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Reproducing kernels for polyharmonic polynomials

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Abstract. The reproducing kernel of the space of all homogeneous polynomials of degree k and polyharmonic order m is computed explicitly, solving a question of A. Fryant and M.K. Vemuri.

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1. Introduction

Let U be an open set in the euclidean space $\mathbb{R}^d$. A function $f : U \to \mathbb{C}$ is called *polyharmonic of order* m if f is $2m$-times differentiable and $\Delta^m f(x) = 0$ for all $x \in U$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$$

is the *Laplace operator* and Δ^m is its m-th iterate. For $m = 1$ this class of functions is just the class of all *harmonic* functions, while for $m = 2$ the term *biharmonic* function is used which is important in elasticity theory. Polyharmonic functions have been studied by several mathematicians, see e.g. [20], [21], [22], [23], [31], [35], and classical work is due to E. Almansi [1], M. Nicolesco [33] and N. Aronszajn [4]. Polyharmonicity is an important tool in several areas of mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis, see [6], [24], [25], [26], [28], [32].

In this paper we shall be concerned with a problem posed by A. Fryant and M.K. Vemuri in [18]. Let $\mathcal{P}\left(\mathbb{R}^d\right)$ be the space of all polynomials endowed with the scalar product

$$\langle P, Q \rangle_F := \sum_{|\alpha| \leq N} \alpha! c_\alpha \overline{d_\alpha} \tag{1.1}$$

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for polynomials $P(x)=\sum_{|\alpha|\leq N}c_{\alpha}x^{\alpha}$ and $Q(x)=\sum_{|\alpha|\leq N}d_{\alpha}x^{\alpha}$. An alternative way to define the scalar product (1.1) is the following:

$$\langle P,Q\rangle_F=\left[P\left(\frac{\partial}{\partial x_1},....,\frac{\partial}{\partial x_d}\right)\overline{Q}\right](0).$$

Let $\mathcal{P}_k\left(\mathbb{R}^d\right)$ be the space of all homogeneous polynomials of degree k. Define the Hilbert space of all homogeneous polynomials of degree k which are polyharmonic of order at most m, so we define

$$\mathcal{H}_k^m\left(\mathbb{R}^d\right):=\left\{h\in\mathcal{P}_k\left(\mathbb{R}^d\right):\Delta^m h=0\right\}.$$

Let $Q_k^j(x)$ with $j=1,...,b_d^{k,m}:=\dim\mathcal{H}_k^m\left(\mathbb{R}^d\right)$ be an orthonormal basis of $\mathcal{H}_k^m\left(\mathbb{R}^d\right)$ with respect to the inner product (1.1) and define the *reproducing kernel* Z_k^m of $\mathcal{H}_k^m\left(\mathbb{R}^d\right)$ by

$$Z_k^m(x,y):=\sum_{j=1}^{b_d^{k,m}}Q_k^j(x)\overline{Q_k^j(y)}. \tag{1.2}$$

In [18] it was proved that there exists a constant $\gamma_d^k(m)$, depending only on the dimension d, the integer m and the degree k, such that

$$\sum_{j=1}^{b_d^{k,m}}\left|Q_k^j(x)\right|^2=\gamma_d^k(m)\ \text{ for all } x\in\mathbb{S}^{d-1}, \tag{1.3}$$

where $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d:|x|=1\}$ is the unit sphere and $|x|^2=x_1^2+...+x_d^2$ for $x=(x_1,...,x_d)\in\mathbb{R}^d$. However, the nature of this constant was not further explored. We shall show that

$$\gamma_d^k(m)=\sum_{s=0}^{\min\{k/2,m-1\}}\frac{a_{k-2s}}{2^s s!d(d+2)....(d+2(k-s-1))} \tag{1.4}$$

where a_k is the dimension of $\mathcal{H}_k^1\left(\mathbb{R}^d\right)$, the set of all homogeneous harmonic polynomials of degree k, given by

$$a_k:=\dim\mathcal{H}_k^1\left(\mathbb{R}^d\right)=\frac{(2k+d-2)(k+d-3)!}{k!(d-2)!}, \tag{1.5}$$

see e.g. [2, p. 450]. Furthermore we shall show that the reproducing kernel $Z_k^m(x,y)$ can be described explicitly:

$$Z_k^m(x,y)=\omega_{d-1}\sum_{s=0}^{\min\{[k/2],m-1\}}\frac{|x|^{2s}|y|^{2s}Z_{k-2s}(x,y)}{2^s s!d(d+2)....(d+2(k-s-1))}$$

where $Z_k(x,y)$ is the *zonal harmonic of degree* k with pole y (for definition see Section 2). Formula (1.4) allows us to improve a criterion for the convergence of the orthogonal series

$$\sum_{k=0}^{\infty}\sum_{j=1}^{b_d^{k,m}} a_{k,j}Q_k^j\left(x\right)$$

which will be presented in Section 3.

2. The reproducing kernel

The inner product defined in (1.1) is an important tool in the theory of spherical harmonics, see [3], [5], [12], [16], [24], [39]. We note that in [34] and [38] this inner product is called the *Fischer inner product,* in honour of the work of E. Fischer [14]; in [8], [9], [42] it is called the *Bombieri inner product*, and in [18] the *Calderón inner product*. However, it seems that is a classical tool in invariant theory (see [13], [36], [40]) and we shall refer to it as the *apolar inner product*.

The apolar inner product has the following property: for all polynomials f, g

$$\left\langle Q^{*}\left(D\right)f,g\right\rangle_F=\left\langle f,Q\cdot g\right\rangle_F \tag{2.1}$$

where $Q^{*}\left(x\right)$ is the polynomial obtained by conjugation the coefficients of the polynomial Q, and $Q^{*}\left(D\right)$ is the differential operator associated to $Q^{*}\left(x\right)$. Equation (2.1) says that the adjoint of the multiplication operator $g\longmapsto Qg$ is just the differential operator $Q^{*}\left(D\right)$. In passing, we note that the apolar inner product has an integral representation:

$$\left\langle f,g\right\rangle_F=\frac{1}{\pi^d}\int_{\mathbb{C}^d}f\left(z\right)\overline{g\left(z\right)}e^{-\left|z\right|^2}dz$$

where dz is Lebesgue measure on $\mathbb{R}^{2d}$, see [7]. The space of all entire functions $f:\mathbb{C}^n\rightarrow\mathbb{C}$ which satisfy

$$\left\|f\right\|_F^2:=\frac{1}{\pi^d}\int_{\mathbb{C}^d}\left|f\left(z\right)\right|^2e^{-\left|z\right|^2}dz<\infty \tag{2.2}$$

is called the *Bargmann space* $\mathcal{F}_n$ (also called *Fock* or *Fischer space,* see [34]).

For homogeneous polynomials f, g we define the following well-known inner product

$$\left\langle f,g\right\rangle_{\mathbb{S}^{d-1}}:=\int_{\mathbb{S}^{d-1}}f\left(\theta\right)\overline{g\left(\theta\right)}d\theta \tag{2.3}$$

where $\mathbb{S}^{d-1}$ is the unit sphere and $d\theta$ is the rotation-invariant measure on $\mathbb{S}^{d-1}$. The following result follows from Theorem 5.14 in [5]; the result seems to be due to Ü. Kuran [30]:

Theorem 2.1. *For homogeneous harmonic polynomials f, g of degree k one has*

$$\left\langle f,g\right\rangle_F=d\left(d+2\right)...\left(d+2k-2\right)\frac{1}{\omega_{d-1}}\left\langle f,g\right\rangle_{\mathbb{S}^{d-1}}.$$

In the following we need some facts from the theory of spherical harmonics. Let $Y_{k,l}(x), l = 1, ..., a_k$ be a basis of $\mathcal{H}_k^1(\mathbb{R}^d)$ which is orthonormal with respect to the scalar product (2.3). Then

$$Z_k(x,y) := \sum_{l=1}^{a_k} Y_{k,l}(x)\overline{Y_{k,l}(y)} \tag{2.4}$$

is the *reproducing kernel* of $\mathcal{H}_k^1(\mathbb{R}^d)$ with respect to (2.3); the function $x \mapsto Z_k(x,y)$ is also called the *zonal harmonic* of degree k with pole y. The addition theorem says that

$$Z_k(x,y) = \frac{a_k}{\omega_{d-1}} |x|^k \cdot |y|^k P_k\left(\frac{\langle x,y\rangle}{|x|\cdot|y|}\right) \tag{2.5}$$

where P_k is a polynomial of degree k with $P_k(1) = 1$ (see [2, p. 455]) and ω_{d-1} is the surface area of $\mathbb{S}^{d-1}$. The polynomial P_k is up to a factor equal to the ultraspherical polynomial $C_k^{(d-2)/2}(t)$, (see [2, p. 456]). Since $P_k(1) = 1$ one has

$$P_k(t) = \frac{C_k^{(d-2)/2}(t)}{C_k^{(d-2)/2}(1)}.$$

Observe that the property $P_k(1) = 1$ also implies that

$$Z_k(x,x) = \frac{a_k}{\omega_{d-1}} |x|^{2k}. \tag{2.6}$$

We now prove

Theorem 2.2. *Let $Y_{k,l}(x)$, $l = 1, ..., a_k$, be an orthonormal basis of $\mathcal{H}_k^1(\mathbb{R}^d)$ with respect to the scalar product (2.3). Then the polynomials $|x|^{2s} Y_{k,l}(x)$ for $s, k \in \mathbb{N}_0$ and $l = 1, ..., a_k$ are orthogonal with respect to the apolar inner product and*

$$\omega_{d-1} \left\| |x|^{2s} Y_{k,l}(x) \right\|_F^2 = 2^s s! d(d+2) \dots (d + 2(k+s-1)). \tag{2.7}$$

Proof. Let $|x|^{2s} Y_{k,l}$ and $|x|^{2s_1} Y_{k_1,l_1}$ be two basis functions. Without loss of generality we may assume that $s \le s_1$. By property (2.1) we obtain

$$\left\langle |x|^{2s} Y_{k,l}, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = \left\langle \Delta^s \left[|x|^{2s} Y_{k,l}\right], |x|^{2s_1-2s} Y_{k_1,l_1} \right\rangle_F.$$

It is well known and it follows by a quick computation that the following formula

$$\Delta\left(|x|^{2s} h\right) = 2s\left[2s + d - 2 + 2\deg h\right] \cdot |x|^{2s-2} h \tag{2.8}$$

holds for any harmonic homogeneous polynomial h. A simple induction argument shows that for $m \le 2s$

$$\Delta^m\left(|x|^{2s} h\right) = |x|^{2s-2m} \cdot h \cdot (2s) \dots (2s - 2(m-1)) \cdot \tag{2.9}$$

$$[2s + d - 2 + 2\deg h] \dots [2s - 2(m-1) + d - 2 + 2\deg h].$$

In particular, for $m = s$ we obtain that $\Delta^s\left(|x|^{2s} h\right) = d_s(\deg h)\cdot h$ where $d_s(\deg h)$ is the number

$$2^s s! \cdot (2s + d - 2 + 2\deg h)(2s - 2 + d - 2 + 2\deg h) \ldots. (d + 2\deg h). \qquad (2.10)$$

Thus we have

$$\left\langle |x|^{2s} Y_{k,l}\,,\, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = d_s(k) \cdot \left\langle Y_{k,l}\,,\, |x|^{2s_1-2s} Y_{k_1,l_1} \right\rangle_F. \qquad (2.11)$$

If $s_1 > s$ we can use again (2.1) and we see that

$$\left\langle |x|^{2s} Y_{k,l}\,,\, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = d_s(k) \cdot \left\langle \Delta Y_{k,l}, |x|^{2s_1-2s-2} Y_{k_1,l_1} \right\rangle_F = 0.$$

If $s_1 = s$, and $k \neq k_1$ or $l \neq l_1$, we see from (2.11) that $\left\langle |x|^{2s} Y_{k,l}\,,\, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = 0$ since $Y_{k,l}$ and Y_{k_1,l_1} are orthogonal according to Theorem 2.1.

For $(s,k,l) = (s_1,k_1,l_1)$ Theorem 2.1 and (2.11) show that $\omega_{d-1}\left\| |x|^{2s} Y_{k,l} \right\|_F^2$ is equal to the product of $d(d+2)\ldots(d+2k-2)$ and $d_s(k)$ (so (2.10) for $\deg h = k$). This product is equal to

$$2^s s! d(d+2) \ldots. (d + 2(k+s-1)).$$

□

Proposition 2.3. *The system $|x|^{2s} Y_{k-2s,l}(x)$ for $s = 0,1,\ldots,\min\{[k/2], m-1\}$ and $l = 1,\ldots,a_{k-2s}$ is an orthogonal basis for $\mathcal{H}_k^m(\mathbb{R}^d)$.*

Proof. The polynomial $f(x) = |x|^{2s} Y_{k,l}(x)$ satisfies $\Delta^m f = 0$ if and only if $s \leq m-1$. Hence $|x|^{2s} Y_{k-2s,l}(x)$, $s = 0,1,\ldots,\min\{[k/2], m-1\}$ and $l = 1,\ldots,a_{k-2s}$, are homogeneous polynomials of degree k which satisfy $\Delta^m f = 0$, and by Theorem 2.2 these functions are orthogonal. In order to see that it is basis, let $f \in \mathcal{H}_k^m(\mathbb{R}^d)$. Then f can be written uniquely in the form $f = \sum_{s=0}^{[k/2]} |x|^{2s} h_{k-2s}$ with harmonic homogeneous polynomials h_{k-2s} of degree $k-2s$, see [5]. Formula (2.9) shows that $\Delta^m\left(|x|^{2s} h_{k-2s}\right) = C_{m,k,s}|x|^{2s-2m} h_{k-2s}$ for $m \leq s$ and for some nonzero constant $C_{m,k,s}$. The condition $\Delta^m f = 0$ implies that the summation in the last sum ranges only over indices s with $s \leq m-1$. So f is a linear combination of the above basis functions. □

Theorem 2.4. *The reproducing kernel $Z_k^m(x,y)$ for the Hilbert space $\mathcal{H}_k^m(\mathbb{R}^d)$ endowed with the apolar inner product is given by*

$$Z_k^m(x,y) = \omega_{d-1} \sum_{s=0}^{\min\{[k/2],m-1\}} \frac{|x|^{2s}|y|^{2s} Z_{k-2s}(x,y)}{2^s s! d(d+2)\ldots.(d+2(k-s-1))}. \qquad (2.12)$$

Proof. We use formula (1.2) for the system $|x|^{2s} Y_{k-2s,l}(x)$, $l = 1,\ldots,a_{k-2s}$, $s = 0,1,\ldots,\min\{[k/2], m-1\}$, by taking into account the normalization constants given in (2.7). This gives

$$Z_k^m(x,y)=\omega_{d-1}\sum_{s=0}^{\min\{[k/2],m-1\}}\sum_{l=1}^{a_{k-2s}}\frac{|x|^{2s}Y_{k-2s,l}(x)\,|y|^{2s}Y_{k-2s,l}(y)}{2^s s!d(d+2)\ldots.(d+2(k-s-1))}.$$

Now (2.4) completes the proof. □

Corollary 2.5. *The values $Z_k^m(x,x)$ for $|x|=1$ of the reproducing kernel Z_k^m of $\mathcal{H}_k^m(\mathbb{R}^d)$ are constant equal to*

$$\gamma_d^k(m):=\sum_{s=0}^{\min\{[k/2],m-1\}}\frac{a_{k-2s}}{2^s s!d(d+2)\ldots.(d+2(k-s-1))}. \tag{2.13}$$

Proof. Insert $y=x$ in formula (2.12) and use (2.6). □

3. Convergence of orthogonal series

Suppose that $Q_k^j(x)$, $j=1,\ldots,b_d^{k,m}$ is an orthonormal basis of $\mathcal{H}_k^m(\mathbb{R}^d)$ for each $k=0,1,2,\ldots$, and let $a_{k,j}, j=1,\ldots,b_d^{k,m}$ be complex numbers. A. Fryant and M.K. Vemuri discuss in [18] conditions for the numbers $a_{k,j}, j=1,\ldots,b_d^{k,m}$ such that the series

$$f(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{b_d^{k,m}}a_{k,j}Q_k^j(x) \tag{3.1}$$

converges absolutely and uniformly on compact subsets of the open ball B_R with radius R and center 0. It is shown in [18] that the series (3.1) converges compactly in B_R for

$$R^{-1}=\limsup_{k\to\infty}\left(\sqrt{\gamma_d^k(m)}\,\|a_k\|\right)^{1/k}\quad\text{and}\quad\|a_k\|^2:=\sum_{j=1}^{b_d^{k,m}}|a_{k,j}|^2.$$

By the next Theorem we obtain the more precise description

$$R^{-1}=\frac{1}{\sqrt{2}}\limsup_{k\to\infty}\left(\frac{\|a_k\|}{\sqrt{k!}}\right)^{1/k}$$

improving the upper bound for R^{-1} in [18] by a factor $1/\sqrt{2}$.

Theorem 3.1. *Let $M_k, k\in\mathbb{N}_0$, be positive numbers and $\gamma_d^k(m)$ as in (2.13). Then*

$$\limsup_{k\to\infty}\left(\sqrt{\gamma_d^k(m)}M_k\right)^{1/k}=\frac{1}{\sqrt{2}}\limsup_{k\to\infty}\left(\frac{M_k}{\sqrt{k!}}\right)^{1/k}. \tag{3.2}$$

Proof. Let us define $D_k(d,s):=d(d+2)\ldots.(d+2(k-s-1))$. From the identity

$$D_k(d,s)=2^{k-s}\frac{d}{2}\left(\frac{d}{2}+1\right)\ldots.\left(\frac{d}{2}+(k-s-1)\right) \tag{3.3}$$

we see that

$$D_k(d,s) \geq 2^{k-s-1}(k-s-1)! \geq 2^{k-s-1}k!\frac{1}{(k+1)^{s+1}}. \tag{3.4}$$

Observe that the inequality $a_{k-2s} \leq a_k \leq 2(k+1)^{d-2}$ is obtained from rewriting the formula (1.5) for a_k as

$$a_k = 2(k+1)\left(\frac{k}{2}+1\right)\dots\left(\frac{k}{d-3}+1\right)\left(\frac{k}{d-2}+\frac{1}{2}\right)$$

for $d>2$; for $d=2$ it is well known that $a_k = 2$ for all $k \in \mathbb{N}$. Thus we obtain

$$\gamma_d^k(m) \leq \sum_{s=0}^{m-1}\frac{2(k+1)^{d-1+s}}{s!2^{k-1}k!} \leq \frac{(k+1)^{d-2+m}}{2^{k-2}k!}\sum_{s=0}^{\infty}\frac{1}{s!}.$$

Now take the square root, multiply the inequality with M_k, take the k-th root and then the limes superior. Hence the $\leq$ in (3.2) is proved.

For the other inequality we estimate $\gamma_d^k(m)$ below by taking only the summand for $s=0$, so

$$\gamma_d^k(m) \geq \frac{a_k}{d(d+2)\dots.(d+2(k-1))}.$$

Now (3.3) yields $D_k(d,0) \leq 2^k(d+k)! \leq 2^k k!(d+k)^d$. Using that $a_k \geq 1$ we obtain

$$\gamma_d^k(m) \geq \frac{1}{2^k k!(d+k)^d}.$$

Again, take the square root, multiply the inequality with M_k, take the k-th root and then the limes superior. □

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