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ON SUMS OF THREE SQUARES

S.K.K. CHOI, A.V. KUMCHEV, AND R. OSBURN

ABSTRACT. Let $r_3(n)$ be the number of representations of a positive integer n as a sum of three squares of integers. We give two distinct proofs of a conjecture of Wagon concerning the asymptotic value of the mean square of $r_3(n)$.

1. INTRODUCTION

Problems concerning sums of three squares have a rich history. It is a classical result of Gauss that

$$n = x_1^2 + x_2^2 + x_3^2$$

has a solution in integers if and only if n is not of the form $4^a(8k + 7)$ with $a, k \in \mathbb{Z}$. Let $r_3(n)$ be the number of representations of n as a sum of three squares (counting signs and order). It was conjectured by Hardy and proved by Bateman [1] that

$$(1) \quad r_3(n) = 4\pi n^{1/2} \mathfrak{S}_3(n),$$

where the singular series $\mathfrak{S}_3(n)$ is given by (16) with $Q = \infty$.

While in principle this exact formula can be used to answer almost any question concerning $r_3(n)$, the ensuing calculations can be tricky because of the slow convergence of the singular series $\mathfrak{S}_3(n)$. Thus, one often sidesteps (1) and attacks problems involving $r_3(n)$ directly. For example, concerning the mean value of $r_3(n)$, one can adapt the method of solution of the circle problem to obtain the following

$$\sum_{n \leq x} r_3(n) \sim \frac{4}{3} \pi x^{3/2}.$$

Moreover, such a direct approach enables us to bound the error term in this asymptotic formula. An application of a result of Landau [9, pp. 200–218] yields

$$\sum_{n \leq x} r_3(n) = \frac{4}{3} \pi x^{3/2} + O(x^{3/4+\epsilon})$$

for all $\epsilon > 0$, and subsequent improvements on the error term have been obtained by Vinogradov [19], Chamizo and Iwaniec [3], and Heath-Brown [6].

In this note we consider the mean square of $r_3(n)$. Crandall and Wagon [4] (see also [2]) conjectured the following asymptotic formula.

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Theorem. Let $r_3(n)$ be the number of representations of a positive integer n as a sum of three squares of integers. Then

$$(2) \quad \sum_{n \leq x} r_3(n)^2 \sim \frac{8\pi^4}{21\zeta(3)} x^2.$$

Apparently, at the time they proposed this conjecture Crandall and Wagon were unaware of the earlier work of Müller [11, 12]. He obtained a more general result which, in a special case, gives

$$\sum_{n \leq x} r_3(n)^2 = Bx^2 + O(x^{14/9}),$$

where B is a constant. However, since in Müller's work B arises as a specialization of a more general (and more complicated) quantity, it is not immediately clear that $B = \frac{8}{21}\pi^4/\zeta(3)$. The purpose of this paper is to give two distinct proofs of this fact: one that evaluates B in the form given by Müller and a direct proof using the Hardy–Littlewood circle method.

2. A DIRECT PROOF: THE CIRCLE METHOD

Our first proof exploits the observation that the left side of (2) counts solutions of the equation

$$m_1^2 + m_2^2 + m_3^2 = m_4^2 + m_5^2 + m_6^2$$

in integers m_1, \dots, m_6 with $|m_j| \leq x$. This is exactly the kind of problem that the circle method was designed for.

Set $N = \sqrt{x}$ and define

$$f(\alpha) = \sum_{m \leq N} e(\alpha m^2),$$

where $e(z) = e^{2\pi iz}$. Then for an integer $n \leq x$, the number $r^*(n)$ of representations of n as a sum of three squares of *positive* integers is

$$r^*(n) = \int_0^1 f(\alpha)^3 e(-\alpha n) d\alpha.$$

Since $r_3(n) = 8r^*(n) + O(r_2(n))$, where $r_2(n)$ is the number of representations of n as a sum of two squares, we have

$$(3) \quad \sum_{n \leq x} r_3(n)^2 = 64 \sum_{n \leq x} r^*(n)^2 + O(x^{3/2+\epsilon}).$$

Therefore, it suffices to evaluate the mean square of $r^*(n)$. Let

$$P = N/4 \quad \text{and} \quad Q = N^{1/2}.$$

We introduce the sets

$$\mathfrak{M}(q, a) = \{ \alpha \in [PN^{-2}, 1 + PN^{-2}] : |q\alpha - a| \leq PN^{-2} \}$$

and

$$\mathfrak{M} = \bigcup_{q \leq Q} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = [PN^{-2}, 1 + PN^{-2}] \setminus \mathfrak{M}.$$

We have

$$(4) \quad \begin{aligned} r^*(n) &= \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right) f(\alpha)^3 e(-\alpha n) d\alpha \\ &= r^*(n, \mathfrak{M}) + r^*(n, \mathfrak{m}), \quad \text{say.} \end{aligned}$$

We now proceed to approximate the mean square of $r^*(n)$ by that of $r^*(n, \mathfrak{M})$. By (4) and Cauchy's inequality,

$$(5) \quad \sum_{n \leq x} r^*(n)^2 = \sum_{n \leq x} r^*(n, \mathfrak{M})^2 + O((\Sigma_1 \Sigma_2)^{1/2} + \Sigma_2),$$

where

$$\Sigma_1 = \sum_{n \leq x} |r^*(n, \mathfrak{M})|^2, \quad \Sigma_2 = \sum_{n \leq x} |r^*(n, \mathfrak{m})|^2.$$

By Bessel's inequality,

$$(6) \quad |\Sigma_2| = \sum_{n \leq x} \left| \int_{\mathfrak{m}} f(\alpha)^3 e(-\alpha n) d\alpha \right|^2 \leq \int_{\mathfrak{m}} |f(\alpha)|^6 d\alpha.$$

By Dirichlet's theorem of diophantine approximation, we can write any real α as $\alpha = a/q + \beta$, where

$$1 \leq q \leq N^2 P^{-1}, \quad (a, q) = 1, \quad |\beta| \leq P/(qN^2).$$

When $\alpha \in \mathfrak{m}$, we have $q \geq Q$, and hence Weyl's inequality (see Vaughan [18, Lemma 2.4]) yields

$$(7) \quad |f(\alpha)| \ll N^{1+\epsilon} (q^{-1} + N^{-1} + qN^{-2})^{1/2} \ll N^{1+\epsilon} Q^{-1/2}.$$

Furthermore, we have

$$(8) \quad \int_0^1 |f(\alpha)|^4 d\alpha \ll N^{2+\epsilon},$$

because the integral on the right equals the number of solutions of

$$m_1^2 + m_2^2 = m_3^2 + m_4^2$$

in integers $m_1, \dots, m_4 \leq N$. For each choice of m_1 and m_2 , this equation has $\ll N^\epsilon$ solutions. Combining (6)–(8) and replacing ϵ by $\epsilon/3$, we obtain

$$(9) \quad \Sigma_2 \ll N^{4+\epsilon} Q^{-1}.$$

Furthermore, another appeal to Bessel's inequality and appeals to (8) and to the trivial estimate $|f(\alpha)| \leq N$ yield

$$(10) \quad \Sigma_1 \leq \int_{\mathfrak{M}} |f(\alpha)|^6 d\alpha \leq \int_0^1 |f(\alpha)|^6 d\alpha \ll N^{4+\epsilon}.$$

We now define a function f^* on \mathfrak{M} by setting

$$f^*(\alpha) = q^{-1} S(q, a) v(\alpha - a/q) \quad \text{for } \alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M};$$

here

$$S(q, a) = \sum_{1 \leq h \leq q} e(ah^2/q), \quad v(\beta) = \frac{1}{2} \sum_{m \leq x} m^{-1/2} e(\beta m).$$

Our next goal is to approximate the mean square of $r^*(n, \mathfrak{M})$ by the mean square of the integral

$$R^*(n) = \int_{\mathfrak{M}} f^*(\alpha)^3 e(-an) d\alpha.$$

Similarly to (5),

$$(11) \quad \sum_{n \leq x} r^*(n, \mathfrak{M})^2 = \sum_{n \leq x} R^*(n)^2 + O(\Sigma_3 + (\Sigma_1 \Sigma_3)^{1/2}),$$

where

$$(12) \quad \Sigma_3 = \sum_{n \leq x} \left| \int_{\mathfrak{M}} [f(\alpha)^3 - f^*(\alpha)^3] e(-an) d\alpha \right|^2 \leq \int_{\mathfrak{M}} |f(\alpha)^3 - f^*(\alpha)^3|^2 d\alpha,$$

after yet another appeal to Bessel's inequality. By [18, Theorem 4.1], when $\alpha \in \mathfrak{M}(q, a)$,

$$f(\alpha) = f^*(\alpha) + O(q^{1/2+\epsilon}).$$

Thus,

$$\int_{\mathfrak{M}(q,a)} |f(\alpha)^3 - f^*(\alpha)^3|^2 d\alpha \ll q^{1+2\epsilon} \int_{\mathfrak{M}(q,a)} (|f(\alpha)|^4 + q^{2+4\epsilon}) d\alpha,$$

whence

$$\int_{\mathfrak{M}} |f(\alpha)^3 - f^*(\alpha)^3|^2 d\alpha \ll Q^{1+2\epsilon} \int_0^1 |f(\alpha)|^4 d\alpha + PQ^{4+6\epsilon}N^{-2}.$$

Bounding the last integral using (8) and substituting the ensuing estimate into (12), we obtain

$$(13) \quad \Sigma_3 \ll QN^{2+2\epsilon} + PQ^4N^{-2+3\epsilon} \ll QN^{2+2\epsilon}.$$

Combining (5), (9)–(11), and (13), we deduce that

$$(14) \quad \sum_{n \leq x} r^*(n)^2 = \sum_{n \leq x} R^*(n)^2 + O(N^{4+\epsilon}Q^{-1/2} + N^{3+\epsilon}Q^{1/2}).$$

We now proceed to evaluate the main term in (14). We have

$$\int_{\mathfrak{M}(q,a)} f^*(\alpha)^3 e(-an) d\alpha = q^{-3}S(q, a)^3 e(-an/q) \int_{\mathfrak{M}(q,0)} v(\beta)^3 e(-\beta n) d\beta,$$

so

$$R^*(n) = \sum_{q \leq Q} A(q, n)I(q, n),$$

where

$$A(q, n) = \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} q^{-3}S(q, a)^3 e(-an/q), \quad I(q, n) = \int_{\mathfrak{M}(q,0)} v(\beta)^3 e(-\beta n) d\beta.$$

Hence,

$$(15) \quad \sum_{n \leq x} R^*(n)^2 = \sum_{n \leq x} I(n)^2 \mathfrak{S}_3(n, Q)^2 + O((\Sigma_4 \Sigma_5)^{1/2} + \Sigma_5),$$

where

$$(16) \quad \mathfrak{S}_3(n, Q) = \sum_{q \leq Q} A(q, n), \quad I(n) = \int_{-1/2}^{1/2} v(\beta)^3 e(-\beta n) d\beta,$$

$$\Sigma_4 = \sum_{n \leq x} I(n)^2 \left(\sum_{q \leq Q} |A(q, n)| \right)^2, \quad \Sigma_5 = \sum_{n \leq x} \left(\sum_{q \leq Q} |A(q, n)(I(n) - I(q, n))| \right)^2.$$

By [18, Theorem 2.3] and [18, Theorem 4.2],

$$(17) \quad I(n) = \Gamma(3/2)^2 \sqrt{n} + O(1) = \frac{\pi}{4} \sqrt{n} + O(1), \quad A(q, n) \ll q^{-1/2}.$$

Furthermore, since $A(q, n)$ is multiplicative in q , [18, Lemma 4.7] yields

$$(18) \quad \sum_{q \leq Q} |A(q, n)| \leq \prod_{p \leq Q} (1 + |A(p, n)| + |A(p^2, n)| + \dots)$$

$$\ll \prod_{p \leq Q} (1 + c_1(p, n)p^{-3/2} + 3c_1p^{-1}) \ll (nQ)^\epsilon,$$

where $c_1 > 0$ is an absolute constant. In particular, we have

$$(19) \quad \Sigma_4 \ll N^{4+\epsilon}.$$

We now turn to the estimation of Σ_5 . By Cauchy's inequality and the second bound in (17),

$$\Sigma_5 \ll (\log Q) \sum_{n \leq x} \sum_{q \leq Q} |I(n) - I(n, q)|^2$$

Another application of Bessel's inequality gives

$$\sum_{n \leq x} |I(n) - I(n, q)|^2 \leq \int_{P/qN^2}^{1/2} |v(\beta)|^6 d\beta.$$

Using [18, Lemma 2.8] to estimate the last integral, we deduce that

$$\Sigma_5 \ll \log Q \sum_{q \leq Q} (q^2 N^4 P^{-2} + 1) \ll N^2 Q^{3+\epsilon}.$$

Substituting this inequality and (19) into (15), we conclude that

$$(20) \quad \sum_{n \leq x} R^*(n)^2 = \sum_{n \leq x} I(n)^2 \mathfrak{S}_3(n, Q)^2 + O(N^{3+\epsilon} Q^{3/2}).$$

We then use (17) and (18) to replace $I(n)$ on the right side of (20) by $\frac{\pi}{4} \sqrt{n}$. We get

$$\sum_{n \leq x} I(n)^2 \mathfrak{S}_3(n, Q)^2 = \frac{\pi^2}{16} \sum_{n \leq x} n \mathfrak{S}_3(n, Q)^2 + O(N^{3+\epsilon}).$$

Together with (14) and (20), this leads to the asymptotic formula

$$(21) \quad \sum_{n \leq x} r^*(n)^2 = \frac{\pi^2}{16} \sum_{n \leq x} n \mathfrak{S}_3(n, Q)^2 + O(N^{4+\epsilon} Q^{-1/2} + N^{3+\epsilon} Q^{3/2}).$$

Finally, we evaluate the sum on the right side of (21). On observing that $\mathfrak{S}_3(n, Q)$ is in fact a real number, we have

$$\sum_{n \leq t} \mathfrak{S}_3(n, Q)^2 = \sum_{q_1, q_2 \leq Q} \sum_{\substack{1 \leq a_1 \leq q_1 \\ (a_1, q_1)=1}} \sum_{\substack{1 \leq a_2 \leq q_2 \\ (a_2, q_2)=1}} (q_1 q_2)^{-3} S(q_1, a_1)^3 S(q_2, -a_2)^3 \sum_{n \leq t} e((a_1/q_1 - a_2/q_2)n).$$

As the sum over n equals $t + O(1)$ when $a_1 = a_2$ and $q_1 = q_2$ and $O(q_1 q_2)$ otherwise, we get

$$\sum_{n \leq t} \mathfrak{S}_3(n, Q)^2 = t \sum_{q \leq Q} \sum_{\substack{1 \leq a \leq q \\ (a, q)=1}} q^{-6} |S(q, a)|^6 + O(\Sigma_6^2),$$

where

$$\Sigma_6 = \sum_{q \leq Q} \sum_{\substack{1 \leq a \leq q \\ (a, q)=1}} q^{-2} |S(q, a)|^3 \ll Q^{3/2}.$$

We find that

$$\sum_{n \leq t} \mathfrak{S}_3(n, Q)^2 = B_1 t + O(tQ^{-1} + Q^3),$$

with

$$B_1 = \sum_{q=1}^{\infty} \sum_{\substack{1 \leq a \leq q \\ (a, q)=1}} q^{-6} |S(q, a)|^6.$$

Thus, by partial summation,

$$\sum_{n \leq x} n \mathfrak{S}_3(n, Q)^2 = (B_1/2)x^2 + O(x^2 Q^{-1} + xQ^3).$$

Combining this asymptotic formula with (21), we deduce that

$$\sum_{n \leq x} r^*(n)^2 = \frac{\pi^2}{32} B_1 x^2 + O(x^{15/8+\epsilon}).$$

Recalling (3), we see that (2) will follow if we show that

$$B_1 = \frac{8\zeta(2)}{7\zeta(3)}.$$

This, however, follows easily from the well-known formula (see [7, §7.5])

$$(22) \quad |S(q, a)| = \begin{cases} \sqrt{q} & \text{if } q \equiv 1 \pmod{2}, \\ \sqrt{2q} & \text{if } q \equiv 0 \pmod{4}, \\ 0 & \text{if } q \equiv 2 \pmod{4}. \end{cases}$$

Indeed, (22) yields

$$B_1 = \frac{4}{3} \sum_{q \text{ odd}} q^{-3} \phi(q) = \frac{8\zeta(2)}{7\zeta(3)},$$

where the last step uses the Euler product of $\zeta(s)$. This completes the proof of our theorem.

3. SECOND PROOF OF THEOREM

Rankin [13] and Selberg [17] independently introduced an important method which allows one to study the analytic behavior of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where $a(n)$ are Fourier coefficients of a holomorphic cusp form for some congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$. Originally the method was for holomorphic cusp forms. Zagier [20] extended the method to cover forms that are not cuspidal and may not decay rapidly at infinity. Müller [11, 12] considered the case where $a(n)$ is the Fourier coefficient of non-holomorphic cusp or non-cusp form of real weight with respect to a Fuchsian group of the first kind. It is this last approach we wish to discuss. Note that if we apply a Tauberian theorem to the above Dirichlet series, we then gain information on the asymptotic behavior of the partial sum

$$\sum_{n \leq x} a(n).$$

We now discuss Müller's elegant work. For details regarding discontinuous groups and automorphic forms, see [8, 10, 11, 14, 15, 16]. Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane and $G = SL(2, \mathbb{R})$ the special linear group of all 2×2 matrices with determinant 1. G acts on \mathbb{H} by

$$z \mapsto gz = \frac{az + b}{cz + d}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We write $y = y(z) = \Im(z)$. Thus we have

$$y(gz) = \frac{y}{|cz + d|^2}.$$

Let $dx dy$ denote the Lebesgue measure in the plane. Then the measure

$$d\mu = \frac{dx dy}{y^2}$$

is invariant under the action of G on \mathbb{H} . A discrete subgroup Γ of G is called a *Fuchsian group of the first kind* if its fundamental domain $\Gamma \backslash \mathbb{H}$ has finite volume. Let Γ be a Fuchsian group of the first kind containing $\pm I$ where I is the identity matrix. Let $\mathcal{F}(\Gamma, \chi, k, \lambda)$ denote the space of (non-holomorphic) automorphic forms of real weight k , eigenvalue $\lambda = \frac{1}{4} - \rho^2$, $\Re(\rho) \geq 0$, and multiplier system χ . For $k \in \mathbb{R}$, $g \in SL(2, \mathbb{R})$ and $f : \mathbb{H} \rightarrow \mathbb{C}$, we define the stroke operator $|_k$ by

$$(f|_k g)(z) := \left(\frac{cz + d}{|cz + d|} \right)^{-k} f(gz)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The transformation law for $f \in \mathcal{F}(\Gamma, \chi, k, \lambda)$ is then

$$(f|_k g)(z) = \chi(g)f(z)$$

for all $g \in \Gamma$. Automorphic forms $f \in \mathcal{F}(\Gamma, \chi, k, \lambda)$ have a Fourier expansion at every cusp κ of Γ , namely

$$A_{\kappa,0}(y) + \sum_{n \neq 0} a_{\kappa,n} W_{(sgn n)\frac{k}{2}, \rho}(4\pi|n + \mu_\kappa|y)e((n + \mu_\kappa)x),$$

where μ_κ is the cusp parameter and $a_{\kappa,n}$ are the Fourier coefficients of f at κ . The functions $W_{\alpha, \rho}$ are Whittaker functions (see [11, §3]), $A_{\kappa,0}(y) = 0$ if $\mu_\kappa \neq 0$ and

$$A_{\kappa,0}(y) = \begin{cases} a_{\kappa,0}y^{1/2+\rho} + b_{\kappa,0}y^{1/2-\rho} & \text{if } \mu_\kappa = 0, \rho \neq 0, \\ a_{\kappa,0}y^{1/2} + b_{\kappa,0}y^{1/2} \log y & \text{if } \mu_\kappa = 0, \rho = 0. \end{cases}$$

An automorphic form f is called a cusp form if $a_{\kappa,0} = b_{\kappa,0} = 0$ for all cusps κ of Γ . Now consider the Dirichlet series

$$S_\kappa(f, s) = \sum_{n>0} \frac{|a_{\kappa,n}|^2}{(n + \mu_\kappa)^s}.$$

This series is absolutely convergent for $\Re(s) > 2\Re(\rho)$ and has been shown [12] to have meromorphic continuation in the entire complex plane. In what follows, we will only be interested in the case f is not a cusp form. If f is not a cusp form and $\Re(\rho) > 0$, then $S_\kappa(f, s)$ has a simple pole at $s = 2\Re(\rho)$ with residue

$$(23) \quad \beta_\kappa(f) = \operatorname{res}_{s=2\Re(\rho)} S_\kappa(f, s) = (4\pi)^{2\Re(\rho)} b^+(k/2, \rho) \sum_{\iota \in K} \varphi_{\kappa,\iota} (1 + 2\Re(\rho)) |a_{\iota,0}|^2,$$

where K denotes a complete set of Γ -inequivalent cusps, $\varphi_{\kappa,\iota} (1 + 2\Re(\rho)) > 0$ and $b^+(\frac{k}{2}, \rho) > 0$ if $\rho + \frac{1}{2} \pm \frac{k}{2}$ is a non-negative integer. For the definition of the functions $\varphi_{\kappa,\iota}$ and b^+ , see Lemma 3.6 and (69) in [12]. This result (23) and a Tauberian argument then provide the asymptotic behaviour of the summatory function

$$\sum_{n \leq x} |a_{\kappa,n}|^2 |n + \mu_\kappa|^r.$$

Precisely, we have (see [11, Theorem 2.1] or [12, Theorem 5.2]) that

$$(24) \quad \sum_{n \leq x} |a_{\kappa,n}|^2 |n + \mu_\kappa|^r = \sum_{z \in R} \operatorname{res}_{s=z} S_\kappa(f, s) \frac{x^{r+s}}{r+s} + O(x^{r+2\Re(\rho)-\gamma} (\log x)^g),$$

where $2\Re(\rho) + r \geq 0$, $R = \{\pm 2\Re(\rho), \pm 2i\Im(\rho), 0, -r\}$, $\gamma = (2+8\Re(\rho))(5+16\Re(\rho))^{-1}$, and $g = \max(0, b-1)$; b denotes the order of the pole of $S_\kappa(f, s)(r+s)^{-1}x^{r+s}$ at $s = 2\Re(\rho)$ ($0 \leq b \leq 5$).

We now consider an application of (24). Let $Q \in \mathbb{Z}^{m \times m}$ be a non-singular symmetric matrix with even diagonal entries and $q(\mathbf{x}) = \frac{1}{2}Q[\mathbf{x}] = \frac{1}{2}\mathbf{x}^T Q \mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^m$, the associated quadratic form in $m \geq 3$ variables. Here we assume that $q(\mathbf{x})$ is positive definite. Let $r(Q, n)$ denote the number of representations of n by the quadratic form Q . Now consider the theta function

$$\theta_Q(z) = \sum_{\mathbf{x} \in \mathbb{Z}^m} e^{\pi i z Q[\mathbf{x}]}.$$

By [11, Lemma 6.1], the Dirichlet series associated with the automorphic form θ_Q is

$$(4\pi)^{-m/4} \zeta_Q\left(\frac{m}{4} + s\right)$$

where

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{\mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}} q(\mathbf{x})^{-s}$$

for $\Re(s) > m/2$. Using (24), Müller proved the following (see [11, Theorem 6.1])

Theorem (Müller). Let $q(\mathbf{x}) = \frac{1}{2}Q[\mathbf{x}] = \frac{1}{2}\mathbf{x}^T Q\mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^m$ be a primitive positive definite quadratic form in $m \geq 3$ variables with integral coefficients. Then

$$\sum_{n \leq x} r(Q, n)^2 = Bx^{m-1} + O\left(x^{(m-1)\frac{4m-5}{4m-3}}\right)$$

where

$$B = (4\pi)^{m/2} \frac{\beta_\infty(\theta_Q)}{m-1}$$

and $\beta_\infty(\theta_Q)$ is given by (23).

We are now in a position to prove our theorem in page 2.

Proof. We are interested in the case $q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$ and so $r(Q, n) = r_3(n)$ counts the number of representations of n as a sum of three squares. By Müller's Theorem above,

$$\sum_{n \leq x} r_3(n)^2 = Bx^2 + O\left(x^{14/9}\right)$$

where B is a computable constant. Specifically, we have by (23) (with $k = 3/2$ and $\rho = 1/4$)

$$B = \frac{4\pi^2}{3-1} b^+(3/4, 1/4) \sum_{\iota \in K} \varphi_{\infty, \iota}(3/2) |a_{\iota, 0}|^2,$$

where K denotes a complete set of $\Gamma_0(4)$ -inequivalent cusps and $a_{\iota, 0}$ is the 0-th Fourier coefficient of $\theta_Q(z)$ at a rational cusp ι . Choose $K = \{1, \frac{1}{2}, \frac{1}{4}\}$. Then by p. 145 and (67) in [11], we have

$$|a_{\iota, 0}|^2 = W_\iota^3 |G(S_\iota)|^2$$

where $\iota = u/w$, $(u, w) = 1$, $w \geq 1$, W_ι is width of the cusp ι , and

$$|G(S_\iota)|^2 = 2^{-3} w^{-3} \left| \sum_{x=1}^w e\left(\frac{u}{w} x^2\right) \right|^6.$$

As $W_{1/4} = W_{1/2} = 1$, $W_1 = 4$, we have $|a_{1,0}|^2 = 1$, $|a_{1/2,0}|^2 = 0$, and $|a_{1/4,0}|^2 = 1$. An explicit description of the functions $\varphi_{\infty, \iota}(s)$ in the case $\Gamma_0(4)$ is given by (see (1.17) and p. 247 in [5])

$$\varphi_{\infty, 1/4}(s) = 2^{1-4s} (1 - 2^{-2s})^{-1} \pi^{1/2} \frac{\Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)},$$

$$\varphi_{\infty, 1/2}(s) = \varphi_{\infty, 1}(s) = 2^{-2s} (1 - 2^{-2s})^{-1} (1 - 2^{1-2s}) \pi^{1/2} \frac{\Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)}.$$

Thus for $s = 3/2$, we have

$$\varphi_{\infty, 1/4}(3/2) = 2^{-5} (1 - 2^{-3})^{-1} \pi^2 \frac{\zeta(2)}{\Gamma(3/2) \zeta(3)},$$

$$\varphi_{\infty, 1/2}(3/2) = \varphi_{\infty, 1}(3/2) = 2^{-3} (1 - 2^{-3})^{-1} (1 - 2^{-2}) \pi^2 \frac{\zeta(2)}{\Gamma(3/2) \zeta(3)}.$$

Now, from p. 65 in [12], we have

$$b^+(3/4, 1/4) = G_{1/4, 1/4}^*(3/2).$$

By Lemma 3.3 and (16) in [12],

$$G_{1/4,1/4}^*(s) = \Gamma(s + 1/2)^{-1}$$

and so $b^+(3/4, 1/4) = \Gamma(2)^{-1}$. In total,

$$\begin{aligned} B &= \frac{(4\pi)^2}{(3-1)\Gamma(2)} \left(2^{-3}(1-2^{-3})^{-1}(1-2^{-2})\pi^{1/2} \frac{\zeta(2)}{\Gamma(3/2)\zeta(3)} \right. \\ &\quad \left. + 2^{-5}(1-2^{-3})^{-1}\pi^{1/2} \frac{\zeta(2)}{\Gamma(3/2)\zeta(3)} \right) \\ &= \frac{8\pi^4}{21\zeta(3)}. \end{aligned}$$

Thus

$$\sum_{n \leq x} r_3(n)^2 \sim \frac{8\pi^4}{21\zeta(3)} x^2.$$

□

Remark. Müller's Theorem can also be used to obtain the mean square value of sums of $N > 3$ squares. Precisely, if $r_N(n)$ is the number of representations of n by $N > 3$ squares, then a calculation similar to the second proof of our theorem yields (compare with Theorem 3.3 in [2])

$$\sum_{n \leq x} r_N(n)^2 = W_N x^{N-1} + O\left(x^{(N-1)\frac{4N-5}{4N-3}}\right)$$

where

$$W_N = \frac{1}{(N-1)(1-2^{-N})} \frac{\pi^N}{\Gamma(N/2)^2} \frac{\zeta(N-1)}{\zeta(N)}.$$

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