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Sets of determination for the Nevanlinna class

Stephen J. Gardiner

Abstract

This paper characterizes the subsets E of the unit disc $\mathbb{D}$ with the property that $\sup_E |f| = \sup_{\mathbb{D}} |f|$ for all functions f in the Nevanlinna class.

1 Introduction

Let $\mathcal{A}$ be a collection of holomorphic functions on the unit disc $\mathbb{D}$, and let $\mathbb{T}$ denote the unit circle. A set $E \subset \mathbb{D}$ is called a *set of determination for* $\mathcal{A}$ if $\sup_E |f| = \sup_{\mathbb{D}} |f|$ for all $f \in \mathcal{A}$. Brown, Shields and Zeller [3] have shown that E is a set of determination for H^∞, the space of bounded holomorphic functions on $\mathbb{D}$, if and only if almost every point of $\mathbb{T}$ can be approached nontangentially by a sequence of points in E. Massaneda and Thomas [6] have observed that the same characterization remains valid when $\mathcal{A}$ is the Smirnov class $\mathcal{N}^+$. However, the situation is more complicated for the Nevanlinna class $\mathcal{N}$, which consists of all holomorphic functions f on $\mathbb{D}$ that satisfy

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta < \infty.$$

This is the main focus of [6], where a variety of conditions are shown to be either necessary or sufficient for E to be a set of determination for $\mathcal{N}$, and some illustrative special cases are examined. (See also Stray [7], p.256.) The purpose of this paper is to give a complete characterization of such sets.

First we recall a related result of Hayman and Lyons [5] for the harmonic Hardy space h^1, which consists of those functions on $\mathbb{D}$ that can be expressed as the difference of two positive harmonic functions. For $n \in \mathbb{N}$ and $0 \leq m < 2^{n+4}$ let

$$z_{m,n} = (1 - 2^{-n}) \exp(2\pi i m / 2^{n+4})$$

and

$$S_{m,n} = \left\{ re^{i\theta} : 2^{-n-1} \leq 1 - r \leq 2^{-n} \text{ and } \frac{2\pi m}{2^{n+4}} \leq \theta \leq \frac{2\pi(m+1)}{2^{n+4}} \right\},$$

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and let $E_{m,n} = E \cap S_{m,n}$. The Poisson kernel for $\mathbb{D}$ is given by

$$P(z,w) = \frac{1-|z|^2}{|z-w|^2} \qquad (z \in \mathbb{D}, w \in \mathbb{T}).$$

Theorem A [5] *Let $E \subset \mathbb{D}$. The following conditions are equivalent:*
(a) $\sup_E h = \sup_{\mathbb{D}} h$ *for all* $h \in h^1$*;*
(b) $\sum\limits_{E_{m,n} \neq \emptyset} 2^{-n} P(z_{m,n}, w) = \infty$ *for every* $w \in \mathbb{T}$.

For any set A which is contained in a disc of radius less than 1, and any $t \geq 0$, we define a capacity-related quantity $\mathcal{Q}(A,t)$ as follows. We put $\mathcal{Q}(A,t) = 0$ if either $t = 0$ or $A = \emptyset$; otherwise,

$$\mathcal{Q}(A,t) = \min\{k \in \mathbb{N} : \exists \xi_1, ..., \xi_k \in \mathbb{C} \text{ such that } \sum_{j=1}^{k} \log \frac{1}{|z-\xi_j|} \geq t \ \ (z \in A)\}.$$

Clearly $\mathcal{Q}(\cdot,t)$ is translation-invariant and $\mathcal{Q}(\{\zeta\},\cdot) = \chi_{(0,\infty)}$ for any $\zeta \in \mathbb{C}$. Also,

$$\mathcal{Q}(\{\zeta_1,\zeta_2\},t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } |\zeta_1 - \zeta_2| \leq 2e^{-t} \text{ and } t > 0 \\ 2 & \text{otherwise} \end{cases}$$

and, if A is a disc of radius of $r < 1$, then $\mathcal{Q}(A,t)$ is the least integer k satisfying $k \geq t/\log(1/r)$. We use $[t]$ to denote the integer part of a non-negative number t, and tA to denote the set $\{tz : z \in A\}$. Our characterization of sets of determination for the Nevanlinna class is as follows.

Theorem 1 *Let $E \subset \mathbb{D}$. The following conditions are equivalent:*
(a) $\sup_E |f| = \sup_{\mathbb{D}} |f|$ *for all* $f \in \mathcal{N}$*;*
(b) $\sum\limits_{m,n} 2^{-n} \mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n}, w)]\right) = \infty$ *for every* $w \in \mathbb{T}$.

Since

$$\log \frac{2^{-n}}{|z - z_{m,n}|} \geq -\frac{1}{2} \log \left(\left(\frac{\pi}{8}\right)^2 + \left(\frac{1}{2}\right)^2 \right) > \frac{1}{3} \qquad (z \in S_{m,n}),$$

we have

$$3P(z_{m,n}, w) \log \frac{2^{-n}}{|z - z_{m,n}|} \geq P(z_{m,n}, w) \qquad (z \in S_{m,n}, w \in \mathbb{T}).$$

By separate consideration of the cases $P(z_{m,n}, w) \geq 1$ and $P(z_{m,n}, w) < 1$, we see that

$$\mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n}, w)]\right) \leq 4P(z_{m,n}, w). \tag{1}$$

Applying this inequality to terms where $E_{m,n} \neq \emptyset$, it is now clear that condition (b) of Theorem 1 implies the corresponding condition of Theorem A. It is not difficult to check that condition (a) of Theorem 1 is equivalent to the assertion that, if $\log|f| \leq h$ on E, where $f \in \mathcal{N}$ and $h \in h^1$, then $\log|f| \leq h$ on all of $\mathbb{D}$ (cf. [6]).

Examples Let $U = \{z : \left|z - \frac{1}{2}\right| < \frac{1}{2}\}$ and $F = U \cap \{z_{m,n}\}$.
(i) The set $E = \mathbb{D} \backslash U$ is not a set of determination (for $\mathcal{N}$) because the series in condition (b) of Theorem A then converges when $w = 1$ (cf. Example 6.2 in [5]).
(ii) Further, even $E \cup F$ is not a set of determination because each of the sets $F_{m,n}$ contains at most 5 points and so

$$\sum_{m,n} 2^{-n} \mathcal{Q}\left(2^n F_{m,n}, [P(z_{m,n}, 1)]\right) \leq 5 \sum_{z_{m,n} \in F} 2^{-n} < \infty$$

(cf. Example 1 in [6]).
(iii) On the other hand, $E \cup [\frac{1}{2}, 1)$ is a set of determination since

$$\mathcal{Q}\left(2^n[1 - 2^{-n}, 1 - 2^{-n-1}], [P(z_{0,n}, 1)]\right) = \mathcal{Q}\left([0, \frac{1}{2}], 2^n\right)$$

and $\inf_n 2^{-n} \mathcal{Q}\left([0, \frac{1}{2}], 2^n\right) > 0$ because $[0, \frac{1}{2}]$ is non-polar.

2 Proof of Theorem 1

Let $G_U(\cdot, \cdot)$ denote the Green function of an open set U, let

$$D_\rho(z) = \{\zeta : |\zeta - z| < \rho(1 - |z|)\} \quad (z \in \mathbb{D}, 0 < \rho < 1),$$

and let $\mathcal{A}(g, z)$ denote the mean value of a function g over the disc $D_{1/8}(z)$. For potential theoretic background we refer to the book [2].

Suppose firstly that condition (b) of Theorem 1 holds and let $f \in \mathcal{N}$. We will assume that $\sup_E |f| < \infty$, for otherwise it is trivially true that $\sup_E |f| = \sup_{\mathbb{D}} |f|$. Further, multiplication by a suitable constant enables us to arrange that $\sup_E |f| \in [0, 1]$. Now let $a \in (-\infty, 0]$ be such that $a \geq \log \sup_E |f|$. We can write

$$\log|f| = h_1 - h_2 - G_{\mathbb{D}}\mu,$$

where h_1 and h_2 are positive harmonic functions and μ is a sum of unit point masses on $\mathbb{D}$ satisfying

$$\int (1 - |z|) d\mu(z) < \infty.$$

Further, by addition to both h_1 and h_2, we may assume that $h_1 \geq 1$. By the Riesz-Herglotz theorem there is a Borel measure ν_1 on $\mathbb{T}$ such that

$$h_1(z) = \int P(z,w)d\nu_1(w) \quad (z \in \mathbb{D}).$$

We know that

$$h_1 - a \leq h_2 + G_{\mathbb{D}}\mu \quad \text{on } E. \tag{2}$$

Also,

$$\begin{aligned} G_{\mathbb{D}}(z,\xi) - \mathcal{A}(G_{\mathbb{D}}(\cdot,\xi),z) &\leq G_{D_{1/8}(z)}(z,\xi) \\ &= \log\frac{(1-|z|)/8}{|z-\xi|} \quad (\xi \in D_{1/8}(z)) \end{aligned} \tag{3}$$

and $G_{\mathbb{D}}(z,\xi) - \mathcal{A}(G_{\mathbb{D}}(\cdot,\xi),z) = 0$ otherwise. Let $\varepsilon \in (0,1)$ and

$$I_\varepsilon = \{(m,n) : G_{\mathbb{D}}\mu \geq \mathcal{A}(G_{\mathbb{D}}\mu,\cdot) + \varepsilon h_1 \ \text{ on } E_{m,n}\},$$

and let I'_ε denote the complementary set of pairs (m,n). (We note that $(m,n) \in I_\varepsilon$ whenever $E_{m,n} = \emptyset$.) If $(m,n) \in I_\varepsilon$, then we see from (3) that

$$\begin{aligned} \varepsilon h_1(z) &\leq G_{\mathbb{D}}\mu(z) - \mathcal{A}(G_{\mathbb{D}}\mu,z) \\ &= \int_{D_{1/8}(z)} (G_{\mathbb{D}}(z,\xi) - \mathcal{A}(G_{\mathbb{D}}(\cdot,\xi),z))\, d\mu(\xi) \\ &\leq \int_{A_{m,n}} \log\frac{2^{-n}}{|z-\xi|} d\mu(\xi) \quad (z \in E_{m,n}), \end{aligned}$$

where

$$A_{m,n} = \{\xi : \operatorname{dist}(\xi, S_{m,n}) < 2^{-n-3}\}.$$

(Here we have used the fact that the diameter of $2^n A_{m,n}$ is less than 1.) By Harnack's inequalities there is an absolute constant $c_1 > 1$ such that $h(\zeta_1) \leq c_1 h(\zeta_2)$ for any positive harmonic function h on $\mathbb{D}$, any points $\zeta_1, \zeta_2 \in S_{m,n}$, and any choice of (m,n). For any $w \in \mathbb{T}$ we thus have

$$P(z_{m,n},w) \leq \frac{c_1}{\varepsilon h_1(z_{m,n})} P(z_{m,n},w) \int_{A_{m,n}} \log\frac{2^{-n}}{|z-\xi|} d\mu(\xi) \quad (z \in E_{m,n}),$$

and so

$$\mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n},w)]\right) \leq \left(\frac{c_1}{\varepsilon h_1(z_{m,n})} P(z_{m,n},w) + 1\right) \mu(A_{m,n}).$$

Integration of the above inequality with respect to $d\nu_1(w)$ yields

$$\int \mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n},w)]\right) d\nu_1(w) \leq \left(\frac{c_1}{\varepsilon} + h_1(0)\right) \mu(A_{m,n}).$$

Since no point of $\mathbb{D}$ can lie in more than 4 of the sets $A_{m,n}$, and $1-|z| > 2^{-n-2}$ when $z \in A_{m,n}$, we see that

$$\int \sum_{(m,n)\in I_\varepsilon} 2^{-n}\mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n}, w)]\right) d\nu_1(w)$$
$$\leq 2^4 \left(\frac{c_1}{\varepsilon} + h_1(0)\right) \int (1-|z|)d\mu(z) < \infty,$$

so

$$\sum_{(m,n)\in I_\varepsilon} 2^{-n}\mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n}, w)]\right) < \infty \quad \text{for } \nu_1\text{-almost every } w \in \mathbb{T},$$

and hence, by hypothesis,

$$\sum_{(m,n)\in I'_\varepsilon} 2^{-n}\mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n}, w)]\right) = \infty \quad \text{for } \nu_1\text{-almost every } w \in \mathbb{T}.$$

In view of (1) we now see that

$$\sum_{(m,n)\in I'_\varepsilon} 2^{-2n} |w - z_{m,n}|^{-2} = \infty \quad \text{for } \nu_1\text{-almost every } w \in \mathbb{T}. \tag{4}$$

For each $(m,n) \in I'_\varepsilon$ we can find $\zeta_{m,n} \in E_{m,n}$ such that

$$G_{\mathbb{D}}\mu(\zeta_{m,n}) < \mathcal{A}(G_{\mathbb{D}}\mu, \zeta_{m,n}) + \varepsilon h_1(\zeta_{m,n}).$$

Let $F = \{\zeta_{m,n} : (m,n) \in I'_\varepsilon\}$. Then

$$(1-\varepsilon)h_1 - a \leq h_2 + \mathcal{A}(G_{\mathbb{D}}\mu, \cdot) \quad \text{on } F, \tag{5}$$

in view of (2). Also, by (4),

$$\int_{F_\rho} |w - z|^{-2}\, d\lambda(z) = \infty \quad (0 < \rho < 1) \tag{6}$$

for ν_1-almost every $w \in \mathbb{T}$, where $F_\rho = \cup_{\zeta \in F} D_\rho(\zeta)$ and λ denotes area measure. At this point we could invoke Theorem 2 of [4], but for the sake of completeness we will extract the relevant reasoning in the next paragraph.

Let $0 < \rho < 1/8$. If $z' \in D_\rho(z)$, then by the mean value inequality

$$\begin{aligned} G_{\mathbb{D}}\mu(z') &\geq \frac{1}{\pi(\rho + 1/8)^2(1-|z|)^2} \int_{\{\zeta : |\zeta - z'| < (\rho+1/8)(1-|z|\}} G_{\mathbb{D}}\mu(\zeta)\, d\lambda(\zeta) \\ &\geq \frac{(1/8)^2}{(\rho + 1/8)^2} \mathcal{A}(G_{\mathbb{D}}\mu, z), \end{aligned}$$

and by Harnack's inequalities

$$\frac{1-\rho}{1+\rho} h_j(z) \leq h_j(z') \leq \frac{1+\rho}{1-\rho} h_j(z) \quad (j = 1, 2),$$

so (5) yields

$$(1-\varepsilon)\frac{1-\rho}{1+\rho}h_1 - a \le \frac{1+\rho}{1-\rho}h_2 + (8\rho+1)^2 G_{\mathbb{D}}\mu \quad \text{on } F_\rho. \tag{7}$$

Condition (6) is known to ensure that the reduced function $R^{F_\rho}_{P(\cdot,w)}$, where

$$R^{F_\rho}_u = \inf\{v : v \text{ is positive and superharmonic on } \mathbb{D} \text{ and } v \ge u \text{ on } F_\rho\},$$

coincides with $P(\cdot,w)$ (see Corollary 7.4.6 in [1]). Since this condition holds ν_1-almost everywhere on $\mathbb{T}$, we have

$$R^{F_\rho}_{h_1} = \int R^{F_\rho}_{P(\cdot,w)} d\nu_1(w) = \int P(\cdot,w) d\nu_1(w) = h_1.$$

Also, $h_1 \ge 1$, so ν_1 majorizes normalized arclength measure on $\mathbb{T}$, and we similarly have $R^{F_\rho}_1 \equiv 1$. Hence, on taking reductions over F_ρ, we see that the inequality in (7) extends to all of $\mathbb{D}$. (Recall that $a \le 0$.) We can now let $\rho \to 0+$ and $\varepsilon \to 0+$ to see that $\log|f| \le a$ on $\mathbb{D}$. It is now clear that (b) implies (a).

Next suppose that condition (b) of Theorem 1 fails. Then there exists $w_0 \in \mathbb{T}$ such that

$$\sum_{m,n} 2^{-n} q_{m,n} < \infty, \quad \text{where } \; q_{m,n} = \mathcal{Q}\left(2^n E_{m,n}, [P(z_{m,n}, w_0)]\right). \tag{8}$$

For each m,n we can choose points $\xi_{k,m,n}$ $(k = 1, ..., q_{m,n})$ such that

$$\sum_{k=1}^{q_{m,n}} \log \frac{2^{-n}}{|z - \xi_{k,m,n}|} \ge P(z_{m,n}, w_0) - 1 \qquad (z \in E_{m,n}), \tag{9}$$

and without loss of generality we can assume that $\xi_{k,m,n}$ lies in the convex hull $\operatorname{conv}(S_{m,n})$ of $S_{m,n}$. In view of (8), the Blaschke product

$$B(z) = \prod_{k,m,n} \frac{|\xi_{k,m,n}|}{\xi_{k,m,n}} \left(\frac{\xi_{k,m,n} - z}{1 - z\overline{\xi}_{k,m,n}} \right)$$

converges on $\mathbb{D}$. There is an absolute constant $c_2 > 0$ such that

$$G_{\mathbb{D}}(z,\xi) \ge c_2 \log \frac{2^{-n}}{|\xi - z|} \qquad (z, \xi \in \operatorname{conv}(S_{m,n}))$$

for any pair (m,n). For a given pair (m_0,n_0) we thus have

$$\begin{aligned} -\log|B(z)| &= \sum_{k,m,n} G_{\mathbb{D}}(z, \xi_{k,m,n}) \ge \sum_{k=1}^{q_{m_0,n_0}} G_{\mathbb{D}}(z, \xi_{k,m_0,n_0}) \\ &\ge c_2 \sum_{k=1}^{q_{m_0,n_0}} \log \frac{2^{-n_0}}{|\xi_{k,m_0,n_0} - z|} \qquad (z \in S_{m_0,n_0}) \end{aligned}$$

so, by (9),

$$c_2 - \log|B(z)| \geq c_2 P(z_{m_0,n_0}, w_0) \geq \frac{c_2}{c_1} P(z, w_0) \qquad (z \in E_{m_0,n_0})\,. \tag{10}$$

Let

$$f(z) = B(z) \exp\left(\frac{c_2}{c_1}\left(\frac{w_0+z}{w_0-z}\right)\right) \qquad (z \in \mathbb{D}).$$

Then $\log|f(z)| \leq (c_2/c_1)P(z, w_0)$, so $f \in \mathcal{N}$, and certainly f is unbounded on $\mathbb{D}$. However, $|f| \leq e^{c_2}$ on E, by (10). Hence condition (a) of Theorem 1 also fails.

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