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The Effect of Vehicle Velocity on the Dynamic Amplification of Two Vehicles crossing a Simply Supported Bridge

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Abstract

The effect of multiple vehicles on a bridge's dynamic amplification is a complex problem. Previously authors have examined multiple vehicle presence by constructing elaborate finite element models or undertaking field tests. Although both these methods give valuable information regarding the magnitude of dynamic amplification, the results tend to be site-specific and give limited insight into how large amplifications occur. This paper examines the dynamic amplification factor of a simply supported bridge being crossed by two loads traveling in both the same and opposing directions. Simple numerical point load models are used to determine the critical load velocities and load positions that result in high amplifications. An experimentally validated finite element model is used to examine the applicability of the conclusions to real bridge/vehicle systems.

1 Introduction

The dynamic amplification resulting from two vehicles crossing a simply supported bridge is a considerably more complex process than for a single vehicle. However, two vehicle events are generally more critical overall than single vehicle crossings and it is this dynamic amplification that is most relevant to bridge design and assessment. A single point force crossing a bridge is considered in a companion paper (Brady et al. 2005). The two main areas considered in this paper are the effect of the spacing of following vehicles, and the effect of meeting position of opposing vehicles. Humer & Kashif (1995) examined the case of two following sprung masses meeting on a plate. They found that, in general, the dynamic amplification for two following forces is less than for a single force. Yang et al. (1995) found that the dynamic amplification for following vehicles were similar to those for opposing vehicles. Zhu & Law (2002) found that the amplification for a following point force is identical to an opposing force, but the amplification factor for two following and two opposing forces were not identical. Kirkegaard et al. (1997) found that the amplification for opposing vehicles is considerably lower than for a single vehicle crossing a bridge.

To the authors' knowledge, little research has been undertaken to determine *why* particular following/meeting scenarios result in high amplifications. This paper uses simple point load models, crossing a simply supported beam, to investigate the effect of load velocity and relative load position on the dynamic amplification for both meeting and opposing loads.

2 Dynamic Amplifications for Following Loads

In the case of two following vehicles, each point load is deemed to represent a separate (uni-axle) vehicle. This is similar to the case considered in Brady et al. (2004), except that two vehicles are being considered here. The loads both travel at the same constant velocity. This is a reasonable assumption as in reality, following vehicles will generally match velocities. Clearly, representing a multi-axle vehicle with a single point load is approximate. As the bridge span increases, the effect of the individual axles will become less significant.

A schematic of the problem is shown in Figure 1. The equation for this system has been described by Frýba (1971) in dimensionless form:

$$\frac{\partial^2 q_{(j)}(\tau)}{\partial \tau^2} = \frac{96}{\pi^2 \alpha^2} \sum_{i=1}^2 \varepsilon_i P_i \sin j\pi\xi_i - j^4 \frac{\pi^2}{\alpha^2} q_{(j)}(\tau) - \frac{\vartheta}{\alpha} \frac{\partial q_{(j)}(\tau)}{\partial \tau}$$

$$j = 1, 2, 3, \dots \quad (1)$$

$q_{(j)}(\tau)$ is the *generalized* dimensionless time coordinate of the beam and τ is the dimensionless time coordinate. P_i is the dimensionless load magnitude, where $i=1$ represents the leading load and $i=2$ represents the following load; the value of ε_i determines if a particular load is on the beam; ξ_i is the dimensionless position of each of the loads; α represents the speed parameter; ϑ is the damping parameter. Equation 1 can be solved numerically using the Runge-Kutta-Nyström method (Kreyszig 1993). Values for $\ddot{q}_{(j)}(\tau)$, $\dot{q}_{(j)}(\tau)$, and $q_{(j)}(\tau)$ can be obtained for each mode of vibration j and each moment in dimensionless time, τ . The dimensionless bending moment, defined by equation 2, can be evaluated using equation 3.

$$M(\xi, \tau) = \frac{M(x, t)}{M_0}, \quad (2)$$

where

$$M_0 = \bar{P}l/8$$

$$M(\xi, \tau) = \sum_{i=1}^2 M_{Ri}(\xi, \tau) + M_{\mu}(\xi, \tau) \quad (3)$$

where:

$$M_{Ri}(\xi, \tau) = \begin{cases} \frac{8}{Pl} \frac{1}{l} \bar{\epsilon}_i \bar{P}_i (l - x_i) x = 4\epsilon_i P_i (1 - \xi_i) \xi & \text{for } \xi_i \geq \xi \\ \frac{8}{Pl} \frac{1}{l} \bar{\epsilon}_i \bar{P}_i (l - x) x_i = 4\epsilon_i P_i (1 - \xi) \xi_i & \text{for } \xi_i \leq \xi \end{cases}$$

and

$$M_{\mu}(\xi, \tau) = -\frac{1}{12} \alpha^2 \sum_{j=1}^{\infty} \frac{1}{j^2} \ddot{q}_{(j)}(\tau) \sin j\pi\xi \quad (4)$$

For the purposes of analysis the dimensionless parameters ‘Inter Load Spacing (ILS) and Frequency Ratio (FR) are introduced. The ILS is defined as the load spacing divided by the beam span length and the FR as the ratio of the load circular frequency ($\pi c_l/l$) to the beam first circular frequency. Figure 2 shows the variation in dynamic amplification factor for mid-span bending moment with ILS and FR. This figure is valid for all beams as the FR is inversely proportional to beam first circular frequency.

For low FR's (low load velocities), the amplification factor is within the range of 1 to 1.5. When the FR reaches a value of 0.45 there is a substantial increase in the dynamic amplification factor, and in general it increases with increasing FR. There is a change in the pattern of amplification factor at an ILS of 0.5. This is due to a difference in how the peak amplification factors occur. For an ILS of less than 0.5, the first load has not reached mid-span as the second load arrives. Therefore a considerable degree of interaction occurs between the two loads. With an ILS of greater than 0.5, the first load is leaving the beam as the second load arrives, thus reducing the extent of interaction.

In practice, FR's in excess of 0.2 generally correspond to unrealistically high highway velocities, e.g., for a 25 m beam with a first circular frequency of 21.86 rad/s, a Frequency Ratio of 0.2 corresponds to 125 km/hr (78 mph). The range of interest, 0.078 to 0.2, is illustrated in Figure 3. It is apparent that for certain ILS's (e.g., around 0.75) maximum amplifications occur. However, these peaks occur at discrete values of FR that are independent of the ILS. These critical FR's are 0.089, 0.096, 0.135 and 0.158. The practical implications of this finding is significant - it means that the critical dynamic amplifications are unaffected by the ILS in excess of 0.5. For ILS's below 0.5, the critical load circular frequencies cannot be determined in this manner. However, by inspection it can be seen that at an FR of approximately 0.17, the dynamic amplification factor rises to between 1.2 and 1.3. The information necessary to determine the load velocities that result in maximum dynamic amplification factors for two following loads is presented in Table 1.

2.1 Formation of Maximum Dynamic Amplification Factors

To explain the reasons for the peaks in dynamic amplification, they are examined in the context of a particular beam. Figure 4 shows the variation of dynamic amplification factor with load circular frequency and Inter Load Spacing for a 25 m beam with a first circular frequency of 21.86 rad/s and 3% damping.

The values of maximum amplification factor occur at the critical load velocities which can be calculated from Table 1: 55.7 km/hr, 60.1 km/hr 84.5 km/hr, 98.9 km/hr and greater than 106.5 km/hr. To explain why high amplifications occur, a number of different ILS's are examined in the following sections.

Inter Load Spacing of 0.1

Two point loads with an ILS of 0.1 traverse the 25 m beam; in this case the distance between the loads is 2.5 m. Figure 5 shows the variation in dynamic amplification factor with load circular frequency. The figure also shows the dynamic amplification factor for a single load crossing the beam for reference (Brady et al. 2004). Maximum dynamic amplifications are found at load frequencies of 3.45 rad/s and 9.35 rad/s (FR's of 0.158 and 0.427 respectively). These correspond to velocities of 99 km/hr and 268 km/hr. These values of maximum dynamic amplification differ from the case of a single load crossing the beam, although there are similarities in the pattern.

To determine why these particular velocities result in high amplifications' the individual bending moment response curves are considered - Figure 6. The mid-span bending moment response for two point loads crossing a simply supported beam together is identical to the superposition of the two individual loads (Wu & Daj 1986;

Chan & O'Connor 1990). This principle of superposition can also be applied to loads separated by a significant distance; the superposition is merely carried out by including the free vibration of the individual loads in the calculation. Figure 6 shows the normalized bending moment response curve for a peak amplification, in this case corresponding to a load circular frequency of 3.45 rad/s. The bending moment responses for the individual loads are reasonably high. Constructive interference in conjunction with individually high bending moments causes the peak amplification factor. Figure 7 shows a low amplification factor caused by the superposition of two single load responses - each of these responses individually represents a low amplification factor. For both of these cases, constructive interference occurs between the individual load responses. These areas of maximum amplification for two loads are generally the same areas of peak amplification factor for single load events.

Inter Load Spacing of 0.4

When the ILS reaches 0.4, there is a marked change in the shape of the dynamic amplification factor versus load circular frequency curve from that of a single load - Figure 8. For load frequencies greater than 3 rad/s, the regions of maximum dynamic amplification for the single load generally correspond to regions of low dynamic amplification factor for the case of two loads. Maximum individual load dynamic amplifications occur at values of load frequency of 2.3 rad/s and 5.5 rad/s respectively (FR's of 0.105 and 0.252). Figure 9 shows the bending moment response for the maximum amplification factor at 5.5 rad/s. The shape of the two-load response is similar to a peak dynamic amplification factor for a single load, i.e., it has three peaks. However, the response is constructed from two single load events, neither of which individually results in a peak amplification. In this case constructive interference occurs between the second peak in the first response and the first peak in the second

response. Figure 10 shows the bending moment response for a low amplification factor with a load circular frequency of 7.5 rad/s. In contrast to Figure 9, the areas of low overall amplification are caused by the superposition of two individual loads, each of which results in a high amplification factor for a single load.

In conclusion, peak amplifications can only occur when the two central peaks in a single load response are of similar magnitude. Both peaks are then effectively summed to give a maximum bending moment, thus resulting in a high dynamic amplification. In the case of low dynamic amplification factors, destructive interference occurs between the individual responses. Thus, when summed, they result in a low amplification factor.

Inter Load Spacing of 0.8

As stated earlier, the manner in which peak amplifications occur, changes for ILS values greater than 0.5. Figure 11 shows the dynamic amplification factor versus load circular frequency for the 25 m bridge (3% damping) with an ILS of 0.8. There are significant differences between the two curves illustrated. The loads are sufficiently separated that they do not affect each other significantly. However, free vibration from the first load now begins to affect the overall response. In practice, this means that the beam is already vibrating as the second load arrives. Figure 12 shows the response for two loads crossing the beam at a load circular frequency of 3.75 rad/s (maximum dynamic amplification factor). The figure also displays the two single load events to illustrate the contribution made by each individual load. As can be seen from the figure, neither of the single load events are at a maximum. The maximum value in the two load response is defined by the second peak in the second load. This maximum occurs because of the superposition of the fourth peak of the first load (free

vibration) and the second peak of the second load. Thus constructive interference occurs between the second load and the free vibration of the first.

3 Dynamic Amplifications for Two Meeting Loads

The mathematical formulation to describe two loads meeting on a beam is similar to that of two following loads. The equation of motion for the beam is the same, except that in this case the positions of the loads are defined differently. Meetings of two vehicles are important as, for many bridges, the critical load effect occurs as two highway vehicles meet near the peak of the influence line.

Figure 13 shows the schematic of two loads meeting on the simply supported beam. A_{loads} is defined as the distance between the two loads when \bar{P}_1 is at mid-span. As the load \bar{P}_1 starts at the left hand side of the beam at $t=0$ and velocities are assumed equal, A_{loads} also represents the length of the approach for \bar{P}_2 . Using these definitions, the motion of the beam can be described by equation 1. The following new dimensionless parameter is defined:

$$A = \frac{A_{loads}}{l} . \tag{5}$$

The beam response for following loads and opposing loads have been found to be identical where A is used in place of Inter Load Spacing (Brady 2004, Wu and Daj 1986). Two loads meeting at mid-span ($A=0$) is an important meeting scenario as it generally corresponds to the maximum static beam response. It can be seen from

Figure 3 that the maximum amplification factors do not occur when the loads meet at the mid-span of the beam. This is a very significant finding as it means that the maximum dynamic amplification does not occur for the same meeting scenario as the maximum static load case. In effect, the maximum dynamic bridge response is not the maximum amplification factor multiplied by the maximum static loading, as the two maxima do not occur simultaneously. This is explained by comparing the dynamic amplification for a single point load and two point loads crossing a beam in opposing directions, meeting at mid-span. Both result in identical dynamic amplification factors. This means that the extra load travelling on the bridge does not cause an increase in the dynamic amplification factor above that of a single load. For other values of A , constructive interference results in higher amplification factors as illustrated, for example Figure 5.

The effect of the level of beam damping was examined for two loads crossing a simply supported beam. It was found that, similar to the case of a single load, the FR's and ILS's at which the maximum dynamic amplification factors occur, are independent of the level of damping. Only the magnitude of the amplification is affected.

3.1 Comparison of Simple Model with Finite Element Model

The validated finite element (FE) model described by Brady et al. (2004) is used to examine the conclusions arrived at from the simple two point load models. The dynamic amplification factor for two following FE vehicle models is compared with the two point loads crossing the bridge, each load deemed to represent a vehicle.

Figure 14 shows dynamic amplification factors for the 32 m long bridge with 3% damping being traversed by two point loads. This is, in effect, a portion of the contour plot of Figure 3. It is clear that a maximum dynamic amplification factor occurs at approximately 106 km/hr with a load spacing of approximately 23 m. Simulations using an elaborate finite element model (Brady et al. 2004) were carried out for this velocity and vehicle spacing range. The F.E. simulations represent two following two-axle vehicles crossing a bridge in Slovenia in Lane 1 for a variety of velocities. Figure 15 shows the dynamic amplification factors for a range of vehicle velocities and vehicle spacings. While the dynamic amplification factors are clearly different, a considerable degree of similarity exists between the simple and more complex models. Each figure has a region of maximum amplification, and the general positions of the maximum and minimum amplifications are similar. Both the simple and complex models have maximum amplifications at a spacing of 23 m. However, the simple point load model was unable to accurately predict the magnitudes of the maximum dynamic amplification factors. The other significant difference is the velocity at which the maximum dynamic amplification factors occur. For the case of the two point loads the maximum occurs at 106 km/hr, while for the two two-axle vehicles, maxima occur at 99 km/hr and 106 km/hr.

The peak amplifications are examined to determine if they are formed in a similar manner. Figure 16 shows the comparison between the bending moment for the two point loads and the two vehicle models. The two point loads are traveling at a velocity of 106 km/hr and a spacing of 22.4 m. The two two-axle vehicles are traveling at 106.34 km/hr with a spacing, front axle of leading vehicle to front axle of following vehicle, of 22.4 m. Also included in the figure are the corresponding individual

load/vehicle events. It is clear in each case that a maximum response occurs due to constructive interference between each of the individual loads. The manner in which this interference occurs is almost identical in each case, i.e., the same number of peaks, the same general shape, etc. It is important to note that in the case of point loads, the individual loads can be superimposed to achieve the overall response. However, in the case of sprung loads (the two-axle vehicle) this is no longer true.

In addition to a maximum amplification factor occurring at 106 km/hr in Figure 15, a maximum also occurs at 99 km/hr. Figure 17 illustrates the amplifications for the F.E. model at 99 km/hr. From a comparison of this with Figure 16(b), it is clear that both maximum responses are caused in the same manner, i.e., constructive interference. Therefore, although the two two-axle vehicle models have two peaks of amplification factor, the shape of each maximum response is very similar. The two-point load model accurately predicts the manner in which these high amplification factors occur. The reason for the two peaks occurring in the F.E. model was examined by reducing the axle spacing of both two-axle vehicles to 1 m. It was found that as the axle spacing of the vehicles was reduced, the distance between the two peaks decreased, converging to approximately the single peak predicted by the simple model. The reason why two discrete maxima occur in the two-vehicle model is due to the axle spacing of the vehicles (Brady 2004).

4 Conclusions

For following loads it has been shown that peak dynamic amplifications occur for a selection of Critical Frequency Ratios and Inter Load Spacings. These values can be

used to determine the load spacings and velocities at which peak dynamic amplifications occur for a particular beam.

The specific case of two point loads traveling at the same velocity and meeting at the mid-span of the bridge was examined. It was found that this load case does not result in the maximum dynamic amplification. Thus, the maximum static load effect and the maximum dynamic amplification do not occur simultaneously.

The effect of beam damping was examined for the cases of two point loads crossing a beam in the same direction and two point loads traveling in opposing directions. It was found that the level of damping does not significantly affect the Critical Frequency Ratios or Inter Load Spacings at which maximum amplification factors occur. However, the damping does affect the magnitude of the individual amplification factors.

The dynamic amplification factors for two two-axle sprung mass models crossing the bridge in the same direction were compared with the results for two point loads crossing the bridge. It was found that the simple model accurately predicted the velocities and the manner in which maximum amplification factors occurred. However, the simple model does not accurately predict the magnitude of these factors.

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Appendix. Notation

A	- dimensionless spacing between meeting loads when leading load is at mid-span,
A_{loads}	- dimensioned spacing between meeting loads when leading load is at mid-span,
D	- load spacing,
P_i	- dimensionless load magnitude, where $i=1$ represents the leading load and $i=2$ represents the following load,
c_1	- velocity of first load,
c_2	- velocity of second load,
j	- mode number,
l	- beam or bridge span,
x_i	- dimensioned load positions,
$q_{(j)}(\tau)$	- the <i>generalized</i> dimensionless time coordinate of the beam,

- τ - dimensionless time coordinate,
- ε_i - determines if a particular load is on the beam,
- ξ_i - dimensionless position of each of the loads,
- α - speed parameter;
- ϑ - damping parameter,
- $M(\xi, \tau)$ - dimensionless bending moment,
- M_0 - static mid-span bending moment,
- $M_{Ri}(\xi, \tau)$ - quasi-static bending moment,
- $M_{\mu}(\xi, \tau)$ - dimensionless bending moment due to inertial forces.

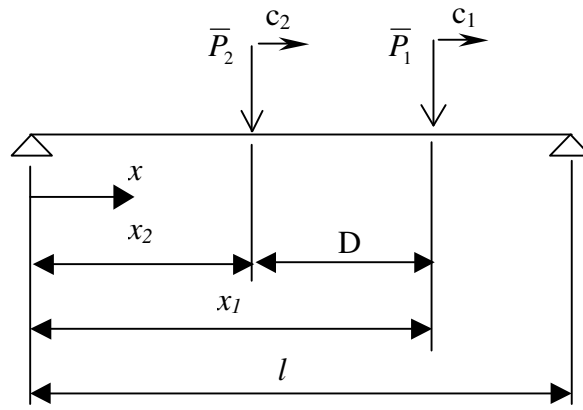


Figure 1 – Schematic of the problem

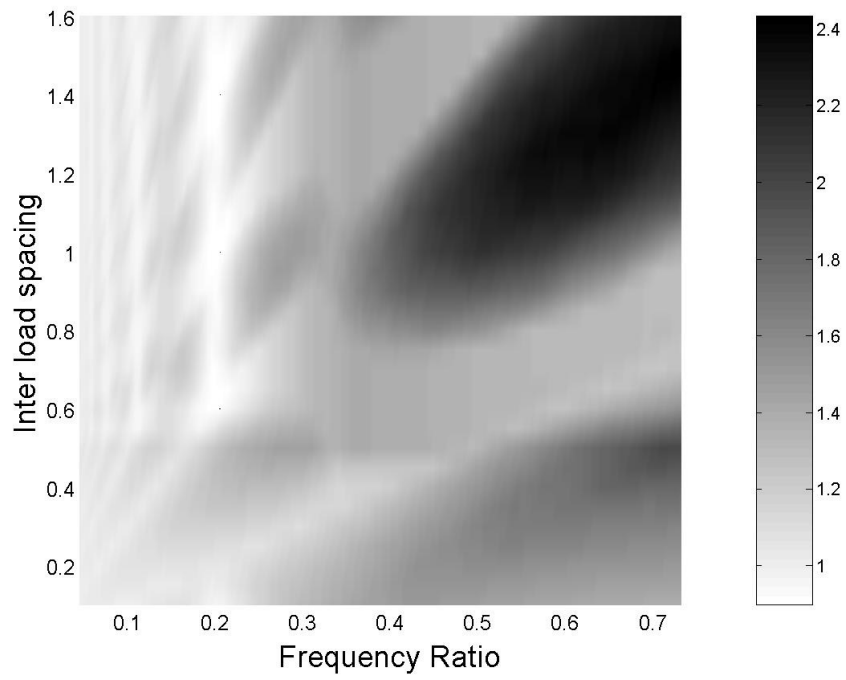


Figure 2 – Contour plot of dynamic amplification factor versus Frequency Ratio and Inter load spacing

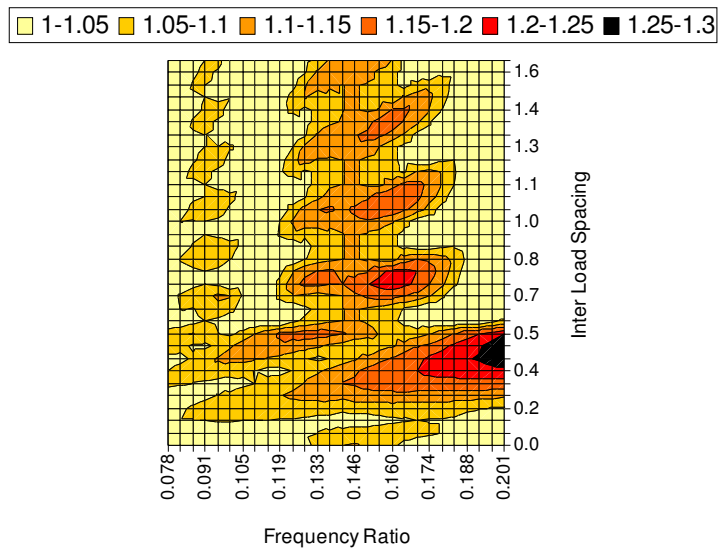


Figure 3 – Dynamic amplification factor versus Frequency Ratio and Inter Load Spacing

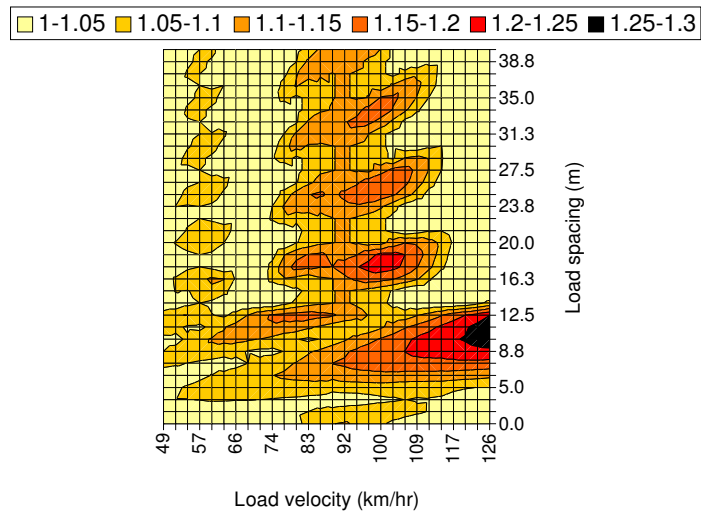


Figure 4 – Dynamic amplification factor versus Inter Load Spacing and load circular frequency for 25 m bridge ($\omega_{(1)}=21.86$ rad/s) with 3% damping

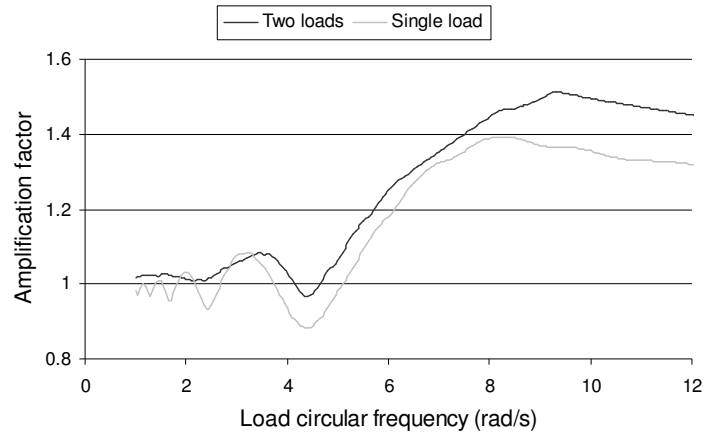


Figure 5 – Dynamic amplification factor versus load frequency for a 25 m beam ($\omega_{(1)}=21.86$ rad/s) being traversed by one load, and by two loads with a load spacing of 2.5 m (ILS=0.1)

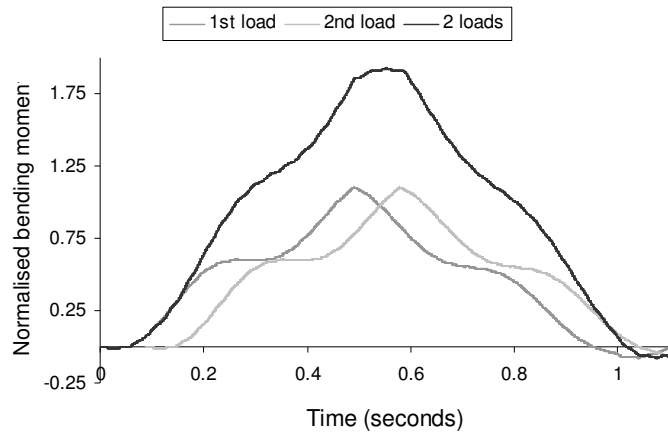


Figure 6 – Peak amplification for load circular frequency of 3.45 rad/s

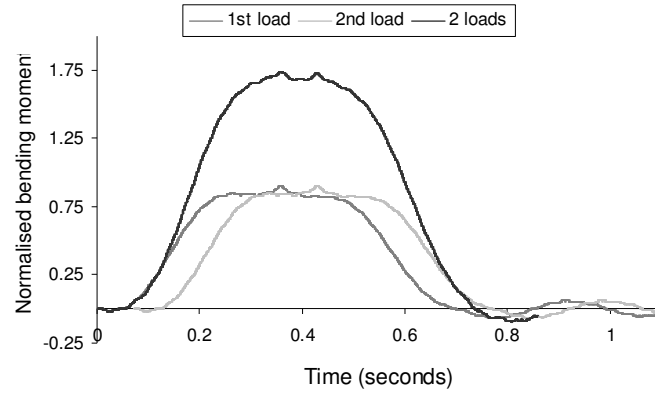


Figure 7 – Low amplification for load circular frequency of 4.4 rad/s

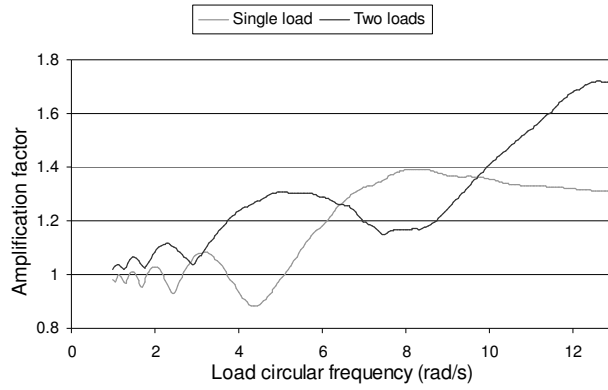


Figure 8 – Graph of dynamic amplification factor versus load frequency for a 25 m beam ($\omega_{(1)}=21.86$ rad/s) being traversed by one load, and by two loads with a load spacing of 10 m (ILS=0.4)

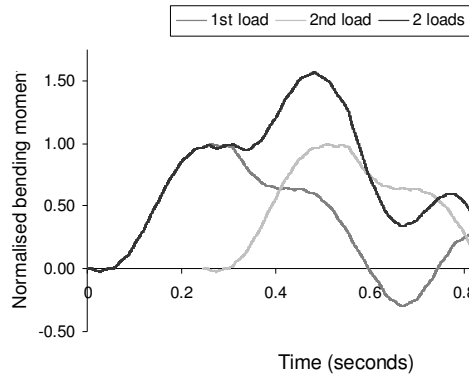


Figure 9 – Load circular frequency of 5.5 rad/s, high amplification factor

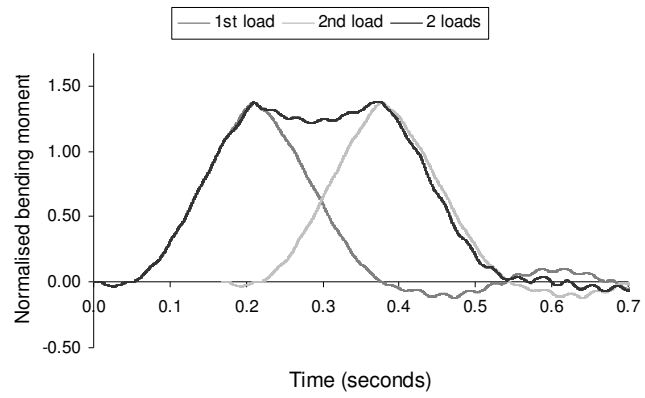


Figure 10 – Low amplification for load circular frequency of 7.5 rad/s

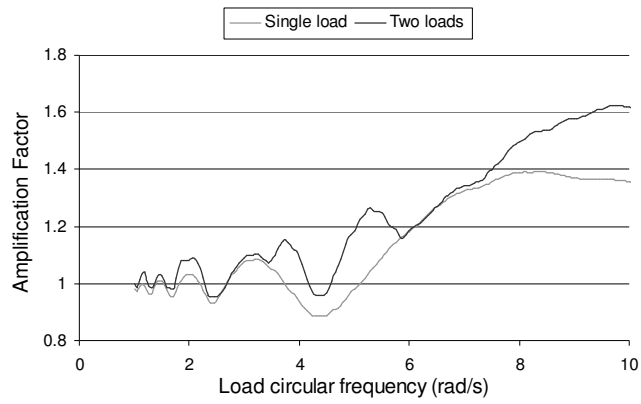


Figure 11 – Graph of dynamic amplification factor versus load frequency for a 25 m beam ($\omega_{(1)}=21.86$ rad/s) being traversed by one load, and by two loads with a load spacing of 20 m (ILS=0.8)

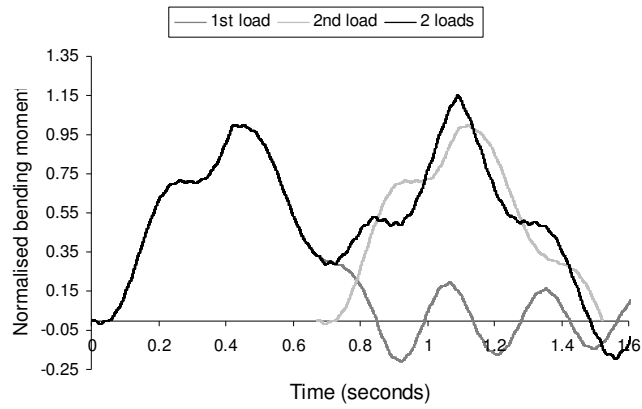


Figure 12 – Peak amplification for 25m bridge, $\omega=3.75$ rad/s,
(ILS=0.8)

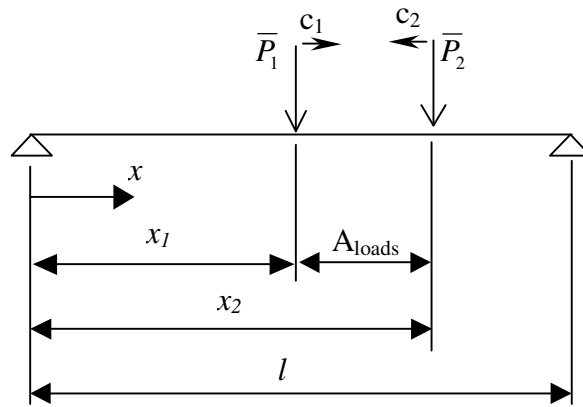


Figure 13 – Schematic of two loads meeting (arrow denotes direction of load)

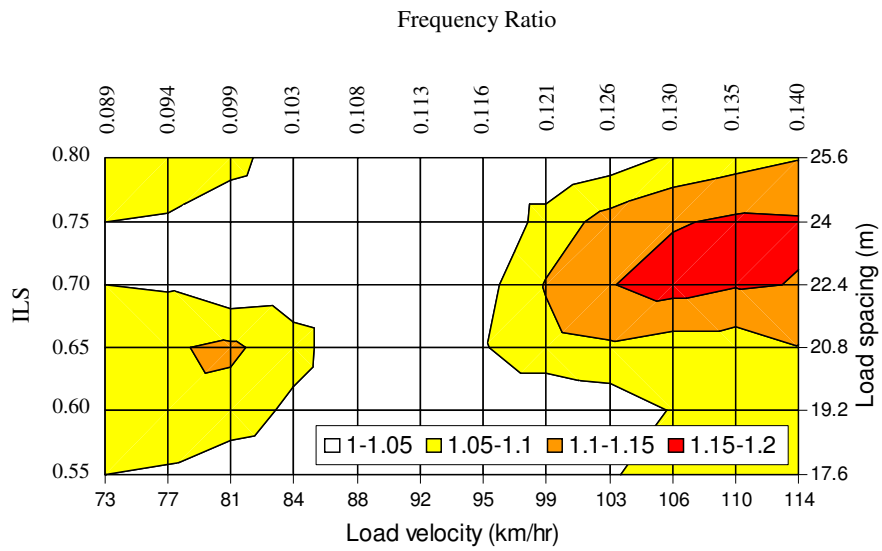


Figure 14 – Contour plot of dynamic amplification factor for two point loads crossing
32 m beam

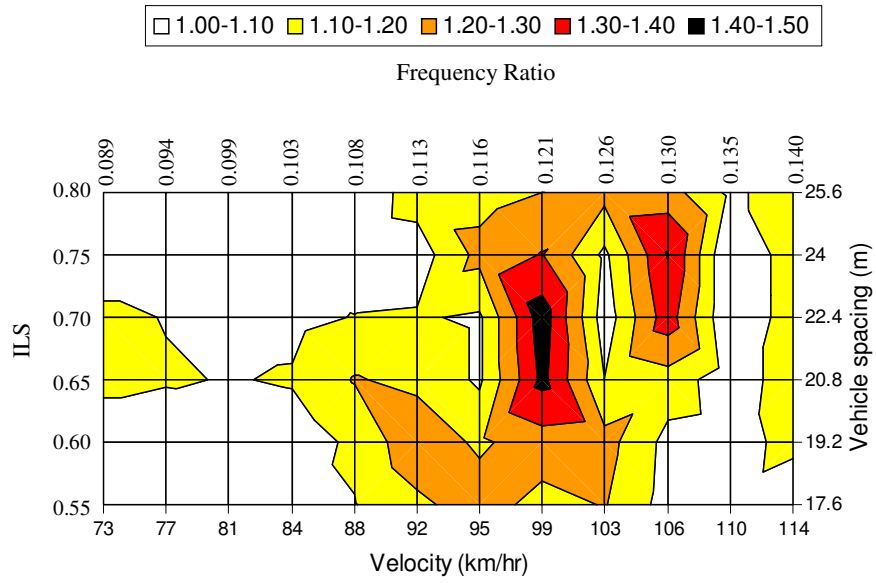
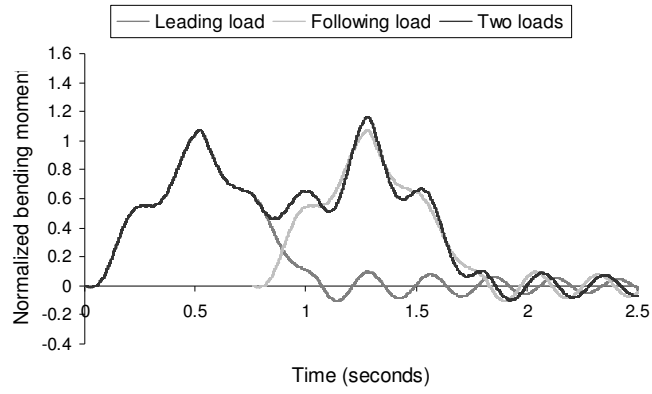
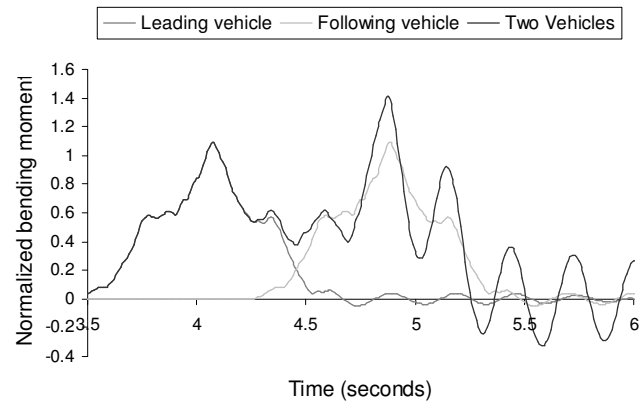


Figure 15 – Contour plot of dynamic amplification factor for the F.E. model vehicles crossing the 32 m long bridge



(a) – Two point loads crossing beam at 106 km/hr



(b) - Two two-axle vehicles traveling at 106.34 km/hr

Figure 16 – Comparison of stress and bending moment of two point loads and two two-axle vehicles

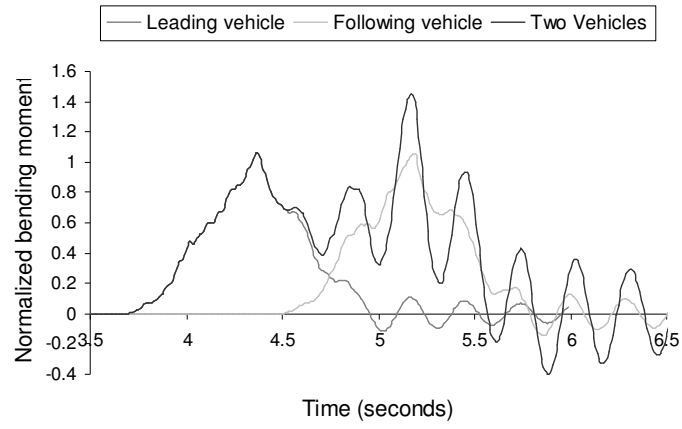


Figure 17 – Two two-axle vehicles traveling at a velocity of 99 km/hr

Table 1 – Critical Frequency Ratios for the formation of maximum dynamic amplification factors, and their magnitude

Critical Frequency Ratio	Inter Load Spacing	Maximum dynamic amplification factor
0.089	≥ 0.5	1.081
0.096	≥ 0.5	1.089
0.135	≥ 0.5	1.162
0.158	≥ 0.5	1.194
0.17 upwards	< 0.5	> 1.2