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ARTICLE TYPE

Optimal solutions for singular linear systems of Caputo fractional differential equations

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Abstract

In this article, we focus on a class of singular linear systems of fractional differential equations with given non-consistent initial conditions (IC). Since the non-consistency of the IC can not lead to a unique solution for the singular system, we use two optimization techniques to provide an optimal solution for the system. A l_2 perturbation to the non-consistent IC which seeks an optimal solution for the system in terms of least squares, and a second order optimization technique at a l_1 minimum perturbation to the non-consistent IC, including appropriate smoothing. Numerical examples are given to justify our theory. We use the Caputo (C) fractional derivative and two recently defined alternative versions of this derivative, the Caputo–Fabrizio (CF) and the Atangana–Baleanu (AB) fractional derivative.

KEYWORDS:
singular system, optimal solutions, fractional derivative, caputo, initial conditions

1 | INTRODUCTION

In the last decade many authors have studied problems of fractional differential–difference equations and have derived interesting results on different type of problems for given initial, or boundary conditions, see[1,4,5,8,10,12,13,19,21,22,23,25,28,29,40]. Focus has also been given in the mathematical modelling of many phenomena by using fractional operators. The theory of fractional differential equations (FDEs) is a promising tool for applications in physics, electrical engineering, control theory, and in applications where the memory effect appears, see[23,26,27,28,30,33,34,36,37,38,39]. In this article we consider the following system of FDEs:

$$FY^{(a)}(t) = GY(t) + V(t). \tag{1}$$

Where $F, G \in \mathbb{R}^{r\times m}$, $Y : [0, +\infty] \to \mathbb{R}^{m\times 1}$, $V : [0, +\infty] \to \mathbb{R}^{r\times 1}$, and $0 < a < 1$. The matrices F, G can be non-square ($r \neq m$) or square ($r = m$) with F singular (det$F$=0). With $Y^{(a)}$ we denote the fractional derivative as defined in the next section.

For given non-consistent IC, it has been proved that even if there exist a solution for (1), its uniqueness is not guaranteed, see[15]. In our opinion, non-consistency and cases such as singularities of certain systems of FDEs have been mostly avoided in the framework of fractional calculus. Hence, explicit and easily testable optimization methods are required in order to provide optimal solutions, such that applied researchers can redesign their models in cases where the fractional operators provide better results than the classical ones.

The article is organised as follows: in Section 2 we give some necessary definitions and present existing results such as conditions under which, there exist solutions for (1). In Section 3 we present our main results. We use two optimization techniques to provide an optimal solution for the system. A l_2 perturbation to the non-consistent IC which seeks an optimal solution for the system in terms of least squares by minimizing a proposed functional and a second order optimization technique at a l_1 minimum perturbation to the non-consistent (IC), including appropriate smoothing. Finally, in Section 4, numerical examples are given to justify our theory.

2 | PRELIMINARIES

In this section we present some existing results that we will use throughout the paper.

Definition 2.1. (see[5]) Let $Y : [0,+\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of continuous and differentiable functions. Then, the Caputo (C) fractional derivative of order a, $0 < a < 1$, is defined by

$$Y_C^{(a)}(t) := Y^{(a)}(t) = \frac{1}{\Gamma(1-a)} \int_0^t \left[(t-x)^{-a} Y'(x)\right] dx.$$

Recently, a new fractional derivative was defined by Caputo and Fabrizio (see[6]) and it was followed by some related theoretical and applied results (see[2],[3], and the references therein). This is an alternative version of the (C) fractional derivative. It replaces the kernel $(t-x)^{-a}$ with an exponential kernel.

Definition 2.2. (see[6],[7]) Let $Y : [0,+\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of continuous and differentiable functions. Then, the Caputo–Fabrizio (CF) fractional derivative of order a, $0 \le a \le 1$, is defined by

$$Y_{CF}^{(a)}(t) := Y^{(a)}(t) = \frac{1}{1-a} \int_0^t \left[e^{-\frac{a}{1-a}(t-x)} Y'(x)\right] dx.$$

Following the question "what is the most accurate kernel which better describes the dynamics of systems with memory effect?", Atangana and Baleanu, see[3], suggested a second alternative (C) fractional derivative which has a non-local kernel.

Definition 2.3. (see[3]) Let $Y : [0,+\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of differentiable functions. Then, the modified Caputo (AB) fractional derivative of order $0 \le a \le 1$, is defined by

$$Y_{AB}^{(a)}(t) := Y^{(a)}(t) = \frac{B(a)}{1-a} \int_0^t E_a\left[-a\frac{(t-x)^a}{1-a}\right] Y'(x) dx.$$

Where $E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+ak)}$, $a, z \in \mathbb{C}$, $Re(a) > 0$ (see[5]). $B(a)$ denotes a normalization function obeying $B(0) = B(1) = 1$.

Throughout the paper with 0_{ij} we will denote the zero matrix of i rows and j columns. With A^* the conjugate transpose of matrix A and with $diag\left(\left[b_i \right]_{1\le i\le m} \right)$ the diagonal matrix with elements b_1, b_2, ...,b_m. Let $B_{n_1} \in \mathbb{C}^{n_1\times n_1}$,$B_{n_2} \in \mathbb{C}^{n_2\times n_2}$, …, $B_{n_r} \in \mathbb{C}^{n_r\times n_r}$. Then with the direct sum $B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$ we will denote the block diagonal matrix $blockdiag\left[B_{n_1} \; B_{n_1} \; \dots \; B_{n_r} \right]$. Finally, $\|\cdot\|_1$ and $\|\cdot\|_2$ will be the l_1 and l_2 norm respectively.

Definition 2.4. Given $F, G \in \mathbb{C}^{r\times m}$, $0 < a < 1$, an arbitrary $s \in \mathbb{C}$ and an inverse function $z = z(s) \in \mathbb{C}$, the matrix pencil $zF - G$ is called:

1. Regular when $r = m$ and $\det(zF - G) \not\equiv 0$;
2. Singular when $r \ne m$ or $r = m$ and $\det(zF - G) \equiv 0$.

In[15] it has been proved that there exists solutions for (1) if the pencil of the system is regular, or, under some conditions that have to hold, if it is singular with $r > m$. Hence, in this article we are interested in the cease of the regular pencil, and the singular with $r > m$. For a *regular pencil* there exist non-singular matrices $P, Q \in \mathbb{C}^{m\times m}$ such that

$$\begin{aligned} PFQ &= I_p \oplus H_q, \\ PGQ &= J_p \oplus I_q. \end{aligned} \tag{2}$$

Where

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

with $P_1 \in \mathbb{C}^{p\times m}$, $P_2 \in \mathbb{C}^{q\times m}$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$. Furthermore, $H_q \in \mathbb{C}^{q\times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, and $J_p \in \mathbb{C}^{p\times p}$ is a Jordan matrix constructed by the finite eigenvalues of the pencil and their algebraic multiplicity and $p + q = m$.

The *singular pencil* with $r > m$ is characterized by the set of the finite–infinite eigenvalues, and the minimal row indices. Let $\mathcal{N}_l$ be the left null space of a matrix respectively. Then the equations $V^T(s)(sF - G) = 0_{1,m}$ have solutions in $V(s)$, which are vectors in the rational vector space $\mathcal{N}_l(sF - G)$. The binary vectors $V^T(s)$ express dependence relationships among the rows of $sF - G$. Note that $V(s) \in \mathbb{C}^{r\times 1}$ are polynomial vectors. Let t=dim$\mathcal{N}_l(sF - G)$. It is known, that $\mathcal{N}_l(sF - G)$ as rational vector spaces, are spanned by minimal polynomial bases of minimal degrees

$$\zeta_1 = \zeta_2 = ... = \zeta_h = 0 < \zeta_{h+1} \le ... \le \zeta_{h+k=t},$$

which is the set of *row minimal indices* of $sF - G$. This means there are t row minimal indices, but $t - h = k$ non-zero row minimal indices. We are interested only in the k non zero minimal indices. To sum up the invariants of a singular pencil with $r > m$ are the finite – infinite eigenvalues of the pencil and the minimal row indices as described above. Following the above given analysis, there exist non-singular matrices P, Q with $P \in \mathbb{C}^{r\times r}$, $Q \in \mathbb{C}^{m\times m}$, such that

$$\begin{aligned} PFQ &= F_K = I_p \oplus H_q \oplus F_\zeta, \\ PGQ &= G_K = J_p \oplus I_q \oplus G_\zeta. \end{aligned} \quad (3)$$

The matrices P, Q can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q & Q_\zeta \end{bmatrix},$$

with $P_1 \in \mathbb{C}^{p\times r}$, $P_2 \in \mathbb{C}^{q\times r}$, $P_3 \in \mathbb{C}^{\zeta_1\times r}$, $\zeta_1 = k + \sum_{i=1}^{k}[\zeta_{h+i}]$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$, $Q_\zeta \in \mathbb{C}^{m\times\zeta_2}$ and $\zeta_2 = \sum_{i=1}^{k}[\zeta_{h+i}]$. Where J_p is the Jordan matrix for the finite eigenvalues, H_q a nilpotent matrix with index q_* which is actually the Jordan matrix of the zero eigenvalue of the pencil $sG - F$. The matrices F_ζ, G_ζ are defined as

$$F_\zeta = \begin{bmatrix} I_{\zeta_{h+1}} \\ 0_{1,\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} I_{\zeta_{h+2}} \\ 0_{1,\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} I_{\zeta_{h+k}} \\ 0_{1,\zeta_{h+k}} \end{bmatrix}, \quad \text{and} \quad G_\zeta = \begin{bmatrix} 0_{1,\zeta_{h+1}} \\ I_{\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} 0_{1,\zeta_{h+2}} \\ I_{\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0_{1,\zeta_{h+k}} \\ I_{\zeta_{h+k}} \end{bmatrix},$$

with $p + q + \sum_{i=1}^{k}[\zeta_{h+i}] + k = r$, $p + q + \sum_{i=1}^{k}[\zeta_{h+i}] = m$.

Let

$$Z_\zeta(t) = \begin{bmatrix} Z_{\zeta_{h+1}}(t) \\ Z_{\zeta_{h+2}}(t) \\ \vdots \\ Z_{\zeta_{h+k}}(t) \end{bmatrix}, \quad Z_{\zeta_{h+i}}(t) \in \mathbb{C}^{(\zeta_{h+i})\times 1}, \quad i = 1, 2, ..., k \quad \text{with} \quad Z_{\zeta_{h+i}}(t) = \begin{bmatrix} Z_{\zeta_{h+i},1}(t) \\ Z_{\zeta_{h+i},2}(t) \\ \vdots \\ Z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix},$$

and

$$P_3 V(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_k(t) \end{bmatrix}, \quad U_i(t) \in \mathbb{C}^{(\zeta_{h+i}+1)\times 1}, \quad i = 1, 2, ..., k \quad \text{with} \quad U_i(t) = \begin{bmatrix} v_{i0} \\ v_{i1} \\ v_{i2} \\ \vdots \\ v_{i\zeta_{h+i}} \end{bmatrix}, \quad i = 1, 2, ..., k.$$

The following Theorem has been proved, see [15].

Theorem 2.1. There exist solutions for the system of FDEs (1) if and only if

(a) The pencil of the system is regular;

(b) The pencil of the system is singular with $r > m$ and the following equivalence holds:

$$\sum_{\rho=0}^{\zeta_{h+i}} v_{i\rho}^{(\rho a)} = 0.$$

Then in the case of (a), the solution is given by

$$Y(t) = Q_p\left[\Phi_0(t)C + \int_0^\infty \Phi(t-\tau)P_1V(\tau)d\tau\right] - Q_q\sum_{i=0}^{q_*-1} H_q^i P_2 V(t), \tag{4}$$

and in the case of (b) by

$$Y(t) = Q_p\left[\Phi_0(t)C + \int_0^\infty \Phi(t-\tau)V(\tau)d\tau\right] - Q_q\sum_{i=0}^{q_*-1} H_q^i P_2 V(t) + Q_\zeta Z_\zeta. \tag{5}$$

In both (a), (b):

(i) If we use the (C) fractional derivative where $\Phi_0(t)$, $\Phi(t)$ are given by:

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{t^{ak}}{\Gamma(ka+1)}J_p^k, \quad \text{and} \quad \Phi(t) = \sum_{k=0}^{\infty}\frac{t^{ak+a-1}}{\Gamma(ka+a)}J_p^k;$$

(ii) If we use the (CF) fractional derivative where $\Phi_0(t)$, $\Phi(t)$ are given by:

$$\Phi_0(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}(1-a)^n a^{k-n}\frac{t^{k-n}}{\Gamma(k+1-n)}J_p^k, \quad \text{and} \quad \Phi(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\binom{k+1}{n}(1-a)^n a^{k+1-n}\frac{t^{k-n}}{\Gamma(k+1-n)}J_p^k;$$

(iii) If we use the (AB) fractional derivative where $\Phi_0(t)$, $\Phi(t)$ are given by:

$$\Phi_0(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}\frac{(1-a)^n a^{k-n}}{B^k(a)}\frac{t^{ak+2-an}}{\Gamma(ak+1-an)}J_p^k, \quad \text{and} \quad \Phi(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\binom{k+1}{n}(1-a)^n a^{k+1-n}\frac{t^{ak+a-an+1}}{\Gamma(ak+a-an)}J_p^k.$$

3 | MAIN RESULTS

Having identified the conditions under which there exists solutions for singular systems in the form of (1), the next step should be to explore the behavior of the system for given IC. Let

$$K = \left\{\begin{array}{ll} Q_q\sum_{i=0}^{q_*-1} H_q^i P_2 V(0), & \text{if the pencil of (1) is regular} \\ Q_q\sum_{i=0}^{q_*-1} H_q^i P_2 V(0) - Q_\zeta Z_\zeta, & \text{if the pencil of (1) is singular} \end{array}\right\}. \tag{6}$$

If there exist solutions for (1), and the given IC are $Y(0) = Y_0$, then by replacing the IC into (4) and (5) respectively, we get:

$$Q_pC = [Y_0 + K]. \tag{7}$$

Note that as defined in the previous section, Q_p is a $m \times p$ matrix with $m > p$ and hence, in respect to C, the above system is overdetermined. It is obvious that since rank$Q_p = p$ (linear independent columns), for

$$Y_0 + K \in colspanQ_p \tag{8}$$

system (7) will have a unique solution. Consequently system (1) will have a unique solution. If (8) holds then the IC will be called consistent IC. If (8) does not hold, then the IC will be called *non-consistent* IC because in this case it would be not possible for C to be identified uniquely.

As mentioned in the previous sections, we are interested in this article for the case of given non-consistent IC. In this case optimization methods are required to get an optimal solution for the system of FDEs (1). We can now state the following Theorem.

Theorem 3.1. We consider system (1) with non-consistent IC, $Y(0) = Y_0$. Then, after a l_2 perturbation to the non-consistent IC accordingly to $min\left\|Y_0-\hat{Y}_0\right\|_2$, and subject to $\hat{Y}_0$ being a consistent IC, an optimal solution of the initial value problem is given by:

$$\hat{Y}(t)=Q_p\Phi_0(t)Q_p(Q_p^*Q_p)^{-1}Q_p^*[Y_0+K]+L. \tag{9}$$

Where

$$L=Q_p\int_0^\infty\Phi(t-\tau)P_1V(\tau)d\tau-Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(t)+\left\{\begin{array}{ll}0, & \text{if the pencil of (1) is regular,}\\ Q_\zeta Z_\zeta, & \text{if the pencil of (1) is singular.}\end{array}\right\}.$$

The matrices Q_p, Q_q, Q_ζ, P_1, P_2 are given by (2), (3). $\Phi_0(t)$, $\Phi(t)$, Z_ζ are given by (4), (5) as parts of the general solution of (1) and K is given by (6).

Proof. The IC Y_0 are assumed non-consistent, i.e. $Y_0+K\notin colspanQ_p$. Thus, system (7) has no solutions. Let $\hat{Y}_0$ be a vector such that $\hat{Y}_0+K\in colspanQ_p$ and let $\hat{C}$ be the unique solution of the system $Q_p\hat{C}=[\hat{Y}_0+K]$. Hence we want to solve the following optimization problem:

$$\text{minimize }\left\|Y_0-\hat{Y}_0\right\|_2^2,\quad\text{subject to:}\quad Q_p\hat{C}=[\hat{Y}_0+K].$$

If we consider that system (1) with a singular pencil, the above expression can be written in the form:

$$min\left\|Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta-Q_p\hat{C}\right\|_2^2.$$

Where $\hat{C}$ is the optimal solution, in terms of least squares (see[9], [17], [20]), of the linear system (12). Thus we seek a solution $\hat{C}$ by minimizing the functional

$$D_1(\hat{C})=\left\|Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta-Q_p\hat{C}\right\|_2^2.$$

Expanding $D_1(\hat{C})$ gives

$$D_1(\hat{C})=(Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta-Q_p\hat{C})^*(Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta-Q_p\hat{C}),$$

or, equivalently,

$$D_1(\hat{C})=[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta]^*[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta]-$$

$$2[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta]^*Q_p\hat{C}+(\hat{C})^*Q_p^*Q_p\hat{C},$$

because

$$[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta]^*Q_p\hat{C}=(\hat{C})^*Q_p^*[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta].$$

Furthermore

$$\frac{\partial}{\partial\hat{C}}D_1(\hat{C})=-2Q_p^*[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta]+2Q_p^*Q_p\hat{C}.$$

Setting the derivative to zero, $\frac{\partial}{\partial\hat{C}}D_1(\hat{C})=0$, we get

$$Q_p^*Q_p\hat{C}=Q_p^*[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta].$$

Since $rankQ_p=p$, the matrix $Q_p^*Q_p$ is invertible and the solution is given by

$$\hat{C}=(Q_p^*Q_p)^{-1}Q_p^*[Y_0+Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V(0)-Q_\zeta Z_\zeta].$$

Note that in the case that we consider (1) with a regular pencil the term $Q_\zeta Z_\zeta$ vanishes and it is easy to observe that $\hat{C}$ takes the form

$$\hat{C} = (Q_p^* Q_p)^{-1} Q_p^* [Y_0 + Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V(0)].$$

Hence we conclude to

$$\hat{C} = \left\{ \begin{array}{c} (Q_p^* Q_p)^{-1} Q_p^* [Y_0 + Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V(0)], \quad \text{if the pencil of (1) is regular,} \\ \\ (Q_p^* Q_p)^{-1} Q_p^* [Y_0 + Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V(0) - Q_\zeta Z_\zeta], \quad \text{if the pencil of (1) is singular.} \end{array} \right\}.$$

and by replacing C with $\hat{C}$ into the general solution (4), (5), an optimal solution of (1) with non-consistent IC Y_0 is given by (9). The proof is completed.

Although this optimization method is easy for use, there can be times where the optimal solution can be the zero vector. Hence we propose another method based on second order optimization and the l_1 norm. Actually, the previous method focused more on finding an optimal solution in terms of least squares to the overdetermined system (7) while the next Theorem will aim into moving under a minimum perturbation from a non-consistent IC to a consistent IC.

Theorem 3.2. We consider system (1) with given non-consistent IC Y_0. Then, after a l_1 perturbation to the non-consistent IC accordingly to $min \left\| Y_0 - \hat{Y}_0 \right\|_1$, and by using a second order optimization method subject to $\hat{Y}_0$ being a consistent IC, we obtain the following optimal consistent IC for (1):

$$\hat{Y}_0 = Y_0 - \left[\gamma \mu^{-1} I_m + (S_q^{-1})^* S_q^{-1}\right]^{-1} \left[(S_q^{-1})^* (S_q^{-1} Y_0 + S_q^{-1} K)\right]. \tag{10}$$

Where γ, μ are a-priori chosen scalars and K is given by (6). If $Q_p^{(i)}$, $i = 1, 2, ..., p$ are p linear independent eigenvectors of the finite eigenvalues and Q_q is a matrix with columns the q linear independent eigenvectors of the infinite eigenvalue, then for $c_i \in \mathbb{R}$, the matrix $S_q \in \mathbb{C}^{m \times q}$ contains q linear independent columns which belong to the set $colspanQ_q + \sum_{i=1}^p c_i Q_p^{(i)}$, i.e. $colspanS_q = colspanQ_q + \sum_{i=1}^p c_i Q_p^{(i)}$. The matrix S_q^{-1} is the left inverse of S_q.

Proof. If Y_0 is a non-consistent IC for (1) then (8) does not hold. Let $\hat{Y}_0$ be a consistent IC. Then

$$\hat{Y}_0 + K \in colspanQ_p.$$

Where K is given by (6), Q_p is a matrix with columns the p linear independent eigenvectors of the pencil of the system. For the non-consistent Y_0 we have, see [14],

$$Y_0 + K \in colspanQ_q + \sum_{i=1}^p c_i Q_p^{(i)}, \quad c_i \in \mathbb{R}.$$

Where $Q_p^{(i)}$, $i = 1, 2, ..., p$ are p linear independent eigenvectors of the finite eigenvalues and Q_q is a matrix with columns the q linear independent eigenvectors of the infinite eigenvalue. Let $S_q \in \mathbb{C}^{m \times q}$ be a matrix with rank equal to q, belonging to the set $colspanQ_q + \sum_{i=1}^p c_i Q_p^{(i)}$, i.e.

$$colspanS_q = colspanQ_q + \sum_{i=1}^p c_i Q_p^{(i)}.$$

Then,

$$Y_0 \in colspanS_q.$$

There always exist S_q^{-1}, left inverse of S_q, such that $S_q^{-1} S_q = I_q$. For the given non-consistent IC Y_0 we aim to find a consistent IC such that they minimize the distance

$$min \left\| Y_0 - \hat{Y}_0 \right\|_1,$$

subject to $\hat{Y}_0$ being consistent IC, i.e. $\hat{Y}_0 + K \in colspanQ_p$, or, equivalently from [14],

$$\hat{Y}_0 + K \in N_r \left\{ colspanS_q^{-1} \right\}.$$

Where N_r is the right kernel of the set $colspanS_q^{-1}$. To sum up, we have the following optimization problem:

$$\text{minimize } \|Y_0 - \hat{Y}_0\|_1, \quad \text{subject to:} \quad S_q^{-1} \hat{Y}_0 = -S_q^{-1} K.$$

In this case, the optimal solution of the following l_1-analysis problem

$$\text{minimize } f_\gamma(\hat{Y}_0) := \gamma \|Y_0 - \hat{Y}_0\|_1 + \frac{1}{2}\|S_q^{-1}\hat{Y}_0 + S_q^{-1}K\|_2^2,$$

is proved to be a good approximation to $\hat{Y}_0$. Where γ is an a-priori chosen positive scalar and $\|\cdot\|_2$ is the Euclidean norm. Let

$$Y_0 = \begin{bmatrix} y_0^{(1)} \\ y_0^{(2)} \\ \vdots \\ y_0^{(m)} \end{bmatrix}, \quad \hat{Y}_0 = \begin{bmatrix} \hat{y}_0^{(1)} \\ \hat{y}_0^{(2)} \\ \vdots \\ \hat{y}_0^{(m)} \end{bmatrix}.$$

Since we will apply second order optimization, we will use derivatives of first and second order. However, the l_1-norm is not differentiable. Many researchers in the literature use first order optimization methods and apply appropriate smoothing into their problem by using the Huber function. In our case this is not possible since the Huber function is differentiable differentiable but not twice differentiable. Hence, we propose to replace the l_1-norm with the Pseudo-Huber function [16],

$$\psi_\mu(Y_0 - \hat{Y}_0) = \sum_{i=1}^{m}((\mu^2 + |y_0^{(i)} - \hat{y}_0^{(i)}|^2)^{\frac{1}{2}} - \mu),$$

where μ controls the quality of approximation, i.e. for $\mu \to 0$ then $\psi_\mu(x)$ tends to the l_1-norm. The Pseudo-Huber function is smooth and has derivatives of all degrees, see Figure 1. It can be derived by perturbing the absolute value function $|x| =$

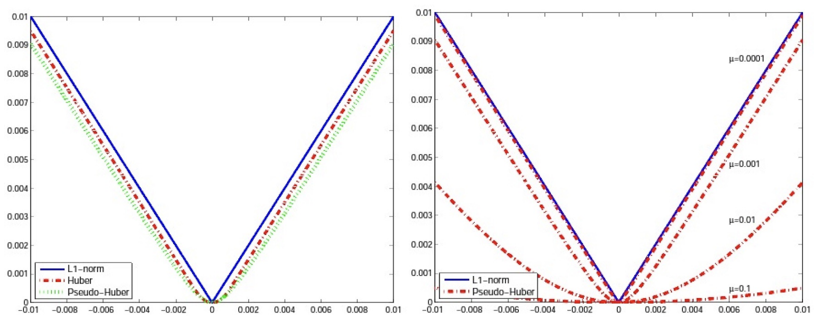

FIGURE 1 On the left a comparison between the l_1 norm, the Huber and the Pseudo-Huber function, and on the right the l_1 norm and the Pseudo-Huber function for different values of μ, see [16].

$\sup\{xz \mid -1 \le z \le 1\}$ with the proximity function $d(z) = 1 - \sqrt{1 - z^2}$ in order to get the smooth function

$$|x|_\mu = \sup\{xz + \mu\sqrt{1 - z^2} - \mu \mid -1 \le z \le 1\} = \sqrt{x^2 + \mu^2} - \mu.$$

Our optimization problem is then approximated by

$$\text{minimize } f_\gamma^\mu(\hat{Y}_0) := \gamma\psi_\mu(Y_0 - \hat{Y}_0) + \frac{1}{2}\|S_q^{-1}\hat{Y}_0 + S_q^{-1}K\|_2^2.$$

Note that $f_\gamma^\mu : \mathbb{R}^m \to \mathbb{R}$. A second order approximation of f_γ^μ at Y_0 is

$$\tilde{f}_\gamma^\mu(\hat{Y}_0) = f_\gamma^\mu(Y_0) + \nabla f_\gamma^\mu(Y_0)^*(\hat{Y}_0 - Y_0) + \frac{1}{2}(\hat{Y}_0 - Y_0)^*\nabla^2 f_\gamma^\mu(Y_0)(\hat{Y}_0 - Y_0).$$

Where $\nabla f_\gamma^\mu(Y_0)$ is $m \times 1$ and $\nabla^2 f_\gamma^\mu(Y_0)$ is $m \times m$. For the optimality condition at $\hat{Y}_0^{opt}$ we set

$$\nabla \tilde{f}_\gamma^\mu(\hat{Y}_0^{opt})^* = 0_{1,m},$$

or, equivalently,

$$\nabla f_\gamma^\mu(Y_0)^* + (\hat{Y}_0 - Y_0)^*\nabla^2 f_\gamma^\mu(Y_0) = 0_{1,m},$$

or, equivalently,

$$\nabla f_\gamma^\mu(Y_0) + \nabla^2 f_\gamma^\mu(Y_0)(\hat{Y}_0 - Y_0) = 0_{m,1},$$

or, equivalently,

$$\gamma\nabla\psi_\mu(0_{m,1}) + [S_q^{-1}]^*(S_q^{-1}Y_0 + S_q^{-1}K) + \left[\gamma\nabla^2\psi_\mu(0_{m,1}) + [S_q^{-1}]^*S_q^{-1}\right](\hat{Y}_0 - Y_0) = 0_{m,1},$$

or, equivalently,

$$\left[\gamma\nabla^2\psi_\mu(0_{m,1})+[S_q^{-1}]^*S_q^{-1}\right](\hat{Y}_0-Y_0)=-[\gamma\nabla\psi_\mu(0_{m,1})+[S_q^{-1}]^*(S_q^{-1}Y_0+S_q^{-1}K)],$$

or, equivalently,

$$\hat{Y}_0-Y_0=-\left[\gamma\nabla^2\psi_\mu(0_{m,1})+[S_q^{-1}]^*S_q^{-1}\right]^{-1}[\gamma\nabla\psi_\mu(0_{m,1})+[S_q^{-1}]^*(S_q^{-1}Y_0+S_q^{-1}K)],$$

or, equivalently,

$$\hat{Y}_0=Y_0-\left[\gamma\nabla^2\psi_\mu(0_{m,1})+[S_q^{-1}]^*S_q^{-1}\right]^{-1}[\gamma\nabla\psi_\mu(0_{m,1})+[S_q^{-1}]^*(S_q^{-1}Y_0+S_q^{-1}K)].$$

The gradient of the pseudo-Huber function $\psi_\mu(Y_0-\hat{Y}_0)$ is given by

$$\nabla\psi_\mu(Y_0-\hat{Y}_0)=\begin{bmatrix}(y_0^{(1)}-\hat{y}_0^{(1)})[\mu^2+(y_0^{(1)}-\hat{y}_0^{(1)})^2]^{-\frac{1}{2}}\\(y_0^{(2)}-\hat{y}_0^{(2)})[\mu^2+(y_0^{(2)}-\hat{y}_0^{(2)})^2]^{-\frac{1}{2}}\\\vdots\\(y_0^{(m)}-\hat{y}_0^{(m)})[\mu^2+(y_0^{(m)}-\hat{y}_0^{(m)})^2]^{-\frac{1}{2}}\end{bmatrix}.$$

Hence,

$$\nabla\psi_\mu(0_{m,1})=0_{m,1}.$$

The Hessian matrix is given by

$$\nabla^2\psi_\mu(Y_0-\hat{Y}_0)=\mu^2 diag\Big(\Big[[\mu^2+(y_0^{(i)}-\hat{y}_0^{(i)})^2]^{-\frac{3}{2}}\Big]_{1\le i\le m}\Big).$$

Hence,

$$\nabla^2\psi_\mu(0_{m,1})=\mu^2 diag\Big(\Big[[\mu^2]^{-\frac{3}{2}}\Big]_{1\le i\le m}\Big),$$

or, equivalently,

$$\nabla^2\psi_\mu(0_{m,1})=\mu^{-1}I_m.$$

Hence

$$\hat{Y}_0=Y_0-\left[\gamma\mu^{-1}I_m+(S_q^{-1})^*S_q^{-1}\right]^{-1}\left[(S_q^{-1})^*(S_q^{-1}Y_0+S_q^{-1}K)\right].$$

The proof is completed.

Remark 3.1. The matrix $\left[\gamma\mu^{-1}I_m+(S_q^{-1})^*S_q^{-1}\right]^{-1}$ is always invertible because γ can be controlled.

Remark 3.2. Q_q has columns the eigenvectors of the infinite eigenvalue, or, equivalently, from[11], the eigenvectors of the zero eigenvalue of the pencil $F-sG$. This means that if F is symmetric then $S_q^*S_q$ is the identity matrix. In many applications which deal with Differential - Algebraic equations F is always symmetric, see[31],[32].

Remark 3.3. In the case that system (1) has a singular pencil with $r>m$ and there exist solutions for the system, if irank$F=m$ then for the matrix F there exists a left inverse F^{-1} with dimension $m\times r$. By multiplying system (1) with the left inverse of F we have:

$$F^{-1}FY^{(a)}(t)=F^{-1}GY(t)+F^{-1}V(t),$$

or, equivalently,

$$Y^{(a)}(t)=F^{-1}GY(t)+F^{-1}V(t),$$

where $F^{-1}G$ is a square matrix. Hence we have an equal system of regular type which has a unique solution for any given IC.

4 | NUMERICAL EXAMPLES

In this Section we provide numerical examples to justify our theory. We consider system (1), assume that the input vector is always zero, and will use the (C) fractional derivative:

Example 1: Let

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}.$$

Then for $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$ and Theorem 2.1, there exist solution given by:

$$Y(t) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{an} c_1}{\Gamma(an+1)} \\ \sum_{n=0}^{\infty} \frac{2^n t^{an} c_2}{\Gamma(an+1)} \end{bmatrix}.$$

The matrix F has linear independent columns, i.e. rank$F$=2 and from Remark 3.3 any IC leads to a unique solution. Hence, in this case we don't require an optimal solution.

Example 2: Let

$$F = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \frac{1}{5} \begin{bmatrix} -5 & 0 & 8 & 5 & -3 \\ -11 & -1 & 14 & 11 & -8 \\ -2 & -2 & 2 & 2 & 0 \\ 11 & 2 & -14 & -11 & 8 \\ -5 & 0 & 10 & 5 & -5 \end{bmatrix}.$$

Then det$(sF - G)$=$s(s - \frac{1}{5})(s - \frac{2}{5})$ and the pencil is regular. The three finite eigenvalues (p=3) of the pencil are 0, $\frac{1}{5}$, $\frac{2}{5}$. Then, the Jordan matrix J_p, and Q_p, the matrix with columns the linear independent eigenvectors of the finite eigenvalues, have respectively the form:

$$J_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}, \quad Q_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, from (4), the general solution is given by

$$Y(t) = \frac{1}{5} \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ak+1)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2^k \\ 0 & 0 & 0 \\ 0 & 0 & 2^k \\ 0 & 0 & 2^k \end{bmatrix} C,$$

where C unknown vector 3×1. Let $Y_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, be given IC. Then, it is easy to observe that the IC are non-consistent and thus from Theorem 3.1:

$$\hat{C} = (Q_p^T Q_p)^{-1} Q_p^T Y_0 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

The optimal solution of the initial value problem is then given by (9),

$$Y(t) = \frac{1}{15} \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ak+1)} \begin{bmatrix} 0 \\ 1 + 2^{k+1} \\ 0 \\ 2^{k+1} \\ 2^{k+1} \end{bmatrix}.$$

Example 3: Let

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad G = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then det$(sF - G)$=$s + \frac{4}{5}$ and the pencil is regular. The finite eigenvalue (p=1) of the pencil is $-\frac{4}{5}$ and the Jordan matrix $J_p = -\frac{4}{5}$, and $Q_p = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. It is easy to observe that

$$Y_0 \in colspan Q_p.$$

Then, the IC are consistent and thus from (4), the general solution is given by

$$Y(t) = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ak+1)} (-\frac{4}{5})^k C \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Where C is the unique solution of the linear system $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} C$, and thus $C = 1$.

Example 4: We assume the matrices as in Example 3 but with different IC. Let

$$Y_0 = \begin{bmatrix} 2.00001 \\ 2.99999 \end{bmatrix}.$$

It is easy to observe that

$$Y_0 \notin colspanQ_p,$$

i.e. the IC are non-consistent. From Theorem 3.1, an optimal solution for C is given by,

$$\hat{C} = (Q_p^T Q_p)^{-1} Q_p^T Y_0 = \begin{bmatrix} 12.99999 \\ 13 \end{bmatrix},$$

and the optimal solution of the initial value problem is given by

$$Y(t) = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ak+1)} (-\frac{4}{5})^k \frac{12.99999}{13} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Example 5: Let

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then from Theorem 2.1, there exists solution given by (4):

$$Y(t) = \begin{bmatrix} -\sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} \\ 0 \end{bmatrix} c.$$

We assume the IC $Y_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The pencil $sF - G$ has one finite eigenvalue, $s = 1$ and one infinite. The column vector spaces of the eigenvectors of the finite eigenvalue, and of the eigenvectors of the infinite eigenvalue, are respectively:

$$colspanQ_p =< \begin{pmatrix} 1 \\ 0 \end{pmatrix} >, \quad colspanQ_q =< \begin{pmatrix} 1 \\ -1 \end{pmatrix} > .$$

Then

$$Y_0 \notin colspanQ_p,$$

which means that Y_0 is a non-consistent IC. We may use Theorem 3.2, to provide an optimal solution for the system. If we use the first method to seek an optimal for c and eventually $Y(t)$, we will end up to $\hat{c} = 0$. Hence, we will use the alternative optimization method as described in Theorem 3.2 using the l_1 norm and the second order optimization method. We have

$$\hat{Y}_0 = Y_0 - \left[\gamma\mu^{-1} I_m + (S_q^{-1})^T S_q^{-1}\right]^{-1} \left[(S_q^{-1})^T S_q^{-1} Y_0\right].$$

or, equivalently,

$$\hat{Y}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \left[\gamma\mu^{-1} I_m + (S_q^{-1})^T S_q^{-1}\right]^{-1} \left[(S_q^{-1})^T S_q^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right].$$

While $\hat{Y}_0$ is assumed a consistent IC, i.e. $\hat{Y}_0 \in< \begin{pmatrix} 1 \\ 0 \end{pmatrix} >$, we may choose $S_q^{-1} = \begin{bmatrix} 0 & -1 \end{bmatrix}$, since in this way, $S_q^{-1} S_q = 1$ and $S_q^{-1} \hat{Y}_0 = 0$. Hence

$$\hat{Y}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \left[\gamma\mu^{-1} I_m + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

or, equivalently, for $\gamma = -2\mu$, $\hat{Y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. An optimal solution of the system is then given by:

$$Y(t) = \begin{bmatrix} -\sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} \\ 0 \end{bmatrix}.$$

CONCLUSIONS

For singular systems of FDEs of Caputo and related fractional derivatives, it has been proved that even if there exist solutions, the uniqueness for given IC is not guaranteed. For this case we provided two optimization methods to obtain an optimal solution of the system. The first method uses a l_2 perturbation to the non-consistent IC which provides an optimal solution to the system in terms of least squares by minimizing a proposed functional. The other alternative method is a second order optimization technique at a $l_1$minimum perturbation to the non-consistent IC, including appropriate smoothing. Numerical examples where given to justify our theory. As a future research we plan to apply the (C), (CF), (AB) fractional derivatives, and the results derived in this work, into Power Systems modeled as: (i) systems of differential–algebraic equations, see[31], and (ii) systems of delayed differential–algebraic equations, see[24], [32].

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