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A generalization of universal Taylor series in simply connected domains

by N. Tsirivas

Abstract: Let Ω be a simply connected proper subdomain of the complex plane and z_0 be a point in Ω. It is known that there are holomorphic functions f on Ω for which the partial sums $(S_n(f, z_0))$ of the Taylor series about z_0 have universal approximation properties outside Ω. In this paper we investigate what can be said for the sequence $(\beta_n S_n(f, z_0))$ when (β_n) is a sequence of non-zero complex numbers. We also study a related analogue of a classical Theorem of Seleznev concerning the case where the radius of convergence of the universal power series is zero.

Key words: Universal series. Cesàro hypercyclicity. Universality. Hypercyclicity. Bernstein - Walsh Theorem. Seleznev Theorem.

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1. Introduction

We begin with the abstract definition of universality [6].

Definition 1.1. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological vector spaces over a field $\mathbb{K}$ and $T_n : X \to Y$, $n = 1, 2, \ldots$ be a sequence of continuous linear operators. We say that the sequence (T_n) is universal if there exists some $x \in X$ such that $Y = \overline{\bigcup_n T_n(x)}$. Any $x \in X$ with the above property is called a universal vector of X with respect to (T_n) and we denote the set of universal vectors of the space X*

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with respect to (T_n) *by*

$$\mathcal{U}(T_n) := \Big\{ x \in X \Big| Y = \overline{\bigcup_n T_n(x)} \Big\}.$$

In the case where $X = Y$ *we call the sequence* (T_n) *hypercyclic.*

Let $(X, \mathcal{T}_X)$ be a topological vector space and $(Y^i, \mathcal{T}_{Y^i})$, $i \in I$, be a family of topological vector spaces over $\mathbb{K}$. For every $i \in I$ let $T_n^i : X \to Y^i$, $n = 1, 2, \ldots$ be a sequence of continuous linear operators. Also let (β_n) be a sequence of complex numbers. Let $\mathcal{U}(\beta_n T_n^i)$, $i \in I$ be the sets of universal vectors in X with respect to the families $(\beta_n T_n^i)$, $i \in I$, as in Definition 1.1. The question is whether the families $(\beta_n T_n^i)$, $i \in I$ share a common universal vector, that is, if $\bigcap_i \mathcal{U}(\beta_n T_n^i) \neq \emptyset$.

This subject is closely related to the notion of Cesàro hypercyclicity [5]. Below we formulate an important particular case of this question and then describe its complete solution.

Let Ω be a simply connected proper subdomain of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denote the space of holomorphic functions on Ω, and let $z_0 \in \Omega$ be fixed. We endow $\mathcal{H}(\Omega)$ with the topology $\mathcal{T}_u$ of uniform convergence on compact subsets of Ω. Let $f \in \mathcal{H}(\Omega)$. We denote by $S_n(f, z_0)$ the n-th partial sum of the Taylor development of f about z_0; that is,

$$S_n(f, z_0)(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \quad n = 0, 1, 2, \ldots, \quad z \in \mathbb{C}.$$

Where no misunderstanding can arise, we write S_n instead of $S_n(f, z_0)$.

Let $\mathcal{M}_{\Omega^c}$ be the collection of compact subsets of Ω^c with connected complement. For every $K \in \mathcal{M}_{\Omega^c}$ we consider the space $A(K)$ of continuous functions on K that are holomorphic in K^0, endowed with the supremum norm, which is a Banach Algebra. Let $\beta = (\beta_n)_{n \in \mathbb{N}_0}$ be a sequence in $\mathbb{C} \setminus \{0\}$. For each $K \in \mathcal{M}_{\Omega^c}$ we consider the sequence of continuous linear operators $S_n^K : \mathcal{H}(\Omega) \to A(K)$, where

$$S_n^K(f)(z) = S_n(f, z_0)(z) \ \text{ for every } \ f \in \mathcal{H}(\Omega), \quad z \in K, \quad n = 0, 1, 2, \ldots.$$

Now we apply the above terminology after Definition 1.1 of universality with $X := \mathcal{H}(\Omega)$, $I := \mathcal{M}_{\Omega^c}$, $Y^K := A(K)$ for every $K \in \mathcal{M}_{\Omega^c}$ and $T_n^K := S_n^K$ for every $K \in \mathcal{M}_{\Omega^c}$ and $n = 0, 1, 2, \ldots$. We define

$$\mathcal{U}(\Omega, z_0, \beta) := \bigcap_{K \in \mathcal{M}_{\Omega^c}} \mathcal{U}(\beta_n T_n^K).$$

Thus a holomorphic function f on Ω belongs to $\mathcal{U}(\Omega, z_0, \beta)$ if, for each $K \in \mathcal{M}_{\Omega^c}$ and $h \in A(K)$, there is a sequence $\lambda = (\lambda_n)$ of natural numbers such that $\beta_{\lambda_n} S^K_{\lambda_n}(f) \to h$ as $n \to \infty$ uniformly on K.

Our main aim in this paper is to completely characterize the sequences β for which $\mathcal{U}(\Omega, z_0, \beta) \neq \emptyset$. The solution to this problem is given below.

Theorem 1.2. *The set $\mathcal{U}(\Omega, z_0, \beta)$ is non empty if and only if $\left(\sqrt[n]{|\beta(n)|}\right)$ has 1 as a limit point. In this case $\mathcal{U}(\Omega, z_0, \beta)$ is a G_δ dense subset of $\mathcal{H}(\Omega)$ that contains a dense vector subspace of $\mathcal{H}(\Omega)$ except 0.* ■

2. Proof of Theorem 1.2

The conclusion of Theorem 1.2 follows easily from known results if (β_n) has a finite non-zero limit ([8], [10], [11]), or if (β_n) has a finite non-zero limit point (see [2, page 420 Theorem 1]).

When these are not the cases new arguments are required. We use the following lemma ([8], [10]).

Lemma 2.1. *There is a sequence $(K_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_{\Omega^c}$ such that, for every $K \in \mathcal{M}_{\Omega^c}$, there exists $m \in \mathbb{N}$ such that $K \subset K_m$.*

The space $(\mathcal{H}(\Omega), \mathcal{T}_u)$ is a complete metric space, so Baire's Category theorem is at our disposal. We will write $\mathcal{U}(\Omega, z_0, \beta)$ in the form $\bigcap_n V_n$, where the sets V_n are open and dense in $\mathcal{H}(\Omega)$. Now we describe the sets V_n.

Let $(f_j)_{j \geq 1}$ be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact sets as in Lemma 2.1. Now for each $j, s, m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we consider the set:

$$E(m, j, s, n) := \left\{ f \in \mathcal{H}(\Omega) \,\middle|\, \left| \beta_n S_n(f) - f_j \right| < \frac{1}{s} \text{ on } K_m \right\}.$$

Lemma 2.2. *With the above notation,*

$$\mathcal{U}(\Omega, z_0, \beta) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} E(m, j, s, n). \quad ■$$

Lemma 2.3. *For each $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the set $E(m, j, s, n)$ is open in the space $(\mathcal{H}(\Omega), \mathcal{T}_u)$.* ■

The above two lemmas hold without any restriction on the sequence β. They can be proved by following the arguments in Lemma 2.4 and Proposition 2.5 of [7], and using Mergelyan's Theorem [13]. We now assume that 1 is a limit point of $(\sqrt[n]{|\beta_n|})$.

Lemma 2.4. *Suppose that 1 is a limit point of $(\sqrt[n]{|\beta_n|})$. For each $m, j, s \in \mathbb{N}$, the set $\bigcup_{n=0}^{\infty} E(m, j, s, n)$ is dense in $(\mathcal{H}(\Omega), \mathcal{T}_u)$.*

Proof. In view of the known cases, and the fact that we need only work with a subsequence of (β_n), it enough to consider what happens when $|\beta_n|$ tends to infinity, or when β_n tends to zero where β_n is different to zero for each n.

Case a) $\lim_{n\to\infty} |\beta_n| = +\infty$

Let $m_0, j_0, s_0 \in \mathbb{N}$, let p_0 be a polynomial, let $\varepsilon_0 > 0$ and $L \subseteq \Omega$ be a compact set. It suffices to find $N_0 \in \mathbb{N}$, and a holomorphic function $f \in \mathcal{H}(\Omega)$, such that

$$|f - p_0| < \varepsilon_0 \quad \text{on} \quad L, \quad \text{and} \quad \left|\beta_{N_0} S_{N_0}(f) - f_{j_0}\right| < \frac{1}{s_0} \quad \text{on} \quad K_{m_0}. \qquad (*)$$

Because Ω is a simply connected domain we can find connected compact sets C_1, C_2 that have connected complements and boundaries that are simple smooth loops (see [4, p. 24]), disjoint open sets G_1, G_2 and simple smooth loops γ_1, γ_2 such that

$$L \subset \overset{\circ}{C}_1 \subset C_1 \subset \mathrm{Int}(\gamma_1) \subset \overline{\mathrm{Int}(\gamma_1)} \subset G_1,$$
$$K_{m_0} \subset \overset{\circ}{C}_2 \subset C_2 \subset \mathrm{Int}(\gamma_2) \subset \overline{\mathrm{Int}(\gamma_2)} \subset G_2.$$

Here $\mathrm{Int}(\gamma_1)$ denotes the interior of the curve γ_1 as usual, and we can further arrange that $\mathrm{Ind}\gamma_1(C_1) = 1$, $\mathrm{Ind}\gamma_1(C_2) = 0$, $\mathrm{Ind}\gamma_2(C_1) = 0$, and $\mathrm{Ind}\gamma_2(C_2) = 1$ [4, Exercise 10.10].

Now let $m \in \mathbb{N}$ and let $F_m : G_1 \cup G_2 \to \mathbb{C}$ be defined by

$$F_m(z) := \begin{cases} p_0(z) & \text{if} \quad z \in G_1 \\ \frac{1}{\beta(m)} f_{j_0}(z) & \text{if} \quad z \in G_2. \end{cases}$$

Also, let $n \in \mathbb{N}$, where $n \geq 2$ and let q_n be a Fekete polynomial of degree at most n for the set $C_3 := C_1 \cup C_2$, (see [12, Definition 5.5.3]). We define the function $p_n(m) : C_3 \to \mathbb{C}$ defined by the formula

$$p_n(m)(w) := \frac{1}{2\pi i} \int_{\gamma_1} \frac{F_m(z)}{q_n(z)} \cdot \frac{q_n(w) - q_n(z)}{w - z} dz + \frac{1}{2\pi i} \int_{\gamma_2} \frac{F_m(z)}{q_n(z)} \cdot \frac{q_n(w) - q_n(z)}{w - z} dz, \quad w \in C_3.$$

Clearly, $p_n(m)$ is a sum of two polynomials of degree at most $n-1$. By the global Cauchy Theorem

$$\begin{aligned} F_m(w) - p_n(m)(w) =& \frac{1}{2\pi i}\int_{\gamma_1} \frac{F_m(z)}{z-w}\cdot\frac{q_n(w)}{q_n(z)}dz \\ &+ \frac{1}{2\pi i}\int_{\gamma_2} \frac{F_m(z)}{z-w}\cdot\frac{q_n(w)}{q_n(z)}dz, \quad w \in C_3. \end{aligned} \tag{1}$$

Using (1) and the fact that the sequence $1/\beta_n$ is bounded we can find a constant $M_0 > 0$, independent of n, m, such that

$$\|F_m - p_n(m)\|_{C_3} < M_0 \cdot \frac{\|q_n\|_{C_3}}{\inf_{\gamma_1\cup\gamma_2}|q_n|} \qquad n, m \in \mathbb{N}, \qquad n \geq 2. \tag{2}$$

Let $G := C_3^c$. The set ∂G is non-polar, so G possesses a Green function g_G. Let r_G be the Harnack distance for G [12, Definition 1.3.4], $c(C_3)$ be the logarithmic capacity of C_3 and $\delta_n(C_3)$ be the n-th diameter of C_3 [12, Definition 5.1.1], for $n \geq 2$. By Bernstein's Lemma [12, Theorem 5.5.7 (b)],

$$\left(\frac{|q_n(z)|}{\|q_n\|_{C_3}}\right)^{1/n} \geq e^{g_G(z,\infty)} \cdot \left(\frac{c(C_3)}{\delta_n(C_3)}\right)^{r_G(z,\infty)} \tag{3}$$

for $n \geq 2$, $z \in G$.

By the Fekete-Szegö Theorem [12, Theorem 5.5.2] we have

$$\lim_{n\to\infty} \delta_n(C_3) = c(C_3). \tag{4}$$

Applying (3) to the curves γ_1 and γ_2 and using (2) and (4) we can find $\theta_1 \in (0,1)$ and $\nu_0 \in \mathbb{N}$ such that

$$\|F_m - p_n(m)\|_{C_3}^{1/n} < \theta_1, \qquad n, m \in \mathbb{N}, \qquad n \geq \nu_0. \tag{5}$$

Using the fact that $L \cup K_{m_0} \subset C_3$, the definition of F_m, the condition $\sqrt[n]{|\beta(n)|} \to 1$ and (5), we can find a natural number N_0 such that

$$\left|p_{N_0}(N_0) - p_0\right| < \varepsilon_0 \;\text{ on }\; L \tag{6}$$

and

$$\left|\beta(N_0)p_{N_0}(N_0) - f_{j_0}\right| < \frac{1}{s_0} \;\text{ on }\; K_{m_0}. \tag{7}$$

We set $f := p_{N_0}(N_0)$. Then $f \in \mathcal{H}(\Omega)$ and $f = S_{N_0}(f)$ because f is a polynomial of degree at most $N_0 - 1$. Thus we have proved the desired inequalities in $(*)$.

Case b) $\lim_{n\to\infty} \beta_n = 0$

The proof is almost the same as in case a). The only change is to replace the constant M_0 in (2) by $(1+\delta)^m$, where $\delta > 0$ is arbitrarily small and $m = n$ is sufficiently large (depending on δ). ■

Now using Lemmas 2.2, 2.3 and 2.4, Baire's Category Theorem and the completeness of the metric space $(\mathcal{H}(\Omega), \mathcal{T}_u)$, we conclude that $\mathcal{U}(\Omega, z_0, \beta)$ is a $G\delta$ dense subset of $(\mathcal{H}(\Omega), \mathcal{T}_u)$.

Now we suppose that $\sqrt[n]{|\beta_n|} \to 1$ and $\beta_n \neq 0 \ \forall\, n \in \mathbb{N}$. The above proof gives us that for every subsequence $(\beta \circ \mu)$ of (β) the set $\mathcal{U}(\Omega, z_0, \beta\circ\mu)$ is a G_δ dense subset of $(\mathcal{H}(\Omega), \mathcal{T}_u)$. Now as in the implication (v)⇒(vi) of Theorem 4.2 of [7] (see also [2], [3]), we see that the set $\mathcal{U}(\Omega, z_0, \beta)$ contains a dense vector subspace of $\mathcal{H}(\Omega)$ except 0. Passing to a subsequence of β, the same holds when 1 is a limit point of $\left(\sqrt[n]{|\beta_n|}\right)$. By the above we have completed the positive cases of Theorem 1.2

Now we examine the negative cases of Theorem 1.2.

Proposition 2.5. *If the number 1 is not a limit point of the sequence* $\left(\sqrt[n]{|\beta(n)|}\right)$, *then* $\mathcal{U}(\Omega, z_0, \phi) = \emptyset$.

Proof. We distinguish three cases.

First case: $\limsup_{n\to\infty} \sqrt[n]{|\beta_n|} < 1$.

We fix $a \in \left(1, 1/\limsup_{n\to\infty} \sqrt[n]{|\beta_n|}\right)$ if $\limsup_{n\to\infty} \sqrt[n]{|\beta_n|} \neq 0$, or else choose an arbitrary number $a > 1$. There exists $n_0 \in \mathbb{N}$ such that

$$|\beta_n| < \frac{1}{a^n}, \qquad n \geq n_0. \tag{1}$$

Let $d := \operatorname{dist}(z_0, \Omega^c) := \inf\{|z - z_0| : z \in \Omega^c\}$ and $\varepsilon_0 \in \left(0, d \cdot \dfrac{a-1}{a+1}\right)$. Let $K \subset \Omega^c$ be a compact set with connected complement such that

$$\max\{|z - z_0| : z \in K\} \leq d + \varepsilon_0.$$

Let $f \in \mathcal{H}(\Omega)$, and (a_n) be the Taylor coefficients of f about z_0. If R is the associated radius of convergence, then $R \geq d$. Thus

$$\sum_{k=0}^{\infty} |a_k| (d - \varepsilon_0)^k = A \in [0, +\infty). \tag{2}$$

From (1) and (2) we see easily that

$$|\beta_n S_n(f,z_0)| \le \left(\frac{d+\varepsilon_0}{d-\varepsilon_0}\cdot\frac{1}{a}\right)^n \cdot A \ \text{ on } \ K \ \text{ for all } \ n\ge n_0,$$

whence

$$\sup_{z\in K}\Big|\beta(n)S_n(f,z_0)(z)\Big|\to 0 \tag{3}$$

as $n\to\infty$ because $\dfrac{d+\varepsilon_0}{d-\varepsilon_0}\cdot\dfrac{1}{a}\in(0,1)$. The convergence in (3) shows that the arbitrary function $f\in\mathcal{H}(\Omega)$ cannot be universal and the result now follows.

Second case: $\liminf\limits_{n\to\infty}\sqrt[n]{|\beta_n|}>1$.

For this proof we use Theorem 1 of [9]. Since Ω is a simply connected domain we can find a compact connected set Γ containing more than one point such that Γ^c is connected, $\Gamma\subset\Omega^c$, and $\operatorname{dist}(\Gamma,z_0)=\operatorname{dist}(z_0,\Omega^c)$.

If Ω is unbounded we consider a sequence $(K_n)_{n\in\mathbb{N}}$, $K_n\in\mathcal{M}_{\Omega^c}$ as in Lemma 2.1 such that $K_n\subseteq K_{n+1}$ for each n and $\Gamma\subset K_1$. In this case we set $E=\Omega^c$.

If Ω is bounded we choose $N_0\in\mathbb{N}$ such that $\Omega\cup\Gamma\subset D(0,N_0)$, and put $K_n:=\Gamma\cup[N_0+1,N_0+1+n]$. In this case we set $E=\bigcup\limits_{n=1}^{\infty}K_n=\Gamma\cup[N_0+1,+\infty)$. In each of these cases the set E is closed and non-thin at ∞. (For the definition of thinness see [1] or [12]).

The proof of this case is similar to that in Proposition 3.7, so it is omitted.

Third case: $\liminf\limits_{n\to\infty}\sqrt[n]{|\beta(n)|}<1<\limsup\limits_{n\to\infty}\sqrt[n]{|\beta(n)|}$.

We consider the same sets Γ,E and the same sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ as in the previous case, where $\Gamma\subset K_1$, and suppose that $\mathcal{U}(\Omega,z_0,\beta)\neq\emptyset$ for the sake of contradiction.

Let $f\in\mathcal{U}(\Omega,z_0,\beta)$. Then we can find a strictly increasing sequence $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ of natural numbers such that

$$\sup_{z\in K_n}\Big|\beta_{\lambda_n}S_{\lambda_n}(f)(z)-1\Big|<\frac{1}{n},\qquad n\in\mathbb{N}. \tag{4}$$

It is easy to see that the sequence $(|\beta_{\lambda_n}|)$, $n=1,2,\dots$ has only two possible limit points, namely 0 and $+\infty$.

Suppose that 0 is a limit point. Let $w_0\in\Gamma$ be such that $|w_0-z_0|=\operatorname{dist}(z_0,\Omega^c)$. Then, as in the proof of the second case, we see that $|\beta_{\mu_n}S_{\mu_n}(f,z_0)(w_0)|\to 0$ as $n\to\infty$, which is a contradiction. Thus the only limit point of $(|\beta_{\lambda_n}|)$ is $+\infty$. By our assumptions on β we have $\liminf_{n\to\infty} \sqrt[\lambda_n]{|\beta_{\lambda_n}|} > 1$. There exists some $\theta_0 \in (0,1)$ and $\nu_0 \in \mathbb{N}$ such that

$$\|S_{\lambda_n}(f)\|_{K_n}^{1/\lambda_n} < \theta_0, \qquad n \geq \nu_0. \tag{5}$$

We can now use the argument in the first case, following (5), to obtain again a contradiction. ■

3. A Theorem of Seleznev

A result of Seleznev [14] gives the first example of a universal Taylor series in the complex plane with radius of convergence zero. A recent extension of it (Theorem 6.2 of [7]) corresponds, roughly speaking, to our Theorem 1.2 in the case where the universal Taylor series have radius of convergence zero. In this paragraph we preserve the original terminology of [7]. Thus we consider a sequence of non-zero complex numbers $(\phi(n))$, where $1/\phi(n)$ will play the same role as β_n did earlier. However, [7] dealt only with the case where $\limsup_{n\to\infty}|\phi(n)| > 0$. In this section we will address the case where $\lim_{n\to\infty} \phi(n) = 0$, to complete the result.

Of course, the condition $\lim_{n\to\infty} \phi(n) = 0$ implies that $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} \leq 1$. We will show that, if $\lim_{n\to\infty} \phi(n) = 0$, then the conclusion of Theorem 6.2 of [7] holds when $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} = 1$ but fails when $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} < 1$. We write $\mathcal{M}_{\{0\}^c}$ for the collection of compact subsets of $\mathbb{C}\setminus\{0\}$ with connected complement, and consider the set $\mathcal{U}(\phi)$ of sequences of $\mathbb{C}^{\mathbb{N}_0}$ such that for all $(K,f) \in \mathcal{M}_{\{0\}^c} \times A(K)$ $\exists\, \lambda = (\lambda_n)_{n\in\mathbb{N}}$ a sequence of natural numbers so that $\dfrac{1}{\phi(\lambda_n)} \sum_{j=0}^{\lambda_n} a_j z^j \to f$ uniformly on K as $n\to\infty$.

Now we consider the space $\mathbb{C}^{\mathbb{N}_0}$ endowed with the Cartesian topology that is induced by the metric $\rho : \mathbb{C}^{\mathbb{N}_0} \times \mathbb{C}^{\mathbb{N}_0} \to \mathbb{R}^+$ with $\rho(a,b) = \sum_{i=0}^{\infty} \dfrac{1}{2^i} \dfrac{|a_i - b_i|}{1 + |a_i - b_i|}$, $(a,b) \in (\mathbb{C}^{\mathbb{N}_0})^2$. We write $(C^{\mathbb{N}_0}, \mathcal{T}_c)$ for the above space.

Theorem 3.1. *Suppose that* $\lim_{n\to\infty} \phi(n) = 0$. *Then the set* $\mathcal{U}(\phi)$ *is non-empty if and only if the number 1 is a limit point of* $\left(\sqrt[n]{|\phi(n)|}\right)$ *or, equivalently* $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} = 1$. *In the case where* $\mathcal{U}(\phi)$ *is non-empty it is also* G_δ *dense in* $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$ *and contains a dense vector subspace of* $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$ *except 0.*

Firstly, we prove the following:

Proposition 3.2. *Let ϕ be a sequence such that $\lim_{n\to\infty} \phi(n) = 0$ and $\lim_{n\to\infty} \sqrt[n]{|\phi(n)|} = 1$. Then the set $\mathcal{U}(\phi)$ is a G_δ-dense subset of $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}\setminus\{0\}$ as in Lemma 2.1. Let f_j, $j = 1, 2, \ldots$ be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$. For every $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ let

$$\widetilde{E}(m, j, s, n) := \left\{ a = (a_0, a_1, \ldots) \in \mathbb{C}^{\mathbb{N}_0} \middle| \sup_{z\in K_m} \left| \frac{1}{\phi(n)} \sum_{i=0}^{n} a_i z^i - f_j(z) \right| < \frac{1}{s} \right\}.$$

We will need the following results.

Lemma 3.3. *For every $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the set $\widetilde{E}(m, j, s, n)$ is open in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$.*

Lemma 3.4. *With the above notation,*

$$\mathcal{U}(\phi) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} \widetilde{E}(m, j, s, n).$$

The proofs of the above two lemmas are similar to those of Lemma 2.4 and Proposition 2.5 of [7] and are omitted.

Proposition 3.5. *For every $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the set $\bigcup_{n=0}^{\infty} \widetilde{E}(m, j, s, n)$ is dense in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$.*

Proof. We fix $m_0, j_0, s_0 \in \mathbb{N}$, and prove that the set $A = \bigcup_{n=0}^{\infty} \widetilde{E}(m_0, j_0, s_0, n)$ is dense in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$. We know that the set c_{00} is dense in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$. It suffices to prove that $S(a, \varepsilon) \cap A \neq \emptyset$ for every $a \in c_{00}$ and $\varepsilon > 0$ where $S(a, \varepsilon) := \{x \in \mathbb{C}^{\mathbb{N}_0} | \rho(a, x) < \varepsilon\}$.

So let $a = (a_0, a_1, a_2, \ldots, a_{\nu_0}, 0, 0, \ldots) \in c_{00}$ where $a_\nu = 0$ where $\nu \geq \nu_0 + 1$ for some fixed $\nu_0 \geq 1$. Also let $\varepsilon_0 > 0$. We will prove that $S(a, \varepsilon_0) \cap A \neq \emptyset$.

This means that we need to find some sequence $b = (b_0, b_1, \ldots, b_n, \ldots) \in \mathbb{C}^{\mathbb{N}_0}$ and a natural number $N_0 \geq 1$ such that:

$$\rho(a, b) < \varepsilon_0 \tag{1}$$

and

$$\sup_{z\in K_{m_0}} \left| \frac{1}{\phi(N_0)} \sum_{i=0}^{N_0} b_i z^i - f_{j_0}(z) \right| < \frac{1}{s_0}. \tag{2}$$

Let $k_0 \in \mathbb{N}$ such that $k_0 > \max\{2, \nu_0\}$ and

$$\sum_{i=k_0+1}^{\infty} \frac{1}{2^i} < \varepsilon_0. \tag{3}$$

We can arrange that the set K_{m_0} is also connected and has a rectifiable curve as its boundary.

Now we can find a bounded simply connected domain $W \subseteq \mathbb{C}\setminus\{0\}$ and a smooth simple loop γ such that $K_{m_0} \subset W$, $\gamma \subset W$, $\gamma \cap K_{m_0} = \emptyset$ and $\text{Ind}\gamma(K_{m_0}) = 1$. So we have $K_{m_0} \subset \overline{\text{Int}(\gamma)} \subset W \subset \mathbb{C}\setminus\{0\}$.

Let $p(z) = a_0 + a_1 z + \cdots + a_{\nu_0} z^{\nu_0}$, and let $m \in \mathbb{N}$. We consider the holomorphic function $F_m : W \to \mathbb{C}$, defined by

$$F_m(z) = \frac{1}{z^{k_0+1}}(\phi(m) f_{j_0}(z) - p(z)), \quad z \in W.$$

Applying a similar proof as Lemma 2.4 previously we have that

$$\|F_m - P_n(m)\|_{K_{m_0}} < \theta_0^n \quad \forall\, n, m \in \mathbb{N}, \quad n \geq \lambda_0 \tag{4}$$

where $\theta_0 \in (0,1)$ and $\lambda_0 \in \mathbb{N}$ are fixed numbers.

Then (4) holds for every $n, m \in \mathbb{N}$, $n \geq \lambda_0$.

We apply (4) for every $m \in \mathbb{N}$, $m > \lambda_0 + k_0$, $n = m - k_0$. We see easily from (4) that there exists a constant C_1 such that

$$\sup_{z \in K_{m_0}} \left| \frac{1}{\phi(m)}(p(z) + z^{k_0+1} p_n(m)(z)) - f_{j_0}(z) \right| < C_1 \cdot \frac{\theta_0^m}{|\phi(m)|}$$
$$\text{for every } n, m \in \mathbb{N},\ n = m - k_0,\ m > \lambda_0 + k_0. \tag{5}$$

Let $\delta_0 \in \left(0, \dfrac{1}{\theta_0} - 1\right)$. Because $\sqrt[n]{|\phi(n)|} \to 1$ we can find $n_1 \in \mathbb{N}$ such that

$$\frac{1}{|\phi(n)|} < (1 + \delta_0)^n, \qquad n \geq n_1. \tag{6}$$

By (5) and (6) we have that:

$$\sup_{z \in K_{m_0}} \left| \frac{1}{\phi(m)}(p(z) + z^{k_0+1} p_n(m)(z)) - f_{j_0}(z) \right| < C_1 \cdot (\theta_0(1 + \delta_0))^m$$
$$\forall\, n, m \in \mathbb{N}, \quad m > \max\{\lambda_0 + k_0, n_1\}, \quad n = m - k_0. \tag{7}$$

Now because $\theta_0(1+\delta_0) \in (0,1)$ by (7) we can find a natural number $N_0 > \max\{\lambda_0 + k_0, n_1\}$ such that

$$\sup_{z\in K_{m_0}} \left| \frac{1}{\phi(N_0)} (p(z) + z^{k_0+1} p_{_{N_1}}(N_0)(z)) - f_{j_0}(z) \right| < \frac{1}{s_0}, \text{ where } N_1 = N_0 - k_0. \quad (8)$$

Now the polynomial $R(z) = p(z) + z^{k_0+1} p_{_{N_1}}(N_0)(z)$ has degree at most N_0. We write $R(z)$ as $\sum_{i=0}^{N_0} b_i z^i$. Then $b_i = a_i$ for $i = 0, 1, \dots, \nu_0$, and $b_i = 0$ for $i = \nu_0 + 1, \nu_0 + 2, \dots, k_0$. Let $b := (b_0, b_1, \dots, b_{N_0}, 0, 0, \dots) \in c_{00}$. Then $b \in \mathbb{C}^{\mathbb{N}_0}$, $\rho(a, b) < \varepsilon_0$ by (3) and

$$\sup_{z\in K_{m_0}} \left| \frac{1}{\phi(N_0)} \sum_{i=0}^{N_0} b_i z^i - f_{j_0}(z) \right| < \frac{1}{s_0}$$

and (1) and (2) are satisfied now and our result follows. ■

Now by Lemmas 3.3, 3.4, Proposition 3.5, Baire's Category Theorem and the fact that $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$ is a complete metric space the proof of Proposition 3.2 is complete. ■

Remark 3.6. *The above argument also yields the classical theorem of Seleznev.*

Proposition 3.7. *If* $\limsup\limits_{n\to\infty} \sqrt[n]{|\phi(n)|} < 1$ *then* $\mathcal{U}(\phi) = \emptyset$.

Proof. Let $\limsup\limits_{n\to\infty} \sqrt[n]{|\phi(n)|} = \theta_0 \in [0,1)$. We suppose, to obtain a contradiction, that $\mathcal{U}(\phi) \neq \emptyset$. Let $a = (a_0, a_1, a_2, \dots) \in \mathcal{U}(\phi)$. We consider the compact subsets $K_n := [1, n]$ for $n = 2, 3, \dots$ of $\mathbb{C}$. We set $E = \bigcup_{n=2}^{\infty} K_n = [1, +\infty)$ which is closed and non-thin at infinity. Let $\theta_1 \in (\theta_0, 1)$. We can find a natural number n_0 such that

$$|\phi(n)| < \theta_1^n \qquad n \geq n_0. \quad (1)$$

We apply now the definition of the set $\mathcal{U}(\phi)$ for the compact set K_2 and the constant function $\mathit{1\!I} : K_2 \to \mathbb{C}$, with formula $\mathit{1\!I}(z) = 1$ for all $z \in K_2$. It follows that there exists a subsequence of natural numbers $\lambda^2 = (\lambda_n^2)_{n\in\mathbb{N}}$ such that

$$\sup_{z\in K_2} \left| \frac{1}{\phi(\lambda_n^2)} \sum_{i=0}^{\lambda_n^2} a_i \cdot z^i - 1 \right| \to 0 \text{ as } n\to\infty.$$

By the above convergence and (1) there exists some natural number $\mu_2 > n_0$ such that

$$\sup_{z\in K_2}\left|\frac{1}{\phi(\mu_2)}\sum_{i=0}^{\mu_2} a_i\cdot z^i - 1\right| < \frac{1}{2} \tag{2}$$

and

$$\sup_{z\in K_2}\left|\sum_{i=0}^{\mu_2} a_i\cdot z^i\right| < \frac{3}{2}\theta_1^{\mu_2}. \tag{3}$$

Inductively we see that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of natural numbers such that

$$\sup_{z\in K_n}\left|\frac{1}{\phi(\mu_n)}\sum_{i=0}^{\mu_n} a_i\cdot z^i - 1\right| < \frac{1}{n} \tag{4}$$

and

$$\sup_{z\in K_n}\left|\sum_{i=0}^{\mu_n} a_i\cdot z^i\right| < \frac{3}{2}\theta_1^{\mu_n} \quad \forall\, n\in\mathbb{N}, \quad n = 2,3,\dots. \tag{5}$$

Now we consider the polynomials $p_n = \sum\limits_{i=0}^{\mu_n} a_i z^i$ for $n = 2,3,\dots$. We have

$$\|p_n\|_{K_n} < \frac{3}{2}\theta_1^{\mu_n}, \qquad n = 2,3,\dots. \tag{6}$$

For the polynomials p_n, $n = 2,3,\dots$, $\Gamma = K_2$, $E = [1,+\infty)$ and $d_n = \mu_n$, $n = 2,3,\dots$ we see that the two conditions (i) and (ii) of Theorem 1 of [9] are satisfied (or Theorem 10 of [15]), and so $\limsup\limits_{n\to\infty}\|p_n\|_K^{1/\mu_n} < 1$ for every compact subset K of $\mathbb{C}$.

We apply this conclusion for

$$K = \widetilde{D}(0,1) = \widetilde{D} = \{z\in\mathbb{C}\,\big|\,|z|\le 1\}.$$

Let $\theta_2 := \limsup\limits_{n\to\infty}\|p_n\|_{\widetilde{D}}^{1/\mu_n}$. Then $\theta_2\in(0,1)$. Let $\theta_3\in(\theta_2,1)$. Then there exists $m_0\in\mathbb{N}$ such that $\|p_n\|_{\widetilde{D}} < \theta_3^{\mu_n}$ for all $n\ge m_0$. Thus $\|p_n\|_{\widetilde{D}}\to 0$ as $n\to\infty$. By the maximum principle we see that $0\le|p_n(0)|\le\|p_n\|_{\widetilde{D}}\to 0$ for all $n\ge 2$. Thus we have $|p_n(0)|\to 0$, whence $a_0 = 0$.

So, we have

$$p_n = \sum_{i=1}^{\mu_n} a_i z^i = z\cdot\sum_{i=0}^{\mu_n - 1} a_{i+1} z^i = z\cdot p_n^1$$

where

$$p_n^1 = \sum_{i=0}^{\mu_n - 1} a_{i+1} z^i \quad n \geq 2.$$

We have $\|p_n\|_{\widetilde{D}} \to 0$ so $\|p_n^1\|_{\widetilde{D}} \to 0$. Thus by the maximum principle we conclude that

$$0 \leq |p_n^1(0)| \leq \|p_n^1\|_{\widetilde{D}} \to 0,$$

which implies that $a_1 = 0$. Inductively we see that $a_n = 0$ for all $n = 0, 1, 2, \ldots$. So $p_n = 0$ for all $n \in \mathbb{N}_0$.

This contradicts (4) and the result now follows. ■

Proof of Theorem 3.1. Suppose that $\lim_{n\to\infty} \phi(n) = 0$. Then $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} \leq 1$. If $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} < 1$ then by Proposition 3.7 we have that $\mathcal{U}(\phi) = \emptyset$.

Now let $\limsup_{n\to\infty} \sqrt[n]{|\phi(n)|} = 1$. Then there exists a sequence of natural numbers $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sqrt[\lambda_n]{|\phi(\lambda_n)|} = 1. \tag{1}$$

Since $\lim_{n\to\infty} \phi(n) = 0$ we see that

$$\lim_{n\to\infty} \phi(\lambda_n) = 0. \tag{2}$$

Then making a proof similar to that in Proposition 3.5 for the sequence $\phi \circ \lambda$ instead of ϕ we can take that $\mathcal{U}(\phi)$ is a G_δ dense subset of $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$. The previous proof holds for every subsequence μ of $\phi \circ \lambda$. Then we argue as at the end of the proof of Lemma 2.4 to complete the proof of Theorem 3.1. ■

Remark 3.8. *Theorem 3.1 tells us that the condition for the function ϕ of Theorem 5.1 of [7] cannot be removed but it is not sharp.*

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