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DENSITIES OF 4-RANKS OF $K_2(\mathcal{O})$

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ABSTRACT. In [1], the authors established a method of determining the structure of the 2-Sylow subgroup of the tame kernel $K_2(\mathcal{O})$ for certain quadratic number fields. Specifically, the 4-rank for these fields was characterized in terms of positive definite binary quadratic forms. Numerical calculations led to questions concerning possible density results of the 4-rank of tame kernels. In this paper, we succeed in giving affirmative answers to these questions.

1. INTRODUCTION

Since the 1960's, relationships between algebraic K-theory and number theory have been intensely studied. For number fields F and their rings of integers \mathcal{O}_F , the K-groups $K_0(\mathcal{O}_F)$, $K_1(\mathcal{O}_F)$, $K_2(\mathcal{O}_F)$, \dots were a main focus of attention. From [8] we have

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \times C(F)$$

where $C(F)$ is the ideal class group of F , and

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^*$$

the group of units of \mathcal{O}_F .

What can we say in general about $K_2(\mathcal{O}_F)$? Garland and Quillen in [3] and [11] showed that $K_2(\mathcal{O}_F)$ is finite. A conjecture of Birch and Tate connects the order of $K_2(\mathcal{O}_F)$ and the value of the zeta function of F at -1 when F is a totally real field. For abelian number fields, this conjecture has been confirmed up to powers of 2 [7]. In [12] a 2-rank formula for $K_2(\mathcal{O}_F)$ was given by Tate. Some results on the 4-rank of $K_2(\mathcal{O}_F)$ were given in [9], [10], and [13]. To gain further insight on the 4-rank of $K_2(\mathcal{O}_F)$, we consider the following specific families of fields.

In this paper we deal with the 4-rank of the Milnor K-group $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \pmod{8}$, $l \equiv 1 \pmod{8}$ with $\left(\frac{l}{p}\right) = 1$. In [1], the authors show that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank } K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$

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4-rank $K_2(\mathcal{O}_F) = 0$ or 1.

Each of the possible values of 4-ranks is then characterized by checking which ones of the quadratic forms X^2+32Y^2 , X^2+2pY^2 , $2X^2+pY^2$ represent a certain power of l over \mathbb{Z} . This approach makes numerical computations accessible. We should note that this approach involves quadratic symbols and determining the matrix rank over \mathbb{F}_2 of 3×3 matrices with Hilbert symbols as entries, see [4]. Fix a prime $p \equiv 7 \pmod{8}$ and consider the set

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \pmod{8} \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1\}.$$

Let

$$\begin{aligned} v &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) \\ \mu &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{2pl})}) \\ \sigma &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) \\ \tau &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2pl})}) \end{aligned}$$

and also consider the sets

$$\begin{aligned} \Omega_1 &= \{l \in \Omega : v = 1\} \\ \Omega_2 &= \{l \in \Omega : v = 2\} \\ \Omega_3 &= \{l \in \Omega : \mu = 1\} \\ \Omega_4 &= \{l \in \Omega : \mu = 2\} \\ \Lambda_1 &= \{l \in \Omega : \sigma = 0\} \\ \Lambda_2 &= \{l \in \Omega : \sigma = 1\} \\ \Lambda_3 &= \{l \in \Omega : \tau = 0\} \\ \Lambda_4 &= \{l \in \Omega : \tau = 1\}. \end{aligned}$$

We have computed the following (see Table 1 in Appendix): For $p = 7$, there are 9730 primes l in Ω with $l \leq 10^6$. Among them, there are 4866 primes (50.01%) in Ω_1 and Ω_3 and 4864 primes (49.99%) in Ω_2 and Ω_4 . Also, there are 4878 primes (50.13%) in Λ_1 and Λ_3 and 4852 primes in Λ_2 and Λ_4 . The goal of this paper is to prove the following theorem.

Theorem 1.1. *For the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in Ω . For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in Ω .*

Now consider the tuple of 4-ranks (v, μ, σ, τ) . By Corollary (5.6) in [1], there are eight possible tuples of 4-ranks. For $p = 7$, among the 9730 primes $l \in \Omega$ with $l \leq 10^6$, the eight possible tuples are realized by 1215, 1213, 1228, 1210, 1210, 1228, 1225, 1201 primes l respectively (see Table 2 in Appendix). And, in fact:

Theorem 1.2. *Each of the eight possible tuples of 4-ranks appear with natural density $\frac{1}{8}$ in Ω .*

2. PRELIMINARIES

Let \mathcal{D} be a Galois extension of \mathbb{Q} , and $G = \text{Gal}(\mathcal{D}/\mathbb{Q})$. Let $Z(G)$ denote the center of G and $\mathcal{D}^{Z(G)}$ denote the fixed field of $Z(G)$. Let p be a rational prime which is unramified in \mathcal{D} and β be a prime of \mathcal{D} containing p . Let $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right)$ denote the Artin symbol of p and $\{g\}$ the conjugacy class containing one element $g \in G$.

Lemma 2.1. $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ for some $g \in Z(G)$ if and only if p splits completely in $\mathcal{D}^{Z(G)}$.

Proof. $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ for some $g \in Z(G)$ if and only if $\left(\frac{\mathcal{D}/\mathbb{Q}}{\beta}\right) = g$ if and only if $\left(\frac{\mathcal{D}^{Z(G)}/\mathbb{Q}}{\beta}\right) = \left(\frac{\mathcal{D}/\mathbb{Q}}{\beta}\right)\Big|_{\mathcal{D}^{Z(G)}} = g|_{\mathcal{D}^{Z(G)}} = \text{Id}_{\text{Gal}(\mathcal{D}^{Z(G)}/\mathbb{Q})}$ if and only if p splits completely in $\mathcal{D}^{Z(G)}$. \square

Thus if we can show that rational primes split completely in the fixed field of the center of a certain Galois group G , then we know the associated Artin symbol is a conjugacy class containing one element. Hence we may identify the Artin symbol with this one element and consider the symbol to be an automorphism which lies in $Z(G)$. Thus determining the order of $Z(G)$ gives us the number of possible choices for the Artin symbol.

Let G_1 and G_2 be finite groups and A a finite abelian group. Suppose $r_1 : G_1 \rightarrow A$ and $r_2 : G_2 \rightarrow A$ are two epimorphisms and $\mathcal{G} \subset G_1 \times G_2$ is the set $\{(g_1, g_2) \in G_1 \times G_2 : r_1(g_1) = r_2(g_2)\}$. Since A is abelian, there is an epimorphism $r : G_1 \times G_2 \rightarrow A$ given by $r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$. Thus $\mathcal{G} = \ker(r) \subset G_1 \times G_2$. One can check that $Z(\mathcal{G}) = \mathcal{G} \cap Z(G_1 \times G_2)$.

Lemma 2.2. (i) If $r_2\Big|_{Z(G_2)}$ is trivial, then $Z(\mathcal{G}) = \ker(r_1\Big|_{Z(G_1)}) \times Z(G_2)$.

(ii) $Z(\mathcal{G}) = Z(G_1) \times Z(G_2) \iff r_1\Big|_{Z(G_1)}$ and $r_2\Big|_{Z(G_2)}$ are both trivial.

Proof. (i) Suppose $(g_1, g_2) \in Z(\mathcal{G}) \subset \ker(r)$ where $g_1 \in Z(G_1)$, $g_2 \in Z(G_2)$. Thus $1 = r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$ and so $r_1(g_1) = r_2(g_2)$. But $r_2(g_2) = 1$ which yields $r_1(g_1) = 1$. Thus $g_1 \in \ker(r_1\Big|_{Z(G_1)})$. The other inclusion is clear.

(ii) Take $(g_1, 1), (1, g_2) \in Z(G_1) \times Z(G_2) = Z(\mathcal{G}) \subset \ker(r)$, respectively to obtain that $r_1\Big|_{Z(G_1)}$ and $r_2\Big|_{Z(G_2)}$ are both trivial. The converse follows from part (i). \square

We will use the following definition throughout this paper.

Definition 2.3. For primes $p \equiv 7 \pmod{8}$, $l \equiv 1 \pmod{8}$ with $\left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1$, $\mathcal{K} = \mathbb{Q}(\sqrt{-2p})$, and $h(\mathcal{K})$ the class number of \mathcal{K} , we say:

l satisfies $\langle 1, 32 \rangle$ if and only if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$

l satisfies $\langle 2, p \rangle$ if and only if $l^{\frac{h(\mathcal{K})}{4}} = 2n^2 + pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{l}$

l satisfies $\langle 1, 2p \rangle$ if and only if $l^{\frac{h(\mathcal{K})}{4}} = n^2 + 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{l}$.

3. THREE EXTENSIONS

In this section, we consider three degree eight field extensions of \mathbb{Q} . The idea will be to study composites of these fields and relate Artin symbols to 4-ranks. Rational primes which split completely in a degree 64 extension of \mathbb{Q} will relate to Artin symbols and thus 4-ranks. Therefore calculating the density of these primes will answer density questions involving 4-ranks.

3.1. First Extension. Consider $\mathbb{Q}(\sqrt{2})$ over \mathbb{Q} . Let $\epsilon = 1 + \sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^*$. Then ϵ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm -1 . The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon})$ over \mathbb{Q} has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1})$. Set

$$N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1}).$$

Note that N_1 is the splitting field of the polynomial $x^4 - 2x^2 - 1$ and so has degree 8 over \mathbb{Q} . Therefore $\text{Gal}(N_1/\mathbb{Q})$ is the dihedral group of order 8. Note that the automorphism induced by sending $\sqrt{\epsilon}$ to $-\sqrt{\epsilon}$ commutes with every element of $\text{Gal}(N_1/\mathbb{Q})$. Thus $Z(\text{Gal}(N_1/\mathbb{Q})) = \text{Gal}(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))$.

Observe that only the prime 2 ramifies in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\epsilon})$, and so only the prime 2 ramifies in the compositum N_1 over \mathbb{Q} . Now as $l \in \Omega$ is unramified in N_1 over \mathbb{Q} , the Artin symbol $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ is defined for primes β of \mathcal{O}_{N_1} containing l . Let $\left(\frac{N_1/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ in $\text{Gal}(N_1/\mathbb{Q})$. The primes $l \in \Omega$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and $N_1^{Z(\text{Gal}(N_1/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Thus by Lemma 2.1, we have that $\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{g\} \subset Z(\text{Gal}(N_1/\mathbb{Q}))$. As $Z(\text{Gal}(N_1/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_1/\mathbb{Q}}{l}\right)$. Combining this statement with Addendum (3.4) from [1], we have

Remark 3.1.

$$\begin{aligned} \left(\frac{N_1/\mathbb{Q}}{l}\right) = \{id\} &\iff l \text{ splits completely in } N_1 \\ &\iff l \text{ satisfies } \langle 1, 32 \rangle. \end{aligned}$$

3.2. Second and Third Extension. Consider the fixed prime $p \equiv 7 \pmod{8}$. Note p splits completely in $\mathcal{L} = \mathbb{Q}(\sqrt{2})$ over \mathbb{Q} and so

$$p\mathcal{O}_{\mathcal{L}} = \mathfrak{B}\mathfrak{B}'$$

for some primes $\mathfrak{B} \neq \mathfrak{B}'$ in \mathcal{L} . The field \mathcal{L} has narrow class number $h^+(\mathcal{L}) = 1$ as $h(\mathcal{L}) = 1$ and $N_{\mathcal{L}/\mathbb{Q}}(\epsilon) = -1$ where $\epsilon = 1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$, see [5]. From [1],

Lemma 3.2. *The prime \mathfrak{B} which occurs in the decomposition of $p\mathcal{O}_{\mathcal{L}}$ has a generator $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$, unique up to a sign and to multiplication by the square of a unit in $\mathcal{O}_{\mathcal{L}}^*$ for which $N_{\mathcal{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = -p$.*

The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$ over \mathbb{Q} has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p})$ as $N_{\mathcal{L}/\mathbb{Q}}(\pi) = -p$. Set

$$N_2 = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p}).$$

Then N_2 is Galois over \mathbb{Q} and $[N_2 : \mathbb{Q}] = 8$. Such an extension N_2 exists since the 2-Sylow subgroup of the ideal class group of $\mathbb{Q}(\sqrt{-2p})$ is cyclic of order divisible by 4 [2]. Thus the Hilbert class field of $\mathbb{Q}(\sqrt{-2p})$ contains a unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{-2p})$. By Lemma 2.3 in [1], N_2 is the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{-2p})$. Also compare [6]. Similar to arguments in Section 2.1, $\text{Gal}(N_2/\mathbb{Q})$ is the dihedral group of order 8. Note that the automorphism induced by sending $\sqrt{\pi}$ to $-\sqrt{\pi}$ commutes with every element of $\text{Gal}(N_2/\mathbb{Q})$. Thus $Z(\text{Gal}(N_2/\mathbb{Q})) = \text{Gal}(N_2/\mathbb{Q}(\sqrt{2}, \sqrt{-p}))$.

Proposition 3.3. *If $l \in \Omega$, then l is unramified in N_2 over \mathbb{Q} .*

Proof. Since $p \equiv 7 \pmod{8}$, the discriminant of $\mathbb{Q}(\sqrt{-2p})$ is $-8p$. For $l \in \Omega$, we have $\left(\frac{-2p}{l}\right) = 1$ and so l is unramified in $\mathbb{Q}(\sqrt{-2p})$. By Lemma 2.3 in [1], we have l is unramified in N_2 over \mathbb{Q} . \square

As $l \in \Omega$ is unramified in N_2 over \mathbb{Q} , the Artin symbol $\left(\frac{N_2/\mathbb{Q}}{\beta}\right)$ is defined for primes β of \mathcal{O}_{N_2} containing l . Let $\left(\frac{N_2/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_2/\mathbb{Q}}{\beta}\right)$ in $\text{Gal}(N_2/\mathbb{Q})$. The primes $l \in \Omega$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-p})$ and $N_2^{Z(\text{Gal}(N_2/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-p})$. By Lemma 2.1, we have that $\left(\frac{N_2/\mathbb{Q}}{l}\right) = \{h\} \subset Z(\text{Gal}(N_2/\mathbb{Q}))$ for some $h \in Z(\text{Gal}(N_2/\mathbb{Q}))$. As $Z(\text{Gal}(N_2/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_2/\mathbb{Q}}{l}\right)$. Combining this statement and Lemmas (3.3) and (3.4) from [1], we have

Remark 3.4.

$$\begin{aligned} \left(\frac{N_2/\mathbb{Q}}{l}\right) = \{id\} &\iff l \text{ splits completely in } N_2 \\ &\iff l \text{ satisfies } \langle 1, 2p \rangle. \end{aligned}$$

$$\begin{aligned} \left(\frac{N_2/\mathbb{Q}}{l}\right) \neq \{id\} &\iff l \text{ does not split completely in } N_2 \\ &\iff l \text{ satisfies } \langle 2, p \rangle. \end{aligned}$$

Finally, for $l \in \Omega$, l splits completely in $\mathbb{Q}(\zeta_{16}) \iff l \equiv 1 \pmod{16}$ This yields

Remark 3.5.

$$\begin{aligned} \left(\frac{\mathbb{Q}(\zeta_{16})/\mathbb{Q}}{l}\right) = \{id\} &\iff l \text{ splits completely in } \mathbb{Q}(\zeta_{16}) \\ &\iff l \equiv 1 \pmod{16}. \end{aligned}$$

4. THE COMPOSITE AND TWO THEOREMS

In this section we consider the composite field $N_1N_2\mathbb{Q}(\zeta_{16})$. Set

$$L = N_1N_2\mathbb{Q}(\zeta_{16}).$$

Note that $[L : \mathbb{Q}] = 64$. As N_1 , N_2 , and $\mathbb{Q}(\zeta_{16})$ are normal extensions of \mathbb{Q} , L is a normal extension of \mathbb{Q} .

For $l \in \Omega$, l is unramified in L as it is unramified in N_1 , N_2 , and $\mathbb{Q}(\zeta_{16})$. The Artin symbol $\left(\frac{L/\mathbb{Q}}{\beta}\right)$ is now defined for some prime β of \mathcal{O}_L containing l . Let $\left(\frac{L/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{L/\mathbb{Q}}{\beta}\right)$ in $Gal(L/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p}) \subset L$, we prove

Lemma 4.1. $Z(Gal(L/\mathbb{Q})) = Gal(L/M)$ is elementary abelian of order 8.

Proof. For $\sigma \in Gal(L/M)$, σ can only change the sign of $\sqrt{\epsilon}$, $\sqrt{\pi}$, and $\sqrt{\zeta_8}$ as $\epsilon \in M$. Since $L = M(\sqrt{\epsilon}, \sqrt{\pi}, \sqrt{\zeta_8})$, $Gal(L/M)$ is elementary abelian of order 8. Now consider the restrictions $r_1 : G_1 \rightarrow Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and $r_2 : G_2 \rightarrow Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ where $G_1 = Gal(N_1/\mathbb{Q})$ and $G_2 = Gal(N_2/\mathbb{Q})$. Clearly $r_1 \Big|_{Z(G_1)}$ and

$r_2 \Big|_{Z(G_2)}$ are both trivial. Then by Lemma 2.2 part (ii), $Z(\mathcal{G})$ is elementary abelian of order 4 where $\mathcal{G} = Gal(N_1N_2/\mathbb{Q})$. Now consider the restrictions $R_1 : Gal(\mathbb{Q}(\zeta_{16})/\mathbb{Q}) \rightarrow Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ and $R_2 : \mathcal{G} \rightarrow Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q})$. Note that $ker(R_1)$ is cyclic of order 2 and $Z(\mathcal{G}) = Gal(M/\mathbb{Q})$. Thus $R_2 \Big|_{Z(\mathcal{G})}$

is trivial and so by the above and Lemma 2.2 part (i), $Z(\text{Gal}(L/\mathbb{Q})) \cong \mathbb{Z}/2\mathbb{Z} \times Z(\mathcal{G}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $Z(\text{Gal}(L/\mathbb{Q})) = \text{Gal}(L/M)$. \square

Now for $l \in \Omega$, l splits completely in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{-p})$ and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. From Lemma 4.1, $L^{Z(\text{Gal}(L/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. So by Lemma 2.1, we have $\left(\frac{L/\mathbb{Q}}{l}\right) = \{k\} \subset Z(\text{Gal}(L/\mathbb{Q}))$ for some $k \in \text{Gal}(L/\mathbb{Q})$. As $Z(\text{Gal}(L/\mathbb{Q}))$ has order 8, there are eight possible choices for $\left(\frac{L/\mathbb{Q}}{l}\right)$. Using Remarks 3.1, 3.4, and 3.5, we now make the following one to one correspondences.

Remark 4.2. (i) $\left(\frac{L/\mathbb{Q}}{l}\right) = \{id\} \iff l$ splits completely in $L \iff$

$$\left\{ \begin{array}{l} l \text{ splits completely in } N_1, \\ N_2, \text{ and } \mathbb{Q}(\zeta_{16}) \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, 2p \rangle \\ l \equiv 1 \pmod{16} \end{array} \right\}.$$

(ii) $\left(\frac{L/\mathbb{Q}}{l}\right) \neq \{id\} \iff l$ does not split completely in L . Now there are seven cases.

1. $\left\{ \begin{array}{l} l \text{ splits completely in } N_1, \\ \text{but does not in } N_2 \text{ or } \mathbb{Q}(\zeta_{16}) \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 2, p \rangle \\ l \equiv 9 \pmod{16} \end{array} \right\}$
2. $\left\{ \begin{array}{l} l \text{ splits completely in } N_1 \\ \text{and } N_2, \text{ but does not in } \mathbb{Q}(\zeta_{16}) \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, 2p \rangle \\ l \equiv 9 \pmod{16} \end{array} \right\}$
3. $\left\{ \begin{array}{l} l \text{ splits completely in} \\ N_2, \text{ but does not in } N_1 \\ \text{or } \mathbb{Q}(\zeta_{16}) \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, 2p \rangle \\ l \equiv 9 \pmod{16} \end{array} \right\}$
4. $\left\{ \begin{array}{l} l \text{ splits completely in} \\ N_2 \text{ and } \mathbb{Q}(\zeta_{16}), \\ \text{but does not in } N_1 \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, 2p \rangle \\ l \equiv 1 \pmod{16} \end{array} \right\}$
5. $\left\{ \begin{array}{l} l \text{ splits completely in } N_1 \\ \text{and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_2 \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 2, p \rangle \\ l \equiv 1 \pmod{16} \end{array} \right\}$
6. $\left\{ \begin{array}{l} l \text{ splits completely in} \\ \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_1 \\ \text{or } N_2 \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 2, p \rangle \\ l \equiv 1 \pmod{16} \end{array} \right\}$
7. $\left\{ \begin{array}{l} l \text{ does not split completely} \\ \text{in } N_1, N_2, \text{ or } \mathbb{Q}(\zeta_{16}) \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 2, p \rangle \\ l \equiv 9 \pmod{16} \end{array} \right\}.$

Now using Theorems (5.2), (5.3), (5.4), and (5.5) from [1], we relate each Artin symbol $\left(\frac{L/\mathbb{Q}}{l}\right)$ to each of the eight possible tuples of 4-ranks.

Remark 4.3. *From Remark 4.2, case (i) occurs if and only if we have (2, 2, 1, 1). For case (ii),*

- (1) *occurs if and only if we have (1, 2, 0, 1)*
- (2) *occurs if and only if we have (2, 1, 1, 0)*
- (3) *occurs if and only if we have (2, 1, 0, 1)*
- (4) *occurs if and only if we have (2, 2, 0, 0)*
- (5) *occurs if and only if we have (1, 1, 0, 0)*
- (6) *occurs if and only if we have (1, 1, 1, 1)*
- (7) *occurs if and only if we have (1, 2, 1, 0).*

We can now prove Theorem 1.2.

Proof. Consider the set $X = \{l \text{ prime} : l \text{ is unramified in } L \text{ and } \left(\frac{L/\mathbb{Q}}{l}\right) = \{k\} \subset Z(\text{Gal}(L/\mathbb{Q}))\}$ for some $k \in \text{Gal}(L/\mathbb{Q})$. By the Čebotarev Density Theorem, the set X has natural density $\frac{1}{64}$ in the set of all primes. Recall

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \pmod{8} \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1\}$$

for some fixed prime $p \equiv 7 \pmod{8}$. By Dirichlet's Theorem on primes in arithmetic progressions, Ω has natural density $\frac{1}{8}$ in the set of all primes. Thus X has natural density $\frac{1}{8}$ in Ω . By Remark 4.2 and 4.3, each of the eight choices for $\left(\frac{L/\mathbb{Q}}{l}\right)$ is in one to one correspondence with each of the possible tuples of 4-ranks. Thus each of the eight possible tuples of 4-ranks appear with natural density $\frac{1}{8}$ in Ω . □

Now we can prove Theorem 1.1

Proof. We see from Remark 4.3, 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1$ in cases (ii), parts (1), (5), (6), and (7), 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{2pl})}) = 2$ in case (i) and case (ii) parts (1), (4), and (7), 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 0$ in case (ii) parts (1), (3), (4), and (5), 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2pl})}) = 1$ in case (i) and case (ii) parts (1), (3), and (6). As each of the 4-rank tuples occur with natural density $\frac{1}{8}$, we have for the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $4 \cdot \frac{1}{8} = \frac{1}{2}$ in Ω . For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $4 \cdot \frac{1}{8} = \frac{1}{2}$ in Ω . □

APPENDIX

The following tables motivated possible density results of 4-ranks of tame kernels. We consider primes $l \in \Omega$ with $l \leq N$ for a fixed prime $p \equiv 7 \pmod{8}$ and positive integer N . For Table 1, we consider the sets $\Omega_1, \dots, \Omega_4$ and $\Lambda_1, \dots, \Lambda_4$ as in the Introduction. For Table 2, we consider the sets

$$\begin{aligned} I_1 &= \{l \in \Omega : 4\text{-rank tuple is } (1,1,0,0)\} \\ I_2 &= \{l \in \Omega : 4\text{-rank tuple is } (1,1,1,1)\} \\ I_3 &= \{l \in \Omega : 4\text{-rank tuple is } (2,1,1,0)\} \\ I_4 &= \{l \in \Omega : 4\text{-rank tuple is } (2,1,0,1)\} \\ I_5 &= \{l \in \Omega : 4\text{-rank tuple is } (1,2,1,0)\} \\ I_6 &= \{l \in \Omega : 4\text{-rank tuple is } (1,2,0,1)\} \\ I_7 &= \{l \in \Omega : 4\text{-rank tuple is } (2,2,0,0)\} \\ I_8 &= \{l \in \Omega : 4\text{-rank tuple is } (2,2,1,1)\}. \end{aligned}$$

TABLE 1

Primes	$p = 7$		$p = 23$		$p = 31$	
Cardinality	$N = 1000000$	%	$N = 1000000$	%	$N = 1000000$	%
$ \Omega $	9730		9742		9754	
$ \Omega_1 $	4866	50.01	4905	50.35	4916	50.40
$ \Omega_2 $	4864	49.99	4837	49.65	4838	49.60
$ \Omega_3 $	4866	50.01	4911	50.41	4851	49.73
$ \Omega_4 $	4864	49.99	4831	49.59	4903	50.27
$ \Lambda_1 $	4878	50.13	4912	50.42	4930	50.54
$ \Lambda_2 $	4852	49.87	4830	49.58	4824	49.46
$ \Lambda_3 $	4878	50.13	4876	50.05	4943	50.68
$ \Lambda_4 $	4852	49.87	4866	49.95	4811	49.32

TABLE 2

Primes	$p = 7$		$p = 23$		$p = 31$	
Cardinality	$N = 1000000$	%	$N = 1000000$	%	$N = 1000000$	%
$ \Omega $	9730		9742		9754	
I_1	1215	12.49	1246	12.79	1246	12.77
I_2	1213	12.46	1229	12.62	1203	12.33
I_3	1228	12.62	1211	12.43	1214	12.45
I_4	1210	12.44	1225	12.57	1188	12.18
I_5	1210	12.44	1204	12.36	1227	12.58
I_6	1228	12.62	1226	12.58	1240	12.71
I_7	1225	12.59	1215	12.47	1256	12.88
I_8	1201	12.34	1186	12.17	1180	12.10

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