \beta$ . It follows that  $m_j \leq m + (2k - \alpha)j$  and  $m_j \geq m + j(2k - \beta)$  for all  $j$ .

Further (32) implies that

$$\frac{\|\varphi_{m_j}\|}{\sqrt{m_j!}} \geq b_j \frac{1}{(l_P CD)^j} \|\varphi_m\|$$

where we have defined

$$b_j := \frac{m_0^k m_1^k \dots m_{j-1}^k}{\sqrt{m_j!}} \prod_{r=1}^j (m_{r-1} + n^* + 2k)^{\frac{1}{2}(m_r - m_{r-1}) - k}.$$

Note that for  $j \geq 2$

$$\begin{aligned} b_j &= b_{j-1} \frac{m_{j-1}^k}{\sqrt{(m_{j-1} + 1) \dots m_j}} (m_{j-1} + n^* + 2k)^{\frac{1}{2}(m_j - m_{j-1}) - k} \\ &= b_{j-1} \frac{1}{\left(1 + \frac{n^*}{m_{j-1}} + \frac{2k}{m_{j-1}}\right)^k} \sqrt{\frac{(m_{j-1} + n^* + 2k)^{m_j - m_{j-1}}}{(m_{j-1} + 1) \dots m_j}}. \end{aligned}$$

Since  $(m_{j-1} + n^* + 2k)^{m_j - m_{j-1}} \geq (m_{j-1} + 1) \dots m_j$  we have

$$b_j \geq b_{j-1} \left(1 + \frac{n^*}{m_{j-1}} + \frac{2k}{m_{j-1}}\right)^{-k}.$$

Let now  $\varepsilon > 0$ . Then there exists  $j_0 \in \mathbb{N}$  such that  $1 + \frac{n^*}{m_{j-1}} + \frac{2k}{m_{j-1}} \leq (1 + \varepsilon)$  for all  $j \geq j_0$ .

It follows that  $b_j \geq (1 + \varepsilon)^{-k(j-j_0)} b_{j_0}$  and

$$\frac{\|\varphi_{m_j}\|}{\sqrt{m_j!}} \geq (1 + \varepsilon)^{-k(j-j_0)} b_{j_0} \frac{1}{(l_P CD)^j} \|\varphi_m\|$$

for  $j \geq j_0$ . Since  $m_j \rightarrow \infty$  it is clear that  $m_j \sqrt{\|\varphi_m\|} \rightarrow 1$ . Since  $m_j/j$  is a bounded sequence there exists a subsequence such that  $\gamma := \lim m_{j_r}/j_r$ . Clearly  $2k - \beta \leq \gamma \leq 2k - \alpha$ . Then

$$\limsup_{r \rightarrow \infty} m_{j_r} \sqrt{\frac{\|\varphi_{m_{j_r}}\|}{\sqrt{m_{j_r}!}}} \geq (1 + \varepsilon)^{-\frac{k}{\gamma}} \left( \frac{1}{l_P C D} \right)^{1/\gamma}.$$

Proposition 12 implies  $R^{-1} \geq (1 + \varepsilon)^{-k/\gamma} \left( \frac{1}{l_P C D} \right)^{1/\gamma}$ . Now let  $\varepsilon \rightarrow 0$ .  $\square$

It is instructive to consider the special case  $P = P_{2k} - 1$ . Then  $l_P = 1$  and  $D = 1$ , and  $\alpha = \beta = 0$ . Then we obtain the following result:

**Theorem 16.** *Let  $P$  be a homogeneous polynomial of degree  $2k$  such that  $CP(x) \geq |x|^{2k}$  for all  $x \in \mathbb{R}^n$ . Suppose that  $\varphi \in A(B_R)$ ,  $\varphi \neq 0$ , satisfies the equation  $(P - 1)\varphi = u$  for some function  $u$  with  $\Delta^k u = 0$  in a neighborhood of 0. Then  $R^{2k} \leq C$ .*

## 6. INHOMOGENEOUS FISCHER PAIRS

In the following it is convenient to introduce a new notation: If  $(P, Q(D))$  is a Fischer pair there exist for each polynomial  $f$  unique polynomials  $q$  and  $r$  such that  $f = Pq + r$  with  $Q(D)r = 0$ . Since the decomposition is unique we can define operators  $T_P : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[x_1, \dots, x_n]$  and  $R_P : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[x_1, \dots, x_n]$  by putting  $T_P(f) := q$  and  $R_P(f) := r$ . So we write now

$$(33) \quad f = P \cdot T_P(f) + R_P(f).$$

It is easy to see that  $T_P$  and  $R_P$  are *linear* operators. Let

$$(34) \quad P(x) = P_k - \sum_{s=0}^{k-1} P_s$$

be the decomposition into a sum of homogeneous polynomials. We consider the polynomials  $P_s$  in (34) as multiplication operators  $P_s : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[x_1, \dots, x_n]$  defined by  $P_s(f) = P_s f$ .

In the next theorem we describe the operator  $T_P$  via the simpler operator  $T := T_{P_k}$ .

**Theorem 17.** *Let  $Q$  be a homogeneous polynomial of degree  $k$  and  $P$  be a polynomial of degree  $k$  of the form (34) and assume that  $(P_k, Q(D))$  is a Fischer pair. With the above notations the identity*

$$(35) \quad T_P(f_m) = \sum_{j=-1}^m \Lambda_j(f_m)$$

holds for all homogeneous polynomials  $f_m$  of degree  $m$  where

$$(36) \quad \Lambda_j(f_m) := \sum_{s_0=0}^{k-1} \sum_{s_1=0}^{k-1} \dots \sum_{s_j=0}^{k-1} T P_{s_j} \dots T P_{s_0} T f_m$$

with the convention that  $\Lambda_{-1}(f_m) := Tf_m$

*Proof.* The proof follows by induction over the degree  $m$ .  $\square$

**Theorem 18.** *Let  $P$  a be polynomial of degree  $2k$  and  $P = P_{2k} - \dots - P_0$  be its homogeneous decomposition, and assume that  $CP_{2k}(x) \geq |x|^{2k}$  for all  $x \in \mathbb{R}^n$ . With the notations of Theorem 15 assume that  $R$  is so large such that*

$$(37) \quad l_P CD < R^\gamma \text{ for all } \gamma \text{ with } 2k - \beta \leq \gamma \leq 2k - \alpha.$$

Then  $(P, \Delta^k)$  is a Fischer pair for  $A(B_R)$ .

*Proof.* 1. It follows from Theorem 15 that condition (37) implies the uniqueness of the Fischer decomposition. Next we show that (37) implies

$$(38) \quad CD < \frac{R^{2k}}{\sum_{s \in E_P} R^s}.$$

Recall that  $E_P := \{s \in \{0, \dots, 2k - 1\} : P_s \neq 0\}$ , and  $\alpha$  is the smallest and  $\beta$  the largest element of  $E_P$ . If  $R \geq 1$  then  $\sum_{s \in E_P} R^s \leq l_P R^\beta$ . Then

$$\frac{R^{2k}}{\sum_{s \in E_P} R^s} \geq \frac{1}{l_P} R^{2k-\beta} > CD$$

where we have used for the last inequality the condition (37) for  $\gamma = 2k - \beta$ . Similarly, if  $R < 1$ , then  $\sum_{s \in E_P} R^s \leq l_P R^\alpha$ , and as above one proves that (38) is fulfilled as well. In the sequel we shall show that (38) implies the existence of the Fischer decomposition.

2. Let  $f \in A(B_R)$ , and write  $f = \sum_{m=0}^{\infty} f_m$  with homogeneous polynomials  $f_m$  of degree  $m$ . We want to estimate the summands  $T_P(f_m)$ , and we are doing this by estimating norms of the summands occurring in (36). We may assume that  $m \geq 8k$  which clearly implies that

$$(39) \quad \frac{m + n^*}{m - 4k + n^*} \leq 2.$$

Consider the tuple  $(s_j, \dots, s_0)$  as fixed and define

$$(40) \quad d_r := \deg P_{s_r} T \dots P_{s_0} T f_m = m + s_0 + \dots + s_r - 2k(r+1)$$

for  $r = 0, \dots, j$ . Note that  $d_r = d_{r-1} + s_r - 2k$ . Further  $s_r \leq \beta < 2k$  for  $r = 0, \dots, j$ . So it is clear that  $d_j < d_{j-1} < \dots < d_0$ . If  $d_j > 0$  and  $d_j - 2k$  is negative, this means that the polynomial  $T[P_{s_j} T \dots P_{s_0} T f_m]$  is the zero polynomial, so we have only to consider the case that  $d_j - 2k \geq 0$ . Further for  $l = 0, \dots, r$  one has with  $b := 2k - \beta$

$$(41) \quad d_{j-l} \geq 2k + lb,$$

which follows by induction over  $l$ , namely  $d_{j-(l+1)} = d_{j-l} - s_{j-l} + 2k \geq 2k + bl - \beta + 2k$ .

Next we shall prove that for each  $\varepsilon > 0$  there exists a constant  $A_\varepsilon > 0$  (not dependent on  $(s_j, \dots, s_0)$ ) such that for all  $m \geq 8k$

$$(42) \quad \frac{\|TP_{s_j}T\dots P_{s_0}Tf_m\|}{\sqrt{(d_j - 2k)!}} \leq 4C^{j+2}D^{j+1}(m + n^*)^k \frac{2^k A_\varepsilon (1 + \varepsilon)^{k(j+1)}}{\sqrt{(m - 4k)!}} \|f_m\|.$$

3. Let us prove (42). By the fundamental inequality (26) we have

$$\|TP_{s_j}T\dots P_{s_0}Tf_m\| \leq C \frac{I_{2d_j - 4k + n - 1}}{I_{2d_j - 2k + n - 1}} \|P_{s_j}T\dots P_{s_0}Tf_m\|.$$

Moreover

$$\|P_{s_j}T\dots P_{s_0}Tf_m\|^2 \leq D^2 \frac{I_{2d_j + n - 1}}{I_{2(d_j - s_j) + n - 1}} \|TP_{s_{j-1}}T\dots P_{s_0}Tf_m\|^2.$$

It follows that

$$\|TP_{s_j}T\dots P_{s_0}Tf_m\| \leq CDE_j \|TP_{s_{j-1}}T\dots P_{s_0}Tf_m\|$$

where

$$(43) \quad E_j := \frac{I_{2d_j - 4k + n - 1}}{I_{2d_j - 2k + n - 1}} \sqrt{\frac{I_{2d_j + n - 1}}{I_{2(d_j - s_j) + n - 1}}}.$$

Iterating this argument we arrive at the estimate

$$(44) \quad \|TP_{s_j}T\dots P_{s_0}Tf_m\| \leq C^{j+1}D^{j+1}E_j\dots E_0 \|Tf_m\|.$$

Recall that  $\|T(f_m)\| \leq 2C\|f_m\|$ , see (27). If we can prove that for all  $m \geq 8k$

$$(45) \quad B_j := \frac{E_j\dots E_0}{\sqrt{(d_j - 2k)!}} \leq (m + n^*)^k \frac{2^{k+1}A_\varepsilon (1 + \varepsilon)^{k(j+1)}}{\sqrt{(m - 4k)!}}$$

then (44) implies our claim (42).

4. Let us prove (45). Since  $d_r = d_{r-1} + s_r - 2k$  and  $2k \leq d_j < d_r < d_{r-1}$  we first observe that for  $r = 1, \dots, j$

$$(46) \quad (d_{r-1} - 2k)! \leq (d_r - 2k)! \cdot (d_{r-1} - 2k)^{2k - s_r}.$$

Note that  $d_{r-1} - 2k = d_r - s_r$ , hence (46) implies that for  $r = 1, \dots, j$

$$(47) \quad B_r = \frac{\sqrt{(d_{r-1} - 2k)!}}{\sqrt{(d_r - 2k)!}} E_r B_{r-1} \leq (d_r - s_r)^{k - \frac{1}{2}s_r} E_r B_{r-1}.$$

Using (12) we obtain for  $E_r$  defined in (43) the estimate

$$(48) \quad E_r \leq \frac{(d_r - 2k + n^* - 1)!}{(d_r - k + n^* - 1)!} (d_r + n^*)^{\frac{1}{2}s_r}.$$

Since  $(d_r + n^*)/(d_r - s_r) \geq 1$  and  $0 \leq s_r \leq 2k$  we have

$$\frac{(d_r + n^*)^{\frac{1}{2}s_r}}{(d_r - s_r)^{\frac{1}{2}s_r}} \leq \frac{(d_r + n^*)^k}{(d_r - s_r)^k}.$$

Thus (48) and the last inequality imply

$$(49) \quad (d_r - s_r)^{k - \frac{1}{2}s_r} E_r \leq (d_r + n^*)^k \frac{(d_r - 2k + n^* - 1)!}{(d_r - k + n^* - 1)!}.$$

If  $d_r > 2k$  (e.g. for all  $r = 0, \dots, j-1$ ) then  $d_r - 2k + n^* - 1 \geq n^* \geq 0$  and  $d_r - k + n^* - 1 > k + n^* - 1 \geq 0$  for all  $r = 0, \dots, j$ . Hence we can estimate

$$(50) \quad \frac{(d_r - 2k + n^* - 1)!}{(d_r - k + n^* - 1)!} \leq \frac{1}{(d_r - 2k + n^*)^k}.$$

For the special case  $d_j = 2k$  we obtain from (49) that (using  $d_j \leq d_0 \leq m$ )

$$(d_j - s_j)^{k - \frac{1}{2}s_j} E_j \leq \frac{(n^* - 1)!}{(k + n^* - 1)!} (d_j + n^*)^k \leq 2(m + n^*)^k.$$

It follows that in both cases,  $d_j = 2k$  and  $d_j > 2k$  (for the latter case use (49) and the fact that  $d_j - 2k + n^* \geq 1 + n^*$  in (50)), the estimate

$$(d_j - s_j)^{k - \frac{1}{2}s_j} E_j \leq 2(m + n^*)^k$$

holds. For  $r = 0, \dots, j-1$  we have  $d_j > 2k$ , so (47), (49) and (50) imply that

$$(51) \quad B_j \leq 2(m + n^*)^k \prod_{r=1}^{j-1} \frac{(d_r + n^*)^k}{(d_r - 2k + n^*)^k} B_0.$$

Note that  $m - 2k \leq d_0 \leq m$  by (40), so  $d_0 - 2k \geq m - 4k$ . Then the definition of  $B_0$  in (45) and the estimates (48) and (50) for  $E_0$  yields

$$(52) \quad B_0 \leq \frac{1}{\sqrt{(m - 4k)!}} \frac{(m + n^*)^{\frac{1}{2}s_0}}{(m - 4k + n^*)^k}.$$

Now we still have to estimate

$$\pi_j := \prod_{r=1}^{j-1} \frac{(d_r + n^*)^k}{(d_r - 2k + n^*)^k} = \prod_{l=1}^{j-1} \frac{(d_{j-l} + n^*)^k}{(d_{j-l} - 2k + n^*)^k}$$

Since the function  $x \mapsto (x + n^*) / (x - 2k + n^*)$  is decreasing for  $x \geq x_0 := 2k - n^*$ , and  $x_0 \leq 2k \leq d_r$  and  $d_{j-l} \geq 2k + bl$ , see (41), we obtain

$$\pi_j \leq \prod_{l=1}^{j-1} \left( \frac{2k + bl + n^*}{bl} \right)^k.$$

Let now  $\varepsilon > 0$ , and choose  $l_\varepsilon \in \mathbb{N}$  so large such that  $2k + n^* \leq \varepsilon b l_\varepsilon$ . Define

$$A_\varepsilon := \prod_{l=1}^{l_\varepsilon} \left( 1 + \frac{2k + n^*}{bl} \right)^k.$$

Then  $\pi_j \leq A_\varepsilon$  for all  $j = 0, \dots, l_\varepsilon + 1$ ; for  $j - 1 \geq l_\varepsilon$  we have

$$\pi_j \leq A_\varepsilon \cdot \prod_{l=l_\varepsilon}^{j-1} \left( 1 + \frac{2k + n^*}{bl} \right)^k \leq A_\varepsilon (1 + \varepsilon)^{k(j+1)}.$$

Then (51) and (52) show that

$$B_j \leq 2(m + n^*)^k \frac{A_\varepsilon (1 + \varepsilon)^{k(j+1)}}{\sqrt{(m - 4k)!}} \frac{(m + n^*)^{\frac{1}{2}s_0}}{(m - 4k + n^*)^k}.$$

Note that  $s_0 < 2k$  and use now (39) to estimate the last factor by  $2^k$ . Hence we obtain (45).

5. Now we want to estimate  $|TP_{s_j}T\dots P_{s_0}Tf_m(x)|$  for  $x = r\theta$  with  $\theta \in \mathbb{S}^{n-1}$  and  $0 \leq r \leq \rho < R$ . Define  $F_P(r) := \sum_{s \in E_P} r^s$ . By continuity of  $(r, \varepsilon) \mapsto F_P(r)(1 + \varepsilon)^k / r^{2k}$  and (38) one may take  $\rho < R$  so large and  $\varepsilon > 0$  so small such that

$$CD \frac{F(\rho)(1 + \varepsilon)^k}{\rho^{2k}} < 1.$$

By (15) we can estimate for  $|x| = r \leq \rho$

$$\begin{aligned} |TP_{s_j}T\dots P_{s_0}Tf_m(x)| &= r^{d_j - 2k} \max_{\theta \in \mathbb{S}^{n-1}} |TP_{s_j}T\dots P_{s_0}Tf_m(\theta)| \\ &\leq \frac{\sqrt{2\omega_{n-1}} r^{d_j - 2k} (1 + d_j - 2k)^{(n-1)/2}}{\sqrt{I_{2(d_j - 2k) + n - 1}}} \|TP_{s_j}T\dots P_{s_0}T\|. \end{aligned}$$

Note that  $d_j \leq m$ , so  $1 + d_j - 2k \leq 1 + m$ . Further  $I_{2(d_j - 2k) + n - 1} \geq \frac{1}{4} (d_j - 2k - 1)!$  by (11). Since  $1/(s-1)! \leq 2^{\frac{1+s}{s!}}$  for  $s \geq 0$ , we have

$$\frac{1}{\sqrt{I_{2(d_j - 2k) + n - 1}}} \leq 2\sqrt{2} \frac{\sqrt{1 + (d_j - 2k)}}{\sqrt{(d_j - 2k)!}} \leq 2\sqrt{2} \frac{\sqrt{1 + m}}{\sqrt{(d_j - 2k)!}}.$$

Now (42) shows that

$$(53) \quad |TP_{s_j}T\dots P_{s_0}Tf_m(r\theta)| \leq (1 + \varepsilon)^{k(j+1)} C^{j+2} D^{j+1} r^{d_j - 2k} \|f_m\| U_m.$$

where

$$(54) \quad U_m := 4\sqrt{\omega_{n-1}} (1 + m)^{n/2} (m + n^*)^k \frac{2^{k+2} A_\varepsilon}{\sqrt{(m - 4k)!}}.$$

Using (40) it follows that

$$|\Lambda_j f_m(r\theta)| \leq C \left[ (1 + \varepsilon)^k C D \right]^{j+1} r^{m-2k(j+1)} \|f_m\| U_m \sum_{s_0 \in E_P} \dots \sum_{s_j \in E_P} r^{s_0 + \dots + s_j}.$$

Since  $\sum_{s_0 \in E_P} \dots \sum_{s_j \in E_P} r^{s_0 + \dots + s_j} = [F(r)]^{j+1}$  one arrives for  $r \leq \rho$  at

$$|T_P f_m(r\theta)| \leq \sum_{j=-1}^m |\Lambda_j f_m(r\theta)| \leq \|f_m\| U_m C \rho^m H_m(\rho\theta)$$

where we have defined

$$H_m(r\theta) := \sum_{j=-1}^m \left[ \frac{(1 + \varepsilon)^k C D F(r)}{r^{2k}} \right]^{j+1}.$$

Since  $(1 + \varepsilon)^k C D F(\rho) < \rho^{2k}$  there exists a constant  $M > 0$  such that  $H_m(\rho\theta) \leq M$  for all  $\theta \in \mathbb{S}^{n-1}$  and for all  $m \in \mathbb{N}$ . Thus we have for  $0 \leq r \leq \rho$

$$\sum_{m=8k}^{\infty} |T_P f_m(r\theta)| \leq C M \sum_{m=8k}^{\infty} \rho^m \|f_m\| U_m.$$

It is easy to see that the latter sum converges since  $f \in A(B_R)$ , cf. Proposition 12 and the definition of  $U_m$ .  $\square$

The following result illustrates the last theorem:

**Theorem 19.** *Let  $D$  be a real constant and let  $P_{2k}$  be a homogeneous polynomial of degree  $2k$  such that  $C P_{2k}(x) \geq |x|^{2k}$ . If  $R^{2k} > C D$  then  $(P_{2k} - D, \Delta^k)$  is a Fischer pair for  $A(B_R)$ .*

## 7. THE ALMANZI THEOREM REVISITED

In this section we will present an Almansi-type theorem where we replace the polynomial  $|x|^2$  in formula (1) by an elliptic homogeneous polynomial  $P(x)$  of degree  $2k$ ; the coefficient functions  $h_j$  will be solutions of the equation  $\Delta^k h_j = 0$ . We start with the following observation:

**Proposition 20.** *Let  $(P, Q(D))$  be a Fischer pair for a space  $E$ , and let  $T : E \rightarrow E$  and  $R : E \rightarrow E$  be defined by the equation  $f = P T(f) + R(f)$  with  $R(f) \in E$ ,  $Q(D) R(f) = 0$ . Then for each  $f \in E$  and each natural number  $N$  there exists  $h_0, \dots, h_N \in E$  with*

$$(55) \quad f = h_0 + P h_1 + \dots + P^N h_N + P^{N+1} T^{N+1} f$$

and  $Q(D) h_j = 0$  for  $j = 0, \dots, N$ . Furthermore  $h_j = R(T^j f)$  for  $j = 0, \dots, N$ .

*Proof.* For  $N = 0$ , the statement follows from the fact that  $(P, Q(D))$  is a Fischer pair. We apply now the induction principle and assume that the statement is true for  $N$ . Since  $(P, Q(D))$  is a Fischer pair we can write  $T^{N+1}f = PT(T^{N+1}f) + R(T^{N+1}f)$ . Insert this in (55) and define  $h_{N+1} := R(T^{N+1}f)$ .  $\square$

Note that for  $E = \mathbb{C}[x_1, \dots, x_n]$  and  $f \in \mathbb{C}[x_1, \dots, x_n]$  one may take  $N_f$  so large such that  $T^{N_f+1}f = 0$  since  $T$  diminishes the degree of  $f$ . So the last proposition can be seen as a generalized Gauß decomposition.

**Theorem 21.** *Let  $0 < R \leq \infty$  and  $P$  be a homogeneous polynomial of degree  $2k$  such that  $CP(x) \geq |x|^{2k}$  for all  $x \in \mathbb{R}^n$  for some constant  $C > 0$ . Then for each  $f \in A(B_R)$  there exists a sequence of function  $h_j \in A(B_R)$ ,  $j \in \mathbb{N}_0$  such that  $\Delta^k h_j = 0$  for all  $j \in \mathbb{N}_0$  and*

$$f(x) = \sum_{j=0}^{\infty} h_j(x) P^j(x)$$

for all  $|x| < R/\sqrt[2k]{CM}$  where  $M := \max_{\theta \in \mathbb{S}^{n-1}} |P(\theta)|$ .

*Proof.* 1. We define  $h_j$  as in Proposition 20. By the formula (55) it suffices to show that  $|P^N T^N f(x)| \rightarrow 0$  for  $|x| \leq \rho < R/\sqrt[2k]{CM}$  for  $N \rightarrow \infty$ . Let us write  $f = \sum_{m=0}^{\infty} f_m$  with homogeneous polynomials  $f_m$  of degree  $m$ . Now we use results from the proof of Theorem 18. We take the tuple  $(s_0, \dots, s_j) = 0$  and  $P_{s_r} = 1$  for  $r = 0, \dots, j$ . Then  $D := 1$ ,  $d_j = m - 2k(j+1)$  and  $TP_{s_j} T \dots P_{s_0} T f_m = T^{j+2} f_m$ .

2. It is easy to see that  $CM > 1$ . Let  $\rho < R/\sqrt[2k]{CM}$  and  $r \leq \rho$ . Take  $\delta > 0$  such that  $\rho^{2k} < (R - \delta)^{2k}/CM$ , and take  $\varepsilon > 0$  and  $\varepsilon_2 > 0$  so small such that  $w := \rho(1 + \varepsilon_2)/(R - \delta) < 1$  and

$$(56) \quad (1 + \varepsilon)^k CM w^{2k} < 1.$$

By (53) there exists  $A_\varepsilon > 0$  such that for all  $m \geq 8k$

$$|T^{j+2} f_m(r\theta)| \leq (1 + \varepsilon)^{k(j+1)} C^{j+2} r^m r^{-2k(j+2)} \|f_m\| U_m.$$

From the definition of  $U_m$  in (54) we see that  $U_m \sqrt{m!} \leq C_1 m^\alpha$  for all  $m \geq \max\{n^*, 1\}$  where  $\alpha = 3k + \frac{1}{2}n$ . Since  $m^{\frac{\alpha}{m}} \rightarrow 1$  there exists  $m_0$  such that  $m^{\frac{\alpha}{m}} \leq (1 + \varepsilon_2)$  for all  $m \geq m_0$ . Since  $R^{-1} < (R - \delta)^{-1}$  and  $f \in A(B_R)$  there exists  $m_1$  such that  $\frac{\|f_m\|}{\sqrt{m!}} \leq (R - \delta)^{-m}$  for  $m \geq m_1$ . It follows that for all  $m \geq \max\{m_0, m_1\}$

$$|T^{j+2} f_m(r\theta)| \leq C_1 (1 + \varepsilon)^{k(j+1)} C^{j+2} r^m r^{-2k(j+2)} \left( \frac{1 + \varepsilon_2}{R - \delta} \right)^m.$$

Since  $|P(r\theta)| \leq Mr^{2k}$  for all  $r > 0$  and  $\theta \in \mathbb{S}^{n-1}$  it is now easy to see that for  $r \leq \rho$  and  $m \geq \max\{m_0, m_1\}$

$$|P^{j+2}(r\theta) T^{j+2} f_m(r\theta)| \leq C_1 \left[ (1 + \varepsilon)^k CM \right]^{j+2} \rho^m \left( \frac{1 + \varepsilon_2}{R - \delta} \right)^m.$$

3. Let us take now  $j$  so large such that  $2k(j+2) \geq \max\{m_0, m_1\}$ . Since  $T^{j+1}f_m = 0$  for  $m < 2k(j+2)$  it follows that

$$|P^{j+2}(r\theta)T^{j+2}f(r\theta)| \leq \sum_{m=2k(j+1)}^{\infty} |P^{j+2}(r\theta)T^{j+2}f_m(r\theta)|,$$

so the index  $m$  of each summand satisfies  $m \geq \max\{m_0, m_1\}$ . Thus we have with  $w := \rho(1 + \varepsilon_2)/(R - \delta)$

$$\begin{aligned} |P^{j+2}(r\theta)T^{j+2}f(r\theta)| &\leq C_1 \left[ (1 + \varepsilon)^k CM \right]^{j+2} \sum_{m=2k(j+2)}^{\infty} w^m \\ &= C_1 \left[ (1 + \varepsilon)^k CM w^{2k} \right]^{j+2} \frac{1}{1 - w}. \end{aligned}$$

Since  $(1 + \varepsilon)^k CM w^{2k} < 1$  by (56) we see that  $|P^{j+2}(r\theta)T^{j+2}f(r\theta)|$  converges to 0 for all  $r \leq \rho$ . The proof is complete.  $\square$

## 8. FISCHER PAIRS FOR ENTIRE FUNCTIONS

In this section we want to discuss the algebra  $A(B_R)$  from the viewpoint of complex analysis. Let us recall some notations: For  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  we define  $|z|^2 = |z_1|^2 + \dots + |z_n|^2$  and  $q(z) = z_1^2 + \dots + z_n^2$ . The set

$$\widehat{B}_R := \left\{ z \in \mathbb{C}^n : |z|^2 + \sqrt{|z|^4 - |q(z)|^2} < R^2 \right\}$$

is called the *harmonicity hull* of  $B_R$ , see [4]. The set  $\widehat{B}_R$  for  $R = 1$  is identical with the *classical domain*  $\mathcal{R}_{\text{IV}}$  of E. Cartan (see [4, p. 59]), or the *Lie ball* in [38]. It is well known that

$$(57) \quad B_{R/\sqrt{2}}^{\mathbb{C}} := \left\{ z \in \mathbb{C}^n : |z| < R/\sqrt{2} \right\} \subset \widehat{B}_R.$$

As pointed out in [59], the following equality

$$\max\{|f(z)| : z \in \widehat{B}_R\} = \max\{|f(x)| : x \in B_R\}$$

is valid for each homogeneous polynomial  $f(x)$  and the following result holds, see [59]:

**Theorem 22.** *The algebra  $A(B_R)$  is isomorphic to the space of all holomorphic function on  $\widehat{B}_R$ . In particular,  $A(\mathbb{R}^n)$  is isomorphic to  $\mathcal{E}_n$ .*

From this the following is immediate:

**Theorem 23.** *Let  $P$  be an elliptic polynomial of degree  $2k$ . Then  $(P, \Delta^k)$  is a Fischer pair for the algebra  $\mathcal{E}_n$  of all entire functions.*

## 9. HAYMAN'S CONJECTURE FOR ENTIRE FUNCTIONS

In this section we extend the results of Section 3 to polyharmonic functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ . Roughly speaking, analogous results are valid under the stronger assumption that the polynomial  $\psi$  is elliptic.

The following result is the analog of Theorem 6 for analytic functions. The result seems to be part of mathematical folklore and we refer to [31] and [62] for the necessary background in algebraic geometry.

**Theorem 24.** *Let  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  be square-free and assume that each irreducible factor  $\psi_j, j = 1, \dots, r$ , changes sign on the open set  $U_j$  in  $\mathbb{R}^n$  for  $j = 1, \dots, r$ . Suppose that  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  is an entire function which vanishes on  $\{x \in \mathbb{R}^n : \psi_j(x) = 0\} \cap U_j$  for  $j = 1, \dots, r$ . Then there exists an entire function  $q : \mathbb{C}^n \rightarrow \mathbb{C}$  such that  $f = q\psi_1 \dots \psi_r$ .*

Now let us formulate the main result of this Section:

**Theorem 25.** *Let  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  square-free and elliptic and assume that each irreducible factor  $\psi_j, j = 1, \dots, r$ , changes sign on given open sets  $U_j$  for  $j = 1, \dots, r$ . Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  has an entire extension, and that  $f$  vanishes on the sets  $\{x \in \mathbb{R}^n : \psi_j(x) = 0\} \cap U_j$  for  $j = 1, \dots, r$ . If  $\Delta^k f = 0$  with  $k = \frac{1}{2} \deg \psi$  then  $f = 0$ .*

*Proof.* Let  $\tilde{f}$  be the entire extension of  $f$ . By Theorem 24 we can write  $\tilde{f} = q\psi$  for some entire function  $q$ . Then  $f$  and the restriction of  $q$  to  $\mathbb{R}^n$  are clearly in  $A(\mathbb{R}^n)$ . Theorem 15 shows that  $q = 0$ .  $\square$

W. Hayman and B. Korenblum have constructed a non-zero biharmonic function  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  which vanishes on *infinitely many analytic Jordan curves*. So the assumption of *algebraic curves* seems to be crucial for this kind of result.

## 10. APPLICATIONS TO DIRICHLET PROBLEMS

Let  $\Omega$  be a domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega$ . A solution of the Dirichlet problem for a continuous data function  $f : \partial\Omega \rightarrow \mathbb{C}$  is any continuous function  $u : \Omega \rightarrow \mathbb{C}$  which is differentiable of order 2 on  $\Omega$  such that  $\Delta u(x) = 0$  for all  $x \in \Omega$  and  $u(\xi) = f(\xi)$  for all  $\xi \in \partial\Omega$ . If  $\Omega$  is bounded the solution is unique by the maximum principle.

The following theorem was proved in [39] for the case  $R = \infty$  by elementary methods from potential theory; here we identify the space of all entire functions with  $A(\mathbb{R}^n)$ . We give a proof (for arbitrary  $R > 0$ ) by means of the Fischer decomposition, cf. Remark 1 on p. 463 in [39].

**Theorem 26.** *Let  $\Omega$  be the ellipsoid  $\{x \in \mathbb{R}^n : \sum_{j=1}^n a_j^{-2} x_j^2 < 1\}$  with  $a_j > 0$  for  $j = 1, \dots, n$ . If  $f \in A(B_R)$  and*

$$R > \max_{j=1, \dots, n} |a_j|$$

*then the solution  $u$  of the Dirichlet problem for  $f|_{\partial\Omega}$  extends to a harmonic function on  $B_R$ .*

*Proof.* We apply Theorem 19 to  $\psi = \psi_2 - 1$  with  $\psi_2(x) := \sum_{j=1}^n a_j^{-2} x_j^2$ . Then  $C\psi_{2k}(x) \geq |x|^2$  with  $C := \max_{j=1, \dots, n} |a_j|^2$ . Then there exists  $q, u \in A(B_R)$  such that  $f = (\psi_2 - 1)q + u$  and  $\Delta u(x) = 0$  for all  $x \in B_R$ . Since  $\psi(x) = 0$  for  $x \in \partial\Omega$ , it follows that  $f(x) = u(x)$  for all  $x \in \partial\Omega$ . The proof is complete.  $\square$

Since  $(\psi, \Delta)$  is a *degree-preserving* Fischer pair for the ellipsoid  $\Omega$  the following well-known property is easily proved: if  $f \in \mathbb{C}[x_1, \dots, x_n]$  has degree  $\leq m$  then the solution  $u$  of the Dirichlet problem for the data function  $f|_{\partial\Omega}$  is a polynomial of degree  $m$ , see also [7]. It was proved in [39], p. 467, that this property characterizes ellipsoids in the class of all bounded domains  $\Omega$  whose boundary  $\partial\Omega$  is an irreducible algebraic set, see also [18], [33], [34] and the recent developments in [22], [23] for rational or algebraic data functions. We now prove the following stronger result:

**Theorem 27.** *Let  $\psi_1, \dots, \psi_r \in \mathbb{R}[x_1, \dots, x_n]$  be irreducible and assume that  $\psi = \psi_1 \dots \psi_r$  is square-free and nonhyperbolic. Let  $\Omega$  be a domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega$  and assume that for each  $j = 1, \dots, r$  there exists an open set  $U_j$  such that*

$$(58) \quad \{y \in \mathbb{R}^n : \psi_j(y) = 0\} \cap U_j \subset \partial\Omega$$

*and  $\psi_j$  changes sign on  $U_j$ . If  $\deg \psi > 2$  then there is no polynomial solution of the Dirichlet problem for the data function  $|x|^2$  restricted to  $\partial\Omega$ .*

*Proof.* Since  $\psi$  is nonhyperbolic the degree of  $\psi$  must be even, say  $2l$ . Since  $\deg \psi > 2$  we know that  $l > 1$ . Suppose that there exists a harmonic polynomial  $u$  which is equal to  $f(x) := |x|^2$  on  $\partial\Omega$ . It follows that  $f - u$  vanishes on  $Z_{\psi_j} \cap U_j$  for each  $j = 1, \dots, r$ . By Theorem 6 there exists a polynomial  $q$  with  $f - u = \psi q$ . Further  $f - u \neq 0$  since  $f$  is not harmonic. We conclude that  $q \neq 0$ . Since  $l > 1$  we have that  $\Delta^l f = \Delta^l |x|^2 = 0$ , so  $\Delta^l(\psi q) = 0$ . On the other hand, Theorem 3 implies that  $0 \neq \Delta^l(\psi q)$ . This contradiction proves the theorem.  $\square$

As an example, consider the square  $\Omega := (-1, 1) \times (-1, 1)$  in  $\mathbb{R}^2$  and  $\psi(x, y) = (x - 1)(x + 1)(y - 1)(y + 1)$ . Clearly  $\psi$  is square-free and nonhyperbolic, and condition (58) is satisfied. Since  $\deg \psi = 4$  it follows that the solution of the Dirichlet problem for the data function  $x^2 + y^2$  can not be a polynomial.

Now we turn to Problem (I) mentioned in the introduction:

**Theorem 28.** *Let  $\psi_1, \dots, \psi_r \in \mathbb{R}[x_1, \dots, x_n]$  be irreducible and assume that  $\psi = \psi_1 \dots \psi_r$  is square-free and elliptic. Let  $\Omega$  be a domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega$  and assume that for each  $j = 1, \dots, r$  there exists an open set  $U_j$  such that*

$$(59) \quad \{y \in \mathbb{R}^n : \psi_j(y) = 0\} \cap U_j \subset \partial\Omega$$

*and  $\psi_j$  changes sign on  $U_j$ . If  $\deg \psi > 2$  then there is no entire solution of the Dirichlet problem for the data function  $|x|^2$  restricted to  $\partial\Omega$ .*

*Proof.* The proof of Theorem 27 carries over verbatim if one uses now Theorem 24 and 15.  $\square$

The following *variation of the Dirichlet problem* was discussed in [15] (for the case  $n = 2$ ): for  $f \in \mathbb{C}[x_1, \dots, x_n]$  find a differentiable function  $u : \mathbb{R}^n \rightarrow \mathbb{C}$  of order 2 such that

$$(60) \quad \Delta u(x) = 0 \text{ for all } x \in \mathbb{R}^n, \text{ and } u(\xi) = f(\xi) \text{ for all } \xi \in Z_\psi$$

where  $Z_\psi = \{x \in \mathbb{R}^n : \psi(x) = 0\}$  and  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  is square-free and non-degenerate. Note that  $\mathbb{R}^n \setminus Z_\psi$  is in general the union of several disjoint connected components, so  $\mathbb{R}^n \setminus Z_\psi$  is not the boundary of a domain  $\Omega$ . It was asked in [15] which polynomials  $\psi$  solve the following **basic problem**: For any polynomial  $f$  there exists a polynomial solution  $u$  of (60). It was shown in [15] for  $n = 2$  that the only bounded sets  $Z_\psi$  satisfying the basic problem are the ellipses. Now we generalize this result to arbitrary dimension.

**Theorem 29.** *Let  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  be square-free and non-degenerate, and assume that  $Z_\psi$  is compact. Then the following statements are equivalent:*

- a)  $\psi$  is an elliptic polynomial of degree 2.
- b)  $(\psi, \Delta)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$
- c) For every polynomial  $f$  there exists a polynomial solution  $u$  of (60).
- d) For  $f(x) = |x|^2$  there exists a polynomial solution  $u$  of (60).

*Proof.* The implication a)  $\rightarrow$  b) follows from Theorem 3. Assume now b), and let  $f$  be a polynomial. Then we can find polynomials  $q, u$  such that  $f = \psi q + u$  and  $\Delta u = 0$ . Clearly one has that  $f(\xi) = u(\xi)$  for all  $\xi \in Z_\psi$ . The implication c)  $\rightarrow$  d) is trivial. Let us assume now d). Since  $Z_\psi$  is bounded the principal part of  $\psi$  or  $-\psi$  is non-negative, see Lemma 30. So we may assume that  $\psi$  is nonhyperbolic, and the degree of  $\psi$  must be even, say  $2l$ . Now we argue as in the proof of Theorem 27: Suppose that  $\deg \psi > 2$ , then  $l > 1$ . By assumption d) there exists a harmonic polynomial  $u$  which is equal to  $f(x) := |x|^2$  on  $Z_\psi$ . It follows that  $f - u$  vanishes on  $Z_\psi$ . By Theorem 6 there exists a polynomial  $q$  with  $f - u = \psi q$ . Further  $f - u \neq 0$  since  $f$  is not harmonic, so  $q \neq 0$ . Since  $\Delta^l f = 0$  (recall that  $l > 1$ ) we obtain that  $\Delta^l(\psi q) = 0$ . Theorem 3 implies that  $0 \neq \Delta^l(\psi q)$ . This contradiction shows that  $\deg \psi \leq 2$ . If  $\deg \psi \leq 1$  it is clear that  $Z_\psi$  is non-compact. Hence  $\deg \psi = 2$ . Using the compactness of  $Z_\psi$  it is easy to see that  $\psi$  is elliptic.  $\square$

**Lemma 30.** *Let  $P_0, \dots, P_k$  be continuous real-valued functions on the sphere  $\mathbb{S}^{n-1}$  and define  $P(r\theta) := r^k P_k(\theta) + \dots + P_0(\theta)$ . If there exists  $\theta_1, \theta_2 \in \mathbb{S}^{n-1}$  with  $P_k(\theta_1) < 0 < P_k(\theta_2)$  then the set  $\{(r, \theta) : P(r\theta) = 0\}$  is not bounded.*

*Proof.* Left to the reader.  $\square$

Now we consider the polyharmonic equation  $\Delta^k u = 0$ . In analogy to the last results define the following problem: for  $f \in \mathbb{C}[x_1, \dots, x_n]$  find a differentiable function  $u : \mathbb{R}^n \rightarrow \mathbb{C}$  of order  $2k$  such that

$$(61) \quad \Delta^k u(x) = 0 \text{ for all } x \in \mathbb{R}^n, \text{ and } u(\xi) = f(\xi) \text{ for all } \xi \in Z_\psi$$

where  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  is square-free and non-degenerate. Then one can prove with the same methods:

**Theorem 31.** *Let  $k$  be a natural number and assume that  $\psi \in \mathbb{R}[x_1, \dots, x_n]$  is square-free, non-degenerate and nonhyperbolic of degree  $\geq 2k$ . Then the following statements are equivalent:*

- a)  $\psi$  is an elliptic polynomial of degree  $2k$ .
- b)  $(\psi, \Delta^k)$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$
- c) For every polynomial  $f$  there exists a polynomial solution  $u$  of (61).
- d) For  $f(x) = |x|^{2k}$  there exists a polynomial solution  $u$  of (61).

The Dirichlet problem  $\Delta^k u = 0$  of order  $k$  for a domain  $\Omega$  is defined in [58] in the following way: Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be  $k - 1$  times continuously differentiable. Then a function  $u : \bar{\Omega} \rightarrow \mathbb{C}$  is called a solution of the Dirichlet problem for the data function  $f$  if  $u$  is  $k - 1$  times continuously differentiable on  $\bar{\Omega}$  and  $2k$  times continuously differentiable on  $\Omega$  such that  $\Delta^k u(x) = 0$  for all  $x \in \Omega$  and

$$(62) \quad \frac{\partial^\alpha}{\partial x^\alpha} u(x) = \frac{\partial^\alpha}{\partial x^\alpha} f(x) \text{ for all } x \in \partial\Omega \text{ and } |\alpha| \leq k - 1.$$

If the boundary  $\partial\Omega$  is sufficiently smooth then the Dirichlet problem has a unique solution  $u$  for the data function  $f$ , see [58]. For a constructive solution of this problem in case of concentric spheres we refer to [26].

Consider as an example the domain  $\Omega := \{x \in \mathbb{R}^n : \psi_{2k}(x) < 1\}$  for the polynomial  $\psi_{2k}(x) = x_1^{2k} + \dots + x_n^{2k}$ . Theorem 31 shows that for any polynomial  $f$  there exists a unique polyharmonic *polynomial*  $u$  of order  $k$  such that  $u(x) = f(x)$  for all  $x \in \partial\Omega$ . On the other hand, the solution of  $\Delta^k u = f$  for a polyharmonic *function*  $u : \Omega \rightarrow \mathbb{C}$  is only unique if we specify boundary conditions as in (62). Indeed, the polynomial solution  $u$  satisfies the equation  $f(x) = (\psi_{2k} - 1)q(x) + u(x)$  for all  $x \in \mathbb{R}^n$ , and from this one can determine the partial derivatives  $\frac{\partial^\alpha}{\partial x^\alpha} u(x)$ .

An analogous result holds for entire functions by using Theorem 18: for any function  $f \in A(\mathbb{R}^n)$  there exists a *unique* polyharmonic function  $r$  defined on  $\mathbb{R}^n$  of order  $k$  such that  $f(x) = r(x)$  for all  $x \in \partial\Omega$ .

## 11. APPENDIX: FISCHER PAIRS FOR $\mathbb{C}[x_1, \dots, x_n]$

A Fischer pair  $(P, Q(D))$  for  $\mathbb{C}[x_1, \dots, x_n]$  is called *degree preserving* if  $\deg r \leq \deg f$  in (2). By  $\mathcal{P}_n(k)$  we denote the set of all homogeneous polynomials of degree  $k$  with complex coefficients. If  $Q \in \mathcal{P}_n(k)$  it is easy to see that  $Q(D)$  maps  $\mathcal{P}_n(m)$  into  $\{0\}$  for  $m < k$  and into  $\mathcal{P}_n(m - k)$  for  $m \geq k$ . We first cite from p. 168 in [17]:

**Proposition 32.** *If  $Q \in \mathcal{P}_n(k)$  then the map  $Q(D) : \mathcal{P}_n(m) \rightarrow \mathcal{P}_n(m - k)$  is surjective for all  $m \geq k$ . In particular, the map  $Q(D) : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[x_1, \dots, x_n]$  is surjective.*

The next result, due to A. Meril and D.C. Struppa in [49], gives an operator-theoretic characterization of Fischer's pairs:

**Theorem 33.** *Let  $P, Q$  be arbitrary polynomials and let  $E$  either be  $\mathbb{C}[x_1, \dots, x_n]$  or  $\mathcal{E}_n$ . Then  $(P, Q(D))$  is a Fischer pair for  $E$  if and only if the "Fischer operator"  $F_{Q,P} : E \rightarrow E$  defined by  $F_{Q,P}(q) := Q(D)(P \cdot q)$  is a bijection.*

The following example, taken from [49], is very instructive: it shows that the degrees of  $P$  and  $Q$  of a Fischer pair  $(P, Q(D))$  may be different. Moreover in this example the reversed pair  $(Q, P(D))$  is not a Fischer pair (see [49]) showing that the notion of a Fischer pair is not symmetric.

**Example 34.** *Let  $n = 2$  and  $P(x, y) = x - y^2$  and  $Q(x, y) = x$ . Then each  $f \in \mathbb{C}[x, y]$  can be uniquely written in the form  $f = (x - y^2)q + r$  for polynomials  $q$  and  $r$  with  $Q(D)r = 0$ . The function  $r$  is given by  $r(x, y) := f(y^2, y)$ , so it satisfies  $\frac{\partial}{\partial x}r(x, y) = 0$ . It follows that  $(P, Q(D))$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  which is not degree preserving.*

The following lemma is part of mathematical folklore, and we include the proof only for completeness.

**Lemma 35.** *Let  $(P, Q(D))$  be Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  and  $Q$  homogeneous. Then  $\deg P \geq \deg Q$ .*

*Proof.* If  $\deg P < \deg Q$  it follows that  $Q(D)(P) = 0$  since  $Q$  is homogeneous. Then we can write  $P = P \cdot 1 + 0$  and  $P = P \cdot 0 + r$  where  $r = P$  satisfies  $Q(D)r = 0$ . This contradicts to the uniqueness property of a Fischer pair.  $\square$

**Theorem 36.** *Let  $(P, Q(D))$  be a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$  and  $P, Q \neq 0$  homogeneous. Then  $\deg P = \deg Q$ , and there exists a constant  $c \neq 0$  and a polynomial  $r$  such that  $Q^* = cP + r$  and  $Q(D)r = 0$ .*

*Proof.* Let us write  $Q^* = Pq + r$  for some polynomials  $q$  and  $r$  with  $Q(D)r = 0$ . Since  $Q$  is homogeneous it is clear that  $Q(D)(Q^*)$  is a positive constant, say  $c$ . Thus  $c = Q(D)(Pq)$  and therefore  $q \neq 0$ . Suppose that  $\deg(Pq) > \deg Q$ . Then  $Pq$  has the principal part  $Pq_m$  where  $q_m$  is the principal part of  $q$  (note that  $P$  is homogeneous). Since  $Q$  is homogeneous, either  $Q(D)(Pq_m) = 0$  or it is a polynomial of degree  $\deg(Pq_m) - \deg Q > 0$ . The latter is not possible since  $Q(D)(Pq)$  is a constant. Now  $Q(D)(Pq_m) = 0$  implies that  $(P, Q(D))$  is not a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$ . This contradiction yields  $\deg Pq \leq \deg Q$ . Hence  $\deg P \leq \deg Q$ , and by Proposition 35  $\deg P = \deg Q$ , so  $\deg q = 0$ .  $\square$

Now we give equivalent characterizations for a Fischer pair  $(P, Q(D))$ . By the above example, the homogeneity is a crucial assumption in the following theorem:

**Theorem 37.** *Let  $P, Q$  be homogeneous polynomials of the same degree  $k$ . Then the following statements are equivalent:*

- a)  $(P, Q(D))$  is a degree preserving Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$

- b)  $(P, Q(D))$  is a Fischer pair for  $\mathbb{C}[x_1, \dots, x_n]$ .
- c) For  $f \in \mathbb{C}[x_1, \dots, x_n]$  there exist  $q, r \in \mathbb{C}[x_1, \dots, x_n]$  with  $f = Pq + r$  and  $Q(D)r = 0$ .
- d) The map  $F_{Q,P,m} : \mathcal{P}_n(m) \rightarrow \mathcal{P}_n(m)$  defined by  $F_{Q,P,m}(q) := Q(D)(P \cdot q)$  is a bijection for any  $m \in \mathbb{N}_0$ .
- e)  $Q(D)(P \cdot q) \neq 0$  for any non-zero homogeneous polynomial  $q$ .

*Proof.* The implications  $a) \rightarrow b) \rightarrow c)$  are trivial. Assume that  $c)$  holds. Let  $f \in \mathcal{P}_n(m)$  with  $m \in \mathbb{N}_0$ . Let  $g \in \mathcal{P}_n(m+k)$  with  $Q(D)g = f$ . By property  $c)$  there exist polynomials  $q$  and  $r$  such that  $g = qP + r$  and  $Q(D)r = 0$ . Write  $q = \sum_{l=0}^N q_l$  and  $r = \sum_{l=0}^M r_l$  where  $q_l$  and  $r_l$  are homogeneous polynomials of degree  $l$ . Since  $P$  is homogeneous of degree  $k$  we obtain

$$t^{m+k}g(x) = g(tx) = \sum_{l=0}^N t^{k+l}q_l(x)P(x) + \sum_{l=0}^M t^l r_l(x).$$

Clearly this implies  $q_l P + r_{l+k} = 0$  for  $l \neq m$ . Hence  $g = q_m P + r_{m+k}$ . It follows that  $f = Q(D)g = Q(D)(q_m P)$ , so  $F_{Q,P,m}$  is surjective, hence bijective. The equivalence of  $d)$  and  $e)$  is evident, so we only have to prove  $d) \rightarrow a)$ . Clearly  $d)$  implies that  $F_{Q,P}$  is surjective. Suppose that  $F_{Q,P}(q) = 0$  for some polynomial  $q$ . Then  $0 = qP + r$  for some polynomial  $r$  with  $Q(D)r = 0$ . Using the homogeneity argument given in the proof of  $c) \rightarrow d)$  one obtains that  $0 = q_m P + r_{m+k}$  for the homogeneous polynomials  $q_m$  and  $r_m$ . Then  $q_m = 0$  by assumption  $d)$  and we conclude that  $q = 0$ . By Theorem 33  $(P, Q(D))$  is a Fischer pair, and it is degree preserving by property  $d)$ .  $\square$

The extension of the algorithm for polynomials of the form  $P - h$  when  $\deg h < \deg P$  is straightforward: the existence of a decomposition is proved by induction over the total degree, the uniqueness depends on a homogeneity argument. The details are omitted.

**Theorem 38.** *Let  $(P, Q(D))$  be a Fischer pair for homogeneous polynomials  $P, Q$  of the same degree  $k \geq 1$ . Let  $h$  be a polynomial with  $\deg h < \deg P$ . Then  $(P - h, Q(D))$  is a degree preserving Fischer pair.*

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