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Boundary feedback stabilization of a reaction-diffusion equation with Robin boundary conditions and state-delay ⋆

Hugo Lhachemi [a], Robert Shorten [a],

[a] *School of Electrical and Electronic Engineering, University College Dublin, Dublin, Ireland*

Abstract

This paper discusses the boundary feedback stabilization of a reaction-diffusion equation with Robin boundary conditions and in the presence of a time-varying state-delay. The proposed control design strategy is based on a finite-dimensional truncated model obtained via a spectral decomposition. By an adequate selection of the number of modes of the original infinite-dimensional system, we show that the design performed on the finite-dimensional truncated model achieves the exponential stabilization of the original infinite-dimensional system. In the presence of distributed disturbances, we show that the closed-loop system is exponentially input-to-state stable with fading memory.

Key words: Distributed parameter systems, Boundary control, State-delay, Reaction-diffusion equation , Input-to-state stability

1 Introduction

Boundary stabilization of partial differential equations (PDEs) in the presence of delays is an active research topic. A first research direction deals with the feedback stabilization of PDEs by means of delayed boundary control. For example, the cases of the heat [25] and wave [23–25] equations were studied via Lyapunov methods for slowly time-varying delays. A backstepping approach was reported in [14] for the boundary feedback stabilization of an unstable reaction-diffusion equation under large constant input delays. Inspired by the early work [29] and the developments reported in [1,2], such a problem was also investigated in [27] by designing a predictor feedback on a finite-dimensional truncated model capturing the unstable modes of the system. Then, a Lyapunov-based argument was employed to ensure that the control law achieves the stabilization of the full infinite-dimensional system. The same approach was applied in [7] for the boundary stabilization of a linear Kuramoto-Sivashinsky equation. Such an approach was generalized to a class of diagonal infinite-dimensional systems in [15,18] for constant input delays and then in [16] for fast time-varying input delays. A second research direction deals with the boundary feedback stabilization of PDEs in the presence of a state-delay. Such state-delays can be used to model either/both locality or/and inertia of certain physical phenomenon such as heat or mass transfers [26]. Motivated by the success of Linear Matrix Inequalities (LMI)-based approaches for the study of delayed finite-dimensional systems [5], LMI conditions were investigated in [6,30] for the stability analysis of PDEs in the presence of a state-delay. For the boundary control design of state-delayed PDEs, backstepping-based methods were reported in [8–11].

This paper is concerned with the boundary feedback stabilization of a reaction-diffusion equation with Robin boundary conditions in the presence of a time-varying state-delay. A similar setting was investigated in [8] via a backstepping approach in the case of Dirichlet-Dirichlet boundary conditions and for a constant state-delay. This problem was also investigated by means of backstepping control design in [10] for Neumann-Dirichlet boundary conditions with Dirichlet actuation and for a time-varying state-delay. This was then extended in [9] for the boundary feedback stabilization of a cascade PDE-ODE system under either Dirichlet or Neumann actuation. In the context of the reaction-diffusion equation, the contribution of this paper is twofold. First, we study the feedback stabilization of the reaction-diffusion equation with Robin-Robin boundary conditions and under a

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Corresponding author H. Lhachemi.

Email addresses: hugo.lhachemi@ucd.ie (Hugo Lhachemi), robert.shorten@ucd.ie (Robert Shorten).

time-varying state-delay. In particular, the proposed approach allows either one single command input (located at one of the two boundaries of the domain) or two command inputs. Second, we show that the resulting closed-loop system is, in the presence of distributed boundary disturbances, exponentially Input-to-State Stable (ISS) with fading memory [13]. The concept of ISS, originally introduced by Sontag in [31], plays an important role in the robustness assessment of dynamical systems and the stability of interconnected systems [13]. The extension of this notion to infinite-dimensional systems raises many challenges [21,22] and is the topic of active research activities. For a complete review of the ISS theory for infinite-dimensional systems, we refer the reader to [19].

The proposed control design strategy is organized as follows. First, a finite dimensional truncated model is obtained via spectral decomposition. The order of the spectral decomposition is selected to capture all the unstable modes of the original infinite-dimensional system plus a certain number of slow stable modes. In particular, the order is selected to guarantee the robust stability of the residual infinite-dimensional system with respect to exponentially vanishing command inputs exhibiting a prescribed decay rate. We show that this allows the design of the control law based on the finite-dimensional truncated model while ensuring the stability of the full infinite-dimensional closed-loop system.

The remainder of this paper is organized as follows. The problem setting and the control strategy are reported in Section 2. The well-posedness of the closed-loop system is assessed in Section 3. The stability analysis is carried out in Section 4. The effectiveness of the proposed control strategy is illustrated by numerical simulations in Section 5. Concluding remarks are provided in Section 6.

Notation. The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. The real and imaginary parts of a complex number z are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively. The field $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The set of n-dimensional vectors over $\mathbb{K}$ is denoted by $\mathbb{K}^n$ and is endowed with the Euclidean norm $\|x\| = \sqrt{x^* x}$. The set of $n \times m$ matrices over $\mathbb{K}$ is denoted by $\mathbb{K}^{n\times m}$ and is endowed with the induced norm denoted by $\|\cdot\|$. Finally, $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ denotes the set of functions $f:\mathbb{R}_+ \to \mathbb{R}^n$ that are absolutely continuous on any compact interval of $\mathbb{R}_+$.

2 Problem setting and control strategy

2.1 Problem setting

We are concerned with the boundary feedback stabilization of the following reaction-diffusion equation for Robin boundary conditions and with state-delay:

$$y_t(t,x) = a y_{xx}(t,x) + b y(t,x) + c y(t-h(t),x) + d(t,x) \tag{1a}$$

$$\cos(\theta_1) y(t,0) - \sin(\theta_1) y_x(t,0) = u_1(t) \tag{1b}$$

$$\cos(\theta_2) y(t,1) + \sin(\theta_2) y_x(t,1) = u_2(t) \tag{1c}$$

$$y(\tau,x) = \phi(\tau,x), \quad \tau \in [-h_M, 0] \tag{1d}$$

for $t > 0$ and $x \in (0,1)$. Here we have $a > 0$, $b, c \in \mathbb{R}$ with $c \neq 0$, and $\theta_1, \theta_2 \in [0, 2\pi)$. In this setting, $u_1, u_2 : \mathbb{R}_+ \to \mathbb{R}$ are the boundary controls (with possibly one input set identically equal to zero), $h : \mathbb{R}_+ \to \mathbb{R}$ is a time-varying delay, and $d : \mathbb{R}_+ \times (0,1) \to \mathbb{R}$ is a distributed perturbation with the regularity $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(0,1))$, i.e. $d : \mathbb{R}_+ \to L^2(0,1)$ is Bochner measurable and essentially bounded on any compact interval of $\mathbb{R}_+$. We assume that h is continuous and that there exist constants, $0 < h_m < h_M$, such that $h_m \leq h(t) \leq h_M$ for all $t \geq 0$. Finally, $\phi : [-h_M, 0] \times (0,1) \to \mathbb{R}$ represents the initial condition.

In the sequel, we use the following abstract version of (1a-1d) defined over the state-space $\mathcal{H} = L^2(0,1)$ endowed with its usual inner product $\langle f, g \rangle = \int_0^1 f(\xi) g(\xi) \,\mathrm{d}\xi$ and associated L^2-norm that is also denoted, with a slight abuse of notation, $\|\cdot\|$.

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + cX(t-h(t)) + p(t) \tag{2a}$$

$$\mathcal{B}X(t) = u(t) \tag{2b}$$

$$X(\tau) = \Phi(\tau), \quad \tau \in [-h_M, 0] \tag{2c}$$

for $t > 0$ with $\mathcal{A}f = af'' + bf \in \mathcal{H}$ defined on $D(\mathcal{A}) = H^2(0,1)$, $\mathcal{B}f = (\cos(\theta_1)f(0) - \sin(\theta_1)f'(0), \cos(\theta_2)f(1) + \sin(\theta_2)f'(1)) \in \mathbb{R}^2$ defined on $D(\mathcal{B}) = H^2(0,1)$, $X(t) = y(t,\cdot) \in \mathcal{H}$, $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$, $p(t) = d(t,\cdot) \in \mathcal{H}$, and $\Phi(t) = \phi(t,\cdot) \in \mathcal{H}$. The abstract system (2) is a natural extension of the concept of boundary control system reported in [3, Def. 3.3.2]. Indeed, 1) the disturbance-free operator $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$ with $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ generates a C_0-semigroup denoted by $S(t)$ [4]; 2) the operator $L_k \in \mathcal{L}(\mathbb{R}^2, \mathcal{H})$, with $k \geq 2$ an integer such that $\cos(\theta_m) + k\sin(\theta_m) \neq 0$ for any $m \in \{1,2\}$, defined for any $u = (u_1, u_2) \in \mathbb{R}^2$ by

$$[L_k u](x) = \frac{u_1 (1-x)^k}{\cos(\theta_1) + k\sin(\theta_1)} + \frac{u_2 x^k}{\cos(\theta_2) + k\sin(\theta_2)}$$

for $x \in [0,1]$, is a lifting operator in the sense that $R(L_k) \subset D(\mathcal{A})$, $\mathcal{A}L_k$ is bounded, and $\mathcal{B}L_k = I_{\mathbb{R}^2}$. Here $\ker(\mathcal{B})$ denotes the kernel of $\mathcal{B}$ while $R(L_k)$ stands for the range of L_k.

Assuming that [1] $u \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^2)$, $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$,

[1] This regularity of the forthcoming control law, as well as the existence and uniqueness of the mild solutions for the resulting closed-loop system, will be assessed in Section 3

$h \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ with $0 < h_m \leq h \leq h_M$, and $\Phi \in \mathcal{C}^0([-h_M, 0];\mathcal{H})$, the mild solution $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ of (2) is uniquely defined by (see [3, Def. 3.1.4, Lem. 3.1.5, and Sec. 3.3])

$$X(t) = S(t)\{\Phi(0) - L_k u(0)\} + L_k u(t) \quad (3)$$
$$+ \int_0^t S(t-s)\{\mathcal{A}L_k u(s) - L_k \dot{u}(s) + cX(s-h(s)) + p(s)\}\,\mathrm{d}s$$

for $t \geq 0$ and with the initial condition $X(\tau) = \Phi(\tau)$ for all $\tau \in [-h_M, 0]$.

2.2 Proposed control strategy

Introducing the operator $\mathcal{A}_c \triangleq \mathcal{A} + cI_{\mathcal{H}}$ defined on $D(\mathcal{A}_c) = D(\mathcal{A})$, (2a) is equivalent to:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}_c X(t) + c\{X(t-h(t)) - X(t)\} + p(t).$$

We also introduce $\mathcal{A}_{c,0} = \mathcal{A}_0 + cI_{\mathcal{H}}$, defined on $D(\mathcal{A}_{c,0}) = D(\mathcal{A}_0)$, which generates a C_0-semigroup $T(t)$. Then, using [3, Thm. 3.2.1], the mild solution (3) can be equivalently rewritten in function of $T(t)$ as:

$$X(t) = T(t)\{\Phi(0) - L_k u(0)\} + L_k u(t) \quad (4)$$
$$+ \int_0^t T(t-s)\Big\{\mathcal{A}_c L_k u(s) - L_k \dot{u}(s)$$
$$+ c\{X(s-h(s)) - X(s)\} + p(s)\Big\}\,\mathrm{d}s$$

for all $t \geq 0$. From the Sturm-Liouville theory (see, e.g., [28, Sec. 8.6]), it is well known that $\mathcal{A}_{c,0}$ is a self-adjoint operator whose eigenvalues $(\lambda_n)_{n\geq 1}$ are all real and can be sorted to form a strictly decreasing sequence with $\lambda_n \to -\infty$ when $n \to +\infty$. Furthermore, denoting by e_n a unit eigenvector of $\mathcal{A}_{c,0}$ associated with λ_n, $(e_n)_{n\geq 1}$ forms a Hilbert basis of $\mathcal{H} = L^2(0,1)$.

Introducing the notation $x_n(t) = \langle X(t), e_n\rangle$ we have [2] $X(t) = \sum_{n\geq 1} x_n(t) e_n$ and $\|X(t)\|^2 = \sum_{n\geq 1} |x_n(t)|^2$. The projection of (4) onto the Hilbert basis $(e_n)_{n\geq 1}$ with the use of $T(t)z = \sum_{n\geq 1} e^{\lambda_n t}\langle z, e_n\rangle e_n$ (see [3, Thm. 2.3.5] and also [20]) and an integration by parts shows that

$$x_n(t) = e^{\lambda_n t} x_n(0) + \int_0^t e^{\lambda_n(t-s)} g_n(s)\,\mathrm{d}s$$

with $g_n(t) = \big\langle -\lambda_n L_k u(t) + \mathcal{A}_c L_k u(t) + c\{X(t-h(t)) - X(t)\} + p(t), e_n\big\rangle$. As g_n is integrable on any compact interval, we have $x_n \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ and the following ODE (see also [17]) is satisfied for almost all $t \geq 0$:

$$\dot{x}_n(t) = \lambda_n x_n(t) + c\{x_n(t-h(t)) - x_n(t)\} \quad (5)$$
$$- \lambda_n \langle L_k u(t), e_n\rangle + \langle \mathcal{A}_c L_k u(t), e_n\rangle + \langle p(t), e_n\rangle.$$

We denote by $\{f_1, f_2\}$ the canonical basis of $\mathbb{R}^2$. Introducing for integers $n \geq 1$ and $m \in \{1,2\}$ the quantities $b_{n,m} = -\lambda_n\langle L_k f_m, e_n\rangle + \langle \mathcal{A}_c L_k f_m, e_n\rangle \in \mathbb{R}$ and $b_n = \begin{bmatrix} b_{n,1} & b_{n,2}\end{bmatrix} \in \mathbb{R}^{1\times 2}$, we obtain that

$$\dot{x}_n(t) = \lambda_n x_n(t) + c\{x_n(t-h(t)) - x_n(t)\}$$
$$+ b_n u(t) + \langle p(t), e_n\rangle.$$

For a given integer $N_0 \geq 1$, which will be discussed in the sequel, the introduction of

$$A = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{R}^{N_0\times N_0}, \quad (6a)$$
$$B = (b_{n,m})_{1\leq n\leq N_0, 1\leq m\leq 2} \in \mathbb{R}^{N_0\times 2}, \quad (6b)$$
$$Y(t) = \begin{bmatrix} x_1(t) & \ldots & x_{N_0}(t)\end{bmatrix}^\top \in \mathbb{R}^{N_0}, \quad (6c)$$
$$D(t) = \begin{bmatrix} \langle p(t), e_1\rangle & \ldots & \langle p(t), e_{N_0}\rangle\end{bmatrix}^\top \in \mathbb{R}^{N_0}, \quad (6d)$$
$$Y_\Phi(\tau) = \begin{bmatrix} \langle \Phi(\tau), e_1\rangle & \ldots & \langle \Phi(\tau), e_{N_0}\rangle\end{bmatrix}^\top \in \mathbb{R}^{N_0}, \quad (6e)$$

yields $Y \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{N_0})$ with, for almost all $t \geq 0$,

$$\dot{Y}(t) = AY(t) + c\{Y(t-h(t)) - Y(t)\} \quad (7a)$$
$$+ Bu(t) + D(t)$$
$$Y(\tau) = Y_\Phi(\tau), \qquad \tau \in [-h_M, 0] \quad (7b)$$

The control strategy, inspired by the works [1,2,29] in a delay-free context, relies on the following two steps. First, a feedback control $u = KY$ is designed to exponentially stabilize the finite-dimensional truncated model (7) capturing the unstable dynamics plus an adequate number of slow stable modes of (1). This configuration includes the case of a single boundary control input (either at $x = 0$ or $x = 1$) because one can obtain $u_m = 0$ by setting the m-th line of the feedback gain K as $0_{1\times N_0}$. Specifically, the feedback gain $K \in \mathbb{R}^{2\times N_0}$ is tuned such that all the poles of $A_{\mathrm{cl}} = A + BK$ are simple and stable with a sufficiently large decay rate. This procedure is allowed, for either one or two boundary control inputs, by the following Lemma, whose proof is described in Annex A and where $B_m \in \mathbb{R}^{N_0}$ denotes the m-th column of the matrix B.

Lemma 1 *For any given $N_0 \geq 1$, the pairs (A, B), (A, B_1) and (A, B_2) satisfy the Kalman condition [32].*

Remark 2 *Note that $b_{n,m}$ is computed based on the selection of a given lifting operator L_k. Even if such a lifting*

[2] The convergence of the series holds in the norm of the state-space $\mathcal{H} = L^2(0,1)$, i.e. in L^2-norm.

operator is not unique, the resulting quantity $b_{n,m}$ is actually independent of the particularly selected lifting operator. See [15] for details. Consequently, the commandability property stated in Lemma 1 is an intrinsic property of the pair $(\mathcal{A}_c, \mathcal{B})$ associated with (1) in the sense that it does not depend on the selection of a particular lifting operator.

In the second step, we will ensure that the design performed on the finite-dimensional truncated model achieves the exponential stabilization of the full infinite-dimensional system provided the fact that the number of modes N_0 used to obtain the truncated model is large enough. This will then lead to the establishment of the following theorem which is stated below.

Theorem 3 *Let $0 < h_m < h_M$ be arbitrarily given. Let $N_0 \geq 1$ be such that $\lambda_{N_0+1} < -2\sqrt{5}|c|$ and consider the matrices A and B defined by (6a-6b). Let $K \in \mathbb{R}^{2\times N_0}$ be such that $A_{\mathrm{cl}} = A + BK$ is Hurwitz with simple eigenvalues $\mu_1, \dots, \mu_{N_0} \in \mathbb{C}$ satisfying $\operatorname{Re}\mu_n < -3|c|$ for all $1 \leq n \leq N_0$. Then, there exist constants $\kappa, C_0, C_1 > 0$ such that, for any initial condition $\Phi \in \mathcal{C}^0([-h_M, 0]; \mathcal{H})$, any distributed perturbation $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$, and any delay $h \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ with $h_m \leq h \leq h_M$, the mild solution $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ of (2) with $u = KY$ satisfies*

$$\begin{aligned} \|y(t,\cdot)\| \leq\ & C_0 e^{-\kappa t} \sup_{\tau\in[-h_M,0]} \|\phi(\tau,\cdot)\| \\ & + C_1 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\kappa(t-\tau)} \|d(\tau,\cdot)\| \end{aligned} \tag{8}$$

for all $t \geq 0$, with control input

$$\begin{aligned} \|u(t)\| \leq\ & C_0 \|K\| e^{-\kappa t} \sup_{\tau\in[-h_M,0]} \|\phi(\tau,\cdot)\| \\ & + C_1 \|K\| \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\kappa(t-\tau)} \|d(\tau,\cdot)\|, \end{aligned} \tag{9}$$

where $y(t,\cdot) = X(t)$, $\phi(t,\cdot) = \Phi(t)$, and $d(t,\cdot) = p(t)$.

Remark 4 *The derivation of the design constraints $\operatorname{Re}\mu_n < -3|c|$ and $\lambda_{N_0+1} < -2\sqrt{5}|c|$ relies on the derivation of small gain arguments toward the proof of Theorem 3. The first (resp. second) constraint is used in the proof of Lemma 8 (resp. Lemma 10) to ensure the exponential ISS property of the closed-loop truncated model (resp. the residual infinite-dimensional dynamics).*

Remark 5 *ISS estimate (8) is said to have fading memory due to the exponential term in the evaluation of the contribution of the perturbation d. This term shows that, as time increases, the contribution of past disturbances on the current magnitude of the state trajectory is vanishing exponentially.*

Remark 6 *Note that the statement of Theorem 3 is still valid in the case $c = 0$, i.e. in the absence of state-delay. However, a much simpler proof than the one developed here in the case $c \neq 0$, which in particular allows the case of A_{cl} with eigenvalues of arbitrary multiplicity, can be given based on a direct integration and estimation of the ODEs of the spectral reduction.*

The remainder of the paper is devoted to the proof of Theorem 3 and its numerical illustration.

3 Well-posedness of the closed-loop system

In this section, we prove the existence and uniqueness of the mild solutions for (2) placed in closed loop with the feedback law $u = KY$. This ensures the validity of the spectral reduction reported in the previous section.

Lemma 7 *Let $0 < h_m < h_M$, $N_0 \geq 1$, and $K \in \mathbb{R}^{2\times N_0}$ be arbitrary. For any $\Phi \in \mathcal{C}^0([-h_M, 0]; \mathcal{H})$, $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$, and $h \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ with $h_m \leq h \leq h_M$, there exists a unique mild solution $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ of (2) with $u = KY \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^2)$.*

Proof. We show first that $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ such that $Y \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{N_0})$ is a mild solution of (2) with $u = KY$ if and only if $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ and satisfies, for all $t \geq 0$, equation (3) with $u = K\zeta$,

$$\begin{aligned} \zeta(t) =\ & e^{(A_{\mathrm{cl}}-cI)t} Y_\Phi(0) \\ & + \int_0^t e^{(A_{\mathrm{cl}}-cI)(t-\tau)} \{cY(\tau - h(\tau)) + D(\tau)\} \,\mathrm{d}\tau, \end{aligned} \tag{10}$$

and the initial condition $X(\tau) = \Phi(\tau)$ for all $\tau \in [-h_M, 0]$. On one hand, if X is a mild solution of (2) with $u = KY \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^2)$, then the developments of Section 2 show that $Y \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{N_0})$ satisfies (7) and thus $\zeta = Y$. On the other hand, assume that $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ satisfies (3) with $u = K\zeta$ and (10). We note from (10) that $\zeta \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{N_0})$ and

$$\dot{\zeta}(t) = (A_{\mathrm{cl}} - cI)\zeta(t) + cY(t - h(t)) + D(t)$$

for almost all $t \geq 0$. Then $u = K\zeta \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^2)$, showing that (3) indeed makes sense and, reproducing the developments of Section 2,

$$\dot{Y}(t) = (A - cI)Y(t) + cY(t - h(t)) + BK\zeta(t) + D(t)$$

for almost all $t \geq 0$. Consequently we have $\dot{\zeta} - \dot{Y} = (A - cI)(\zeta - Y)$ almost everywhere along with the initial condition $\zeta(0) - Y(0) = Y_\Phi(0) - Y_\Phi(0) = 0$. Thus $\zeta = Y \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{N_0})$, showing that X is a mild solution of (2) with $u = KY$.

To conclude, it remains to show the existence and uniqueness of a function $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ satisfying (3) with $u = K\zeta$ and (10). From the regularity assumptions and noting that for any $k \geq 0$, $0 \leq t \leq (k+1)h_m$ implies that $-h_M \leq t - h(t) \leq kh_m$, the existence and uniqueness of such a $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ is immediate by a classical steps procedure and [3, Lem. 3.1.5]. □

4 Stability analysis

This section is devoted to the proof of the main result of this paper: namely, the stability of the closed-loop system.

4.1 Finite-dimensional truncated model

We first study the problem of state-feedback stabilization of the finite-dimensional truncated system (7).

Lemma 8 *Let $N_0 \geq 1$ and $0 < h_m < h_M$ be arbitrarily given. Consider the matrices A and B defined by (6a-6b). Let $K \in \mathbb{R}^{2\times N_0}$ be such that $A_{\mathrm{cl}} = A + BK$ is Hurwitz with simple eigenvalues $\mu_1,\ldots,\mu_{N_0} \in \mathbb{C}$ such that $\operatorname{Re}\mu_n < -3|c|$ for all $1 \leq n \leq N_0$. Then, there exist constants $\sigma, C_2, C_3 > 0$ such that, for all $Y_\Phi \in \mathcal{C}^0([-h_M,0];\mathbb{R}^{N_0})$, $h \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ with $h_m \leq h \leq h_M$, and $D \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{N_0})$, the trajectory $Y \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{N_0})$ of (7) with command input $u = KY$ satisfies*

$$\|Y(t)\| \leq C_2 e^{-\sigma t} \sup_{\tau\in[-h_M,0]} \|Y_\Phi(\tau)\| + C_3 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|D(\tau)\| \tag{11}$$

for all $t \geq 0$.

The proof is inspired by the small gain-analysis reported in [12, Thm. 2.5] for the study of the robustness of predictor feedback with respect to delay mismatches. Nevertheless, 1) we provide here a refinement of the estimates by taking advantage of the properties of the matrix A_{cl}; 2) we consider the contribution of a disturbance input.

Proof. We define $\alpha = -\max_{1\leq n\leq N_0} \operatorname{Re}\mu_n > 3|c|$. Let $\sigma \in (0,\alpha)$ be arbitrarily given. As the eigenvalues of A_{cl} are simple, there exists $P \in \mathbb{C}^{N_0\times N_0}$ such that $PA_{\mathrm{cl}}P^{-1} = \Lambda \triangleq \operatorname{diag}(\mu_1,\ldots,\mu_{N_0})$. Defining $Z = PY \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{N_0})$, $Z_\Phi = PY_\Phi \in \mathcal{C}^0([-h_M,0];\mathbb{R}^{N_0})$, and $\hat{D} = PD \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{N_0})$, we obtain that

$$\dot{Z}(t) = \Lambda Z(t) + c\{Z(t-h(t)) - Z(t)\} + \hat{D}(t), \tag{12}$$

for almost all $t \geq 0$ with the initial condition $Z(\tau) = Z_\Phi(\tau)$ for $\tau \in [-h_M,0]$. Defining $v(t) = Z(t) - Z(t-h(t))$ for all $t \geq 0$, an integration shows that

$$v(t) = (e^{\Lambda h(t)} - I)Z(t-h(t)) + \int_{t-h(t)}^{t} e^{\Lambda(t-\tau)}\{-cv(\tau) + \hat{D}(\tau)\}\,\mathrm{d}\tau,$$

for all $t \geq h_M$. Noting that $\|e^{\Lambda\tau}\| = e^{-\alpha\tau}$ and $\|e^{\Lambda\tau} - I\| \leq 2$ for all $\tau \geq 0$, we obtain that

$$\|v(t)\| \leq 2\|Z(t-h(t))\| + |c|\int_{t-h_M}^{t} e^{-\alpha(t-\tau)}\|v(\tau)\|\,\mathrm{d}\tau + \int_{t-h_M}^{t} e^{-\alpha(t-\tau)}\|\hat{D}(\tau)\|\,\mathrm{d}\tau,$$

for all $t \geq h_M$. We evaluate the integral terms as follows (ψ denotes either v or $\hat{D}$):

$$\begin{aligned}\int_{t-h_M}^{t} & e^{-\alpha(t-\tau)}\|\psi(\tau)\|\,\mathrm{d}\tau \\ &\leq e^{-\sigma t}\int_{t-h_M}^{t} e^{-(\alpha-\sigma)(t-\tau)}e^{\sigma\tau}\|\psi(\tau)\|\,\mathrm{d}\tau \\ &\leq e^{-\sigma t}\frac{1-e^{-(\alpha-\sigma)h_M}}{\alpha-\sigma}\operatorname*{ess\,sup}_{\tau\in[t-h_M,t]} e^{\sigma\tau}\|\psi(\tau)\|,\end{aligned}$$

which gives, for all $t \geq h_M$,

$$\begin{aligned}\sup_{\tau\in[h_M,t]} e^{\sigma\tau}\|v(\tau)\| \leq{}& 2e^{\sigma h_M}\sup_{\tau\in[0,t-h_m]} e^{\sigma\tau}\|Z(\tau)\| \\ &+ |c|\frac{1-e^{-(\alpha-\sigma)h_M}}{\alpha-\sigma}\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \\ &+ \frac{1-e^{-(\alpha-\sigma)h_M}}{\alpha-\sigma}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|.\end{aligned} \tag{13}$$

Now, integrating (12) on $[0,t]$, we obtain for all $t \geq 0$:

$$\begin{aligned}\|Z(t)\| &\leq e^{-\alpha t}\|Z_\Phi(0)\| + |c|\int_0^t e^{-\alpha(t-\tau)}\|v(\tau)\|\,\mathrm{d}\tau + \int_0^t e^{-\alpha(t-\tau)}\|\hat{D}(\tau)\|\,\mathrm{d}\tau \\ &\leq e^{-\sigma t}\|Z_\Phi(0)\| + \frac{|c|e^{-\sigma t}}{\alpha-\sigma}\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| + \frac{e^{-\sigma t}}{\alpha-\sigma}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|,\end{aligned}$$

hence

$$\sup_{\tau\in[0,t]} e^{\sigma\tau}\|Z(\tau)\| \leq \|Z_\Phi(0)\| + \frac{|c|}{\alpha-\sigma}\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| + \frac{1}{\alpha-\sigma}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|. \tag{14}$$

Introducing $\delta \geq 0$ defined by

$$\delta = \frac{|c|}{\alpha-\sigma}\left\{1 - e^{-(\alpha-\sigma)h_M} + 2e^{\sigma h_M}\right\}, \tag{15}$$

we obtain from (13-14) that, for all $t \geq h_M$,

$$\sup_{\tau\in[h_M,t]} e^{\sigma\tau}\|v(\tau)\| \leq 2e^{\sigma h_M}\|Z_\Phi(0)\| + \delta \sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| + \frac{\delta}{|c|} \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|.$$

Using now the control design constraint $\alpha > 3|c|$, a continuity argument in $\sigma = 0$ shows the existence of a $\sigma \in (0,\alpha)$ such that $\delta < 1$. We fix such a $\sigma \in (0,\alpha)$ for the remaining of the proof. Considering separately the cases where the supremum on $[0,t]$ is achieved either on $[0,h_M]$ or $[h_M,t]$, we obtain for all $t \geq h_M$,

$$\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \leq \frac{2e^{\sigma h_M}}{1-\delta}\|Z_\Phi(0)\| + \sup_{\tau\in[0,h_M]} e^{\sigma\tau}\|v(\tau)\| + \frac{\delta}{|c|(1-\delta)} \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|. \tag{16}$$

We now evaluate the second term on the right-hand side of the above inequality. To do so, we note that, for $0 \leq \tau \leq t \leq h_M$, $\|v(\tau)\| \leq \|Z(\tau)\| + \|Z(\tau - h(\tau))\| \leq 2\sup_{\tau\in[0,t]}\|Z(\tau)\| + \sup_{\tau\in[-h_M,0]}\|Z_\Phi(\tau)\|$. Furthermore, integrating (12), one can show (using e.g. Grönwall's inequality) the existence of constants $\gamma_0,\gamma_1 > 0$, independent of Z_Φ, h, and $\hat{D}$, such that $\|Z(t)\| \leq \gamma_0 \sup_{\tau\in[-h_M,0]}\|Z_\Phi(\tau)\| + \gamma_1 \operatorname*{ess\,sup}_{\tau\in[0,t]}\|\hat{D}(\tau)\|$ for all $0 \leq t \leq h_M$. This yields, for all $0 \leq t \leq h_M$,

$$\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \leq (2\gamma_0+1)e^{\sigma h_M} \sup_{\tau\in[-h_M,0]}\|Z_\Phi(\tau)\| + 2\gamma_1 e^{\sigma h_M} \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|. \tag{17}$$

Combining (16-17), we obtain that

$$\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \leq \gamma_2 \sup_{\tau\in[-h_M,0]}\|Z_\Phi(\tau)\| + \gamma_3 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{D}(\tau)\|.$$

for all $t \geq 0$ with $\gamma_2 = (2\gamma_0 + 1 + 2/(1-\delta))e^{\sigma h_M}$ and $\gamma_3 = 2\gamma_1 e^{\sigma h_M} + \delta/(|c|(1-\delta))$. Using finally (14), and the facts that $\|Y\| \leq \|P^{-1}\|\|Z\|$, $\|Z_\Phi\| \leq \|P\|\|Y_\Phi\|$, and $\|\hat{D}\| \leq \|P\|\|D\|$, the claimed conclusion holds true. ☐

Remark 9 *If $K \in \mathbb{R}^{2\times N_0}$ is selected such that $A_{\mathrm{cl}} = A+BK$ has real and simple eigenvalues $\mu_1,\ldots,\mu_{N_0} \in \mathbb{R}$, then the conclusion of Lemma 8 still holds true under the relaxed assumption $\mu_n < -2|c|$ for all $1 \leq n \leq N_0$. This follows from the fact that, in the corresponding proof, the estimate $\|e^{\Lambda\tau} - I\| \leq 2$ can be in this case replaced by $\|e^{\Lambda\tau} - I\| \leq 1$ where $\tau \geq 0$.*

4.2 Infinite-dimensional part of the system neglected in the control design

We now investigate the robust stability property of the infinite-dimensional part of the system that has been hitherto neglected in our control design. Specifically, the following result holds.

Lemma 10 *Let $0 < h_m < h_M$ and $\sigma, C_4, C_5 > 0$ be arbitrarily given. Let $N_0 \geq 1$ be such that $\lambda_{N_0+1} < -2\sqrt{5}|c|$. Then, there exist constants $\kappa \in (0,\sigma)$ and $C_6, C_7 > 0$ such that, for all $\Phi \in \mathcal{C}^0([-h_M,0];\mathcal{H})$, $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{H})$, $h \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ such that $h_m \leq h \leq h_M$, and $u \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^2)$ such that*

$$\|u(t)\| + \|\dot{u}(t)\| \leq C_4 e^{-\sigma t} \sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\| + C_5 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|p(\tau)\| \tag{18}$$

for almost all $t \geq 0$, the mild solution $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ of (2) satisfies

$$\sum_{n\geq N_0+1} |x_n(t)|^2 \leq C_6 e^{-2\kappa t} \sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 + C_7 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-2\kappa(t-\tau)}\|p(\tau)\|^2 \tag{19}$$

for all $t \geq 0$, where $x_n(t) = \langle X(t), e_n\rangle$.

Proof. Introducing $\beta = -\lambda_{N_0+1}/2 > \sqrt{5}|c| > 0$, let $\kappa \in (0,\min(\beta,\sigma))$ be arbitrarily given. Then $\lambda_n \leq -2\beta < -2\kappa < 0$ for all $n \geq N_0+1$. We introduce, for $t \geq 0$, $z_n(t) = \langle X(t) - L_k u(t), e_n\rangle$, $p_n(t) = \langle p(t), e_n\rangle$,

$$q_n(t) = \langle \mathcal{A}_c L_k u(t), e_n\rangle - \langle L_k \dot{u}(t), e_n\rangle + c\langle L_k[\chi u](t-h(t)), e_n\rangle - c\langle L_k u(t), e_n\rangle,$$

where χ is defined as the characteristic function of the set $[0,+\infty)$. We also introduce the following series: $Z(t) = \sum_{n\geq N_0+1} |z_n(t)|^2 \leq \|X(t) - L_k u(t)\|^2$, $P(t) = \sum_{n\geq N_0+1} |p_n(t)|^2 \leq \|p(t)\|^2$, and $Q(t) = \sum_{n\geq N_0+1} |q_n(t)|^2$ which is such that, for almost all $t \geq 0$,

$$Q(t) \leq \gamma_1 e^{-2\kappa t} \sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 + \gamma_2 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-2\kappa(t-\tau)}\|p(\tau)\|^2, \tag{20}$$

with $\gamma_1 = 8C_4^2\left(\|\mathcal{A}_c L_k\|^2 + \{1 + c^2(1+e^{2\kappa h_M})\}\|L_k\|^2\right)$ and $\gamma_2 = \gamma_1 C_5^2/C_4^2$. For $t \geq h_M$, we define $v_n(t) = z_n(t) - z_n(t-h(t))$ and $V(t) = \sum_{n\geq N_0+1} |v_n(t)|^2$. As $z_n \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, we infer from (5) that, for almost all $t \geq 0$,

$$\dot{z}_n(t) = \lambda_n z_n(t) + c\{[\chi z_n](t-h(t)) - z_n(t)\} \tag{21}$$

$$+ c\langle[(1-\chi)\Phi](t-h(t)), e_n\rangle + q_n(t) + p_n(t).$$

In particular, we have for almost all $t \geq h_M$

$$\dot{z}_n(t) = \lambda_n z_n(t) - c v_n(t) + q_n(t) + p_n(t). \tag{22}$$

In the sequel, we always consider integers $n \geq N_0 + 1$. We introduce for any $t_1 < t_2$ and any real-valued and locally essentially bounded function ψ the notation $\mathcal{I}(\psi, t_1, t_2) = \int_{t_1}^{t_2} e^{-2\beta(t_2-\tau)}|\psi(\tau)|\,\mathrm{d}\tau$. Then, the following inequalities hold:

$$\begin{aligned}\mathcal{I}(\psi, t_1, t_2) &= e^{-2\kappa t_2}\int_{t_1}^{t_2} e^{-2(\beta-\kappa)(t_2-\tau)} \times e^{2\kappa\tau}|\psi(\tau)|\,\mathrm{d}\tau \\ &\leq e^{-2\kappa t_2}\frac{1-e^{-2(\beta-\kappa)(t_2-t_1)}}{2(\beta-\kappa)}\operatorname*{ess\,sup}_{\tau\in[t_1,t_2]} e^{2\kappa\tau}|\psi(\tau)|\end{aligned}$$

and, by Cauchy-Schwarz,

$$\begin{aligned}\mathcal{I}(\psi, t_1, t_2)^2 &\leq \int_{t_1}^{t_2} e^{-2\beta(t_2-\tau)}\,\mathrm{d}\tau \times \mathcal{I}(\psi^2, t_1, t_2) \\ &\leq \frac{1-e^{-2\beta(t_2-t_1)}}{2\beta}\mathcal{I}(\psi^2, t_1, t_2).\end{aligned}$$

For any $t \geq 2h_M$, we obtain via integration of (22) over $[t-h(t), t]$ that

$$\begin{aligned}v_n(t) &= (e^{\lambda_n h(t)} - 1)z_n(t-h(t)) \\ &+ \int_{t-h(t)}^{t} e^{\lambda_n(t-\tau)}\{-c v_n(\tau) + q_n(\tau) + p_n(\tau)\}\,\mathrm{d}\tau,\end{aligned}$$

from which, recalling that $\lambda_n \leq -2\beta$, we obtain the estimate

$$\begin{aligned}|v_n(t)|^2 &\leq 4|z_n(t-h(t))|^2 + 4|c|^2\mathcal{I}(v_n, t-h(t), t)^2 \\ &+ 4\mathcal{I}(q_n, t-h(t), t)^2 + 4\mathcal{I}(p_n, t-h(t), t)^2 \\ &\leq 4|z_n(t-h(t))|^2 + 4|c|^2\gamma_3\mathcal{I}(v_n^2, t-h(t), t) \\ &+ 4\gamma_3\mathcal{I}(q_n^2, t-h(t), t) + 4\gamma_3\mathcal{I}(p_n^2, t-h(t), t),\end{aligned}$$

with $\gamma_3 = (1-e^{-2\beta h_M})/(2\beta)$. From (20) we deduce that, for all $t \geq 2h_M$,

$$\begin{aligned}V(t) &\leq 4Z(t-h(t)) + 4\gamma_3|c|^2\mathcal{I}(V, t-h(t), t) \\ &+ 4\gamma_3\mathcal{I}(Q, t-h(t), t) + 4\gamma_3\mathcal{I}(P, t-h(t), t) \\ &\leq 4e^{2\kappa h_M}e^{-2\kappa h(t)}Z(t-h(t)) \\ &+ 4\gamma_4|c|^2 e^{-2\kappa t}\sup_{\tau\in[t-h(t),t]} e^{2\kappa\tau}V(\tau) \\ &+ 4\gamma_1\gamma_4 e^{-2\kappa t}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 \\ &+ 4(1+\gamma_2)\gamma_4 e^{-2\kappa t}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2\end{aligned}$$

with

$$\gamma_4 = \frac{(1-e^{-2\beta h_M})(1-e^{-2(\beta-\kappa)h_M})}{4\beta(\beta-\kappa)}.$$

Consequently, we have for all $t \geq 2h_M$,

$$\begin{aligned}\sup_{\tau\in[2h_M,t]} e^{2\kappa\tau}V(\tau) &\leq 4e^{2\kappa h_M}\sup_{\tau\in[h_M,t-h_m]} e^{2\kappa\tau}Z(\tau) \\ &+ 4\gamma_4|c|^2\sup_{\tau\in[h_M,t]} e^{2\kappa\tau}V(\tau) \\ &+ 4\gamma_1\gamma_4\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 \\ &+ 4(1+\gamma_2)\gamma_4\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2.\end{aligned} \tag{23}$$

Now, integrating (22) over $[h_M, t]$, we have for $t \geq h_M$

$$\begin{aligned}|z_n(t)| &\leq e^{-2\beta(t-h_M)}|z_n(h_M)| + |c|\mathcal{I}(v_n, h_M, t) \\ &+ \mathcal{I}(q_n, h_M, t) + \mathcal{I}(p_n, h_M, t),\end{aligned}$$

from which we infer

$$\begin{aligned}|z_n(t)|^2 &\leq 4e^{-4\kappa(t-h_M)}|z_n(h_M)|^2 + \frac{2|c|^2}{\beta}\mathcal{I}(v_n^2, h_M, t) \\ &+ \frac{2}{\beta}\mathcal{I}(q_n^2, h_M, t) + \frac{2}{\beta}\mathcal{I}(p_n^2, h_M, t).\end{aligned}$$

Thus, using again (20), we obtain for all $t \geq h_M$

$$\begin{aligned}Z(t) &\leq 4e^{-2\kappa(t-h_M)}Z(h_M) + \frac{2|c|^2}{\beta}\mathcal{I}(V, h_M, t) \\ &+ \frac{2}{\beta}\mathcal{I}(Q, h_M, t) + \frac{2}{\beta}\mathcal{I}(P, h_M, t) \\ &\leq 4e^{2\kappa h_M}e^{-2\kappa t}Z(h_M) + \gamma_5|c|^2 e^{-2\kappa t}\sup_{\tau\in[h_m,t]} e^{2\kappa\tau}V(\tau) \\ &+ \gamma_1\gamma_5 e^{-2\kappa t}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 \\ &+ (1+\gamma_2)\gamma_5 e^{-2\kappa t}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2,\end{aligned}$$

with $\gamma_5 = 1/(\beta(\beta-\kappa))$. Hence, we have for all $t \geq h_M$,

$$\begin{aligned}\sup_{\tau\in[h_M,t]} e^{2\kappa\tau}Z(\tau) &\leq 4e^{2\kappa h_M}Z(h_M) + \gamma_5|c|^2\sup_{\tau\in[h_M,t]} e^{2\kappa\tau}V(\tau) \\ &+ \gamma_1\gamma_5\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 \\ &+ (1+\gamma_2)\gamma_5\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2.\end{aligned} \tag{24}$$

Introducing $\eta \geq 0$ defined by

$$\begin{aligned}\eta &= 4|c|^2(\gamma_4 + \gamma_5 e^{2\kappa h_M}) \\ &= \frac{|c|^2}{\beta(\beta-\kappa)}\left\{(1-e^{-2\beta h_M})(1-e^{-2(\beta-\kappa)h_M}) + 4e^{2\kappa h_M}\right\},\end{aligned} \tag{25}$$

we obtain from (23-24) that, for all $t \geq 2h_M$,

$$\sup_{\tau\in[2h_M,t]} e^{2\kappa\tau}V(\tau) \leq 16e^{4\kappa h_M}Z(h_M) + \eta \sup_{\tau\in[h_M,t]} e^{2\kappa\tau}V(\tau) + \frac{\gamma_1\eta}{|c|^2}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 + \frac{(1+\gamma_2)\eta}{|c|^2}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2.$$

Using now the control design constraint $\beta = -\lambda_{N_0+1}/2 > \sqrt{5}|c|$, a continuity argument in $\kappa = 0$ shows the existence of a $\kappa \in (0, \min(\beta,\sigma))$ such that $\eta < 1$. We fix such a $\kappa \in (0, \min(\beta,\sigma))$ for the remaining of the proof. As all the supremums in the latter estimate are finite, we obtain for all $t \geq 2h_M$ that

$$\begin{aligned}\sup_{\tau\in[2h_M,t]} e^{2\kappa\tau}V(\tau) \leq{}& \frac{16e^{4\kappa h_M}}{1-\eta}Z(h_M) \\ &+ \frac{\eta}{1-\eta}\sup_{\tau\in[h_M,2h_M]} e^{2\kappa\tau}V(\tau) \\ &+ \frac{\gamma_1\eta}{|c|^2(1-\eta)}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 \\ &+ \frac{(1+\gamma_2)\eta}{|c|^2(1-\eta)}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2.\end{aligned} \tag{26}$$

Based on the integration of (21), straightforward estimations and the use of (20) show the existence of constants $\gamma_6, \gamma_7 > 0$, independent of Φ, p, h, and u, such that

$$Z(t) \leq \gamma_6 \sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 + \gamma_7 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2, \tag{27}$$

for all $0 \leq t \leq 2h_M$. Now, noting that, for $h_M \leq \tau \leq t \leq 2h_M$, $V(\tau) \leq 2Z(\tau) + 2Z(\tau - h(\tau))$, we infer that

$$\begin{aligned}\sup_{\tau\in[h_M,t]} e^{2\kappa\tau}V(\tau) &\leq 4e^{4\kappa h_M}\sup_{\tau\in[0,t]} Z(\tau) \\ &\leq 4\gamma_6 e^{4\kappa h_M}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 \\ &\quad + 4\gamma_7 e^{4\kappa h_M}\operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2,\end{aligned} \tag{28}$$

for all $h_M \leq t \leq 2h_M$. Combining (26-28), we obtain the existence of constants $\gamma_8, \gamma_9 > 0$, independent of Φ, p, h, and u, such that

$$\sup_{\tau\in[h_M,t]} e^{2\kappa\tau}V(\tau) \leq \gamma_8 \sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 + \gamma_9 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2, \tag{29}$$

for all $t \geq h_M$. Using (29) into (24) and combining with (27), we obtain the existence of constants $\gamma_{10}, \gamma_{11} > 0$, independent of Φ, p, h, and u, such that

$$\sup_{\tau\in[0,t]} e^{2\kappa\tau}Z(\tau) \leq \gamma_{10} \sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\|^2 + \gamma_{11} \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{2\kappa\tau}\|p(\tau)\|^2, \tag{30}$$

for all $t \geq 0$. Finally, we note that $|x_n(t)| \leq |z_n(t)| + |\langle L_k u(t), e_n\rangle|$ hence, for all $t \geq 0$,

$$\sum_{n\geq N_0+1} |x_n(t)|^2 \leq 2Z(t) + 2\|L_k\|^2\|u(t)\|^2.$$

As u is continuous, the right-hand side of (18) is actually an upper-bound of $\|u(t)\|$ for all $t \geq 0$. Recalling that $0 < \kappa < \sigma$ and using (30) the proof is complete. □

4.3 Proof of the main result and complementary remark

We can now complete the proof of the main result.

Proof of Theorem 3. Let $0 < h_m < h_M$ be arbitrarily given. As $(\lambda_n)_{n\geq 1}$ is strictly decreasing with $\lambda_n \to -\infty$ when $n \to +\infty$, let $N_0 \geq 1$ be an integer such that $\lambda_{N_0+1} < -2\sqrt{5}|c|$. Considering the matrices A and B defined by (6a-6b), let $K \in \mathbb{R}^{2\times N_0}$ be a feedback gain such that $A + BK$ is Hurwitz with simple eigenvalues $\mu_1, \ldots, \mu_{N_0} \in \mathbb{C}$ such that $\operatorname{Re}\mu_n < -3|c|$ for all $1 \leq n \leq N_0$. The existence of such a feedback gain is ensured by Lemma 1. From Lemma 8, we have the existence of constants $\sigma, C_2, C_3 > 0$ such that, for all $\Phi \in \mathcal{C}^0([-h_M, 0]\mathcal{H})$, $p \in L^\infty_{\text{loc}}(\mathbb{R}_+;\mathcal{H})$, $h \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ with $h_m \leq h \leq h_M$, the mild solution $X \in \mathcal{C}^0(\mathbb{R}_+\mathcal{H})$ of (2) with $u = KY$ satisfies, for all $t \geq 0$,

$$\sqrt{\sum_{n=1}^{N_0}|x_n(t)|^2} \leq C_2 e^{-\sigma t}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\| + C_3 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|p(\tau)\|, \tag{31}$$

where $x_n(t) = \langle X(t), e_n\rangle$. This estimate follows from (11) with Y defined by (6c) and using the fact that, based on (6d-6e) and recalling that $(e_n)_{n\geq 1}$ is a Hilbert basis, $\|D(t)\| \leq \|p(t)\|$ and $\|Y_\Phi(\tau)\| \leq \|\Phi(\tau)\|$. From $u = KY$ with $Y \in \text{AC}_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^{N_0})$ satisfying (7), we obtain the existence of constants $C_4, C_5 > 0$, independent of Φ, p, and h, such that the estimate (18) holds for almost all $t \geq 0$. Thus, the application of Lemma 10 yields the existence of constants $\kappa \in (0,\sigma)$ and $C_6, C_7 > 0$, independent of Φ, p, and h, such that (19) holds for all $t \geq 0$. Consequently, as $0 < \kappa < \sigma$, we obtain from (19) and (31) that

$$\|X(t)\| = \sqrt{\sum_{n\geq 1}|x_n(t)|^2} \leq C_0 e^{-\kappa t}\sup_{\tau\in[-h_M,0]}\|\Phi(\tau)\| + C_1 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|p(\tau)\|$$

for all $t \geq 0$, with $C_0 = C_2 + \sqrt{C_6}$ and $C_1 = C_3 + \sqrt{C_7}$ which are constants independent of Φ, p, and h. As $u = KY$, this completes the proof of Theorem 3. $\square$

Remark 11 *Theorem 3 merely ensures the existence of a decay rate $\kappa > 0$ but does not provide design mechanisms to further constrain its value. Nevertheless, the above proofs can be used to refine the control design procedure in order to enforce any desired value of the decay rate $\kappa > 0$. Indeed, assume that $0 < h_m < h_M$ and $\kappa > 0$ are arbitrarily given. From (25), we have $\eta \to 0$ when $\beta \to +\infty$. Recalling that $\beta = -\lambda_{N_0+1}/2$ with $\lambda_n \to -\infty$ when $n \to +\infty$, we can select an integer $N_0 \geq 1$ such that $\beta > \kappa$ is large enough and thus $\eta < 1$. Now let $\sigma > \kappa$ be given. From (15) we have $\delta \to 0$ when $\alpha \to +\infty$. Recalling that $\alpha = - \max_{1 \leq n \leq N_0} \operatorname{Re} \mu_n$, we can select the feedback gain $K \in \mathbb{R}^{2 \times N_0}$ such that $\alpha > \sigma$ is large enough and thus $\delta < 1$. Then, applying the same reasoning as in the proof of Theorem 3, we obtain the following result.*

Theorem 12 *Let $0 < h_m < h_M$ and $\kappa > 0$ be arbitrarily given. There exist $N_0 \geq 1$, $K \in \mathbb{R}^{2 \times N_0}$ and $C_0, C_1 > 0$ such that, for any initial condition $\Phi \in \mathcal{C}^0([-h_M, 0]; \mathcal{H})$, any distributed perturbation $p \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{H})$, and any delay $h \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ with $h_m \leq h \leq h_M$, the mild solution $X \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H})$ of (2) with $u = KY$ satisfies (8), with control input such that (9), for all $t \geq 0$.*

5 Numerical illustration

In this section we propose a numerical illustration of the result of Theorem 3. Considering the case $h_i \triangleq \cot(\theta_i) > 0$ for $i \in \{1, 2\}$, standard computations show that the eigenvalues of $\mathcal{A}_{c,0}$ are given by $\lambda_n = b + c - ar_n^2$ for $n \geq 1$, where $(r_n)_{n \geq 1}$ is the increasing sequence formed by the (strictly) positive solutions r of

$$\left(h_1 h_2 - r^2\right) \sin(r) + (h_1 + h_2) r \cos(r) = 0.$$

The corresponding unit eigenvectors are given by $e_n = \phi_n / \|\phi_n\|$ with $\phi_n(x) = r_n \cos(r_n x) + h_1 \sin(r_n x)$.

For numerical computations, we set $a = 0.2$, $b = 2$, $c = 1$, $\theta_1 = \pi/3$, and $\theta_2 = \pi/10$. The first three eigenvalues of $\mathcal{A}_{c,0}$ are approximately given by $\lambda_1 \approx 2.5561$, $\lambda_2 \approx -0.1186 > -2\sqrt{5}|c|$, and $\lambda_3 \approx -6.2299 < -2\sqrt{5}|c|$. Thus we set $N_0 = 2$ and we compute a feedback gain $K \in \mathbb{R}^{2 \times N_0}$ such that the eigenvalues of $A + BK$ are given by $\mu_1 = -3.5$ and $\mu_2 = -4$ with in particular $\mu_2 < \mu_1 < -3|c|$. For simulations, the initial condition, the time-varying delay, and the distributed disturbance are set to $\Phi(t,x) = (1-t)^2 \{(1-2x)/2 + 20x(1-x)(x-3/5)\}$, $h(t) = 2 + 1.5 \sin(t)$, and $d(t,x) = d_0(t)(1-x)$ with d_0 as depicted in Fig.1, respectively. The employed numerical scheme relies on the modal approximation of the reaction-diffusion equation using its first 30 modes. The time domain evolution of the closed-loop system (with feedback gain K computed accordingly) is depicted in 1) Fig. 2 in the case of two command inputs u_1 and u_2; 2) Fig. 3 in the case of a single command input u_1 with $u_2 = 0$. As shown over the time interval $[0, 8]$ s over which $d(t, \cdot) = 0$, both control strategies achieve the exponential stabilization of the closed-loop system with a similar settling time (because the pole placement is identical for the two actuation schemes). Over the time interval $[8, 20]$ s, the maximum of the perturbation d occurs at time $t = 10$ s and then vanishes as t increases. As expected via the established fading memory estimates (8-9), the impact of the vanishing perturbation is rapidly eliminated for $t > 10$ s. Finally, for times $t \geq 20$ s, we observe the behavior of the closed-loop system in the presence of a non-vanishing disturbance. These results are compliant with the theoretical predictions.

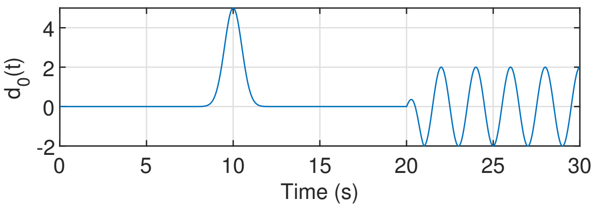
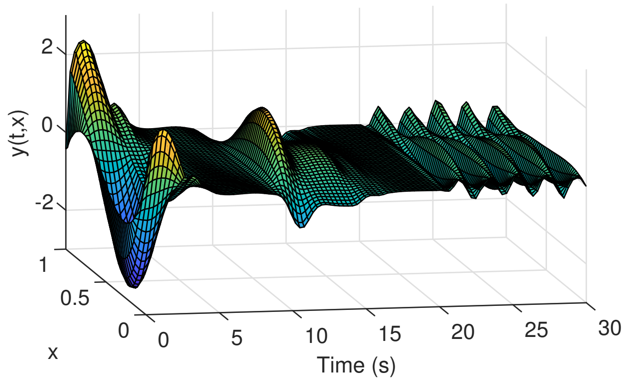
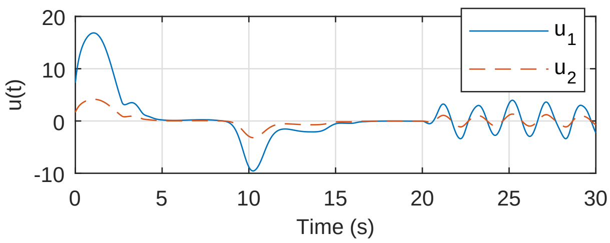


Fig. 1. Time evolution of the temporal component $d_0(t)$ of the distributed perturbation $d(t, x)$

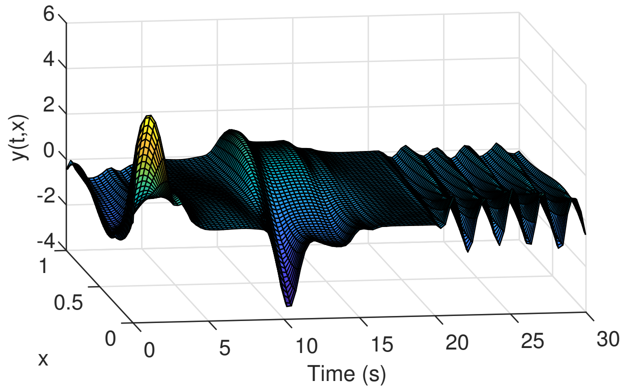


(a) State $y(t, x)$

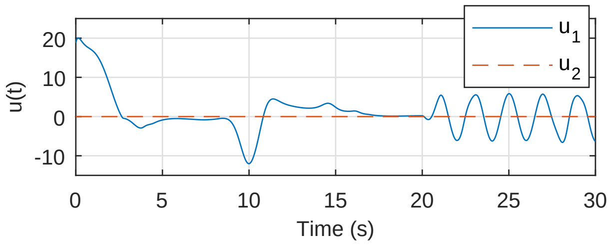


(b) Command inputs u_1 and u_2

Fig. 2. Time evolution of the closed-loop system with two command inputs u_1 and u_2



(a) State $y(t, x)$



(b) Command input u_1

Fig. 3. Time evolution of the closed-loop system with one command input u_1

6 Conclusion

This paper introduced a new method for the feedback stabilization of a reaction-diffusion equation in the presence of a state-delay in the reaction term. The essence of the method relies on the design of the control law on a finite-dimensional truncated model. This LTI model captures the unstable modes and an adequate number of slow stable modes of the original infinite-dimensional system in order to ensure the robust stability of the residual infinite-dimensional dynamics. This technique offers an alternative to backstepping-based methods previously used to tackle this kind of problem. In particular, the developed technique presents the advantage that the feedback is performed on a finite number of modes of the distributed parameter system while the infinite-dimensional residual dynamics is not actively controlled. Future developments of this method include the boundary stabilization of damped wave or beam equations, as well as the derivation of ISS properties with respect to boundary disturbances.

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A Proof of Lemma 1

As A is diagonal with simple eigenvalues, one can see that the Kalman condition is satisfied provided $b_{n,m} \neq 0$. We show that the latter condition always holds true. Let $k_0 \geq 2$ be such that $\cos(\theta_m)+k\sin(\theta_m) \neq 0$ for all $k \geq k_0$ and all $m \in \{1,2\}$. Following Remark 2, we recall that the quantity $b_{n,m}$ is independent of the selected lifting operator L_k. Thus, using $\mathcal{A}_c e_n = a e_n'' + (b+c)e_n = \lambda_n e_n$ and two successive integration by parts, we have:

$$\begin{aligned} b_{n,m} &= -\lambda_n \langle L_k f_m, e_n \rangle + \langle \mathcal{A}_c L_k f_m, e_n \rangle \\ &= (b+c-\lambda_n) \langle L_k f_m, e_n \rangle + a \langle (L_k f_m)'', e_n \rangle \\ &= a \{ - \langle L_k f_m, e_n'' \rangle + \langle (L_k f_m)'', e_n \rangle \} \\ &= a \{ (L_k f_m)'(1) e_n(1) - (L_k f_m)(1) e_n'(1) \} \\ &\quad + a \{ (L_k f_m)(0) e_n'(0) - (L_k f_m)'(0) e_n(0) \}, \end{aligned}$$

for all $k \geq k_0$, and then

$$b_{n,1} = a \frac{e_n'(0) + k e_n(0)}{\cos(\theta_1) + k\sin(\theta_1)}, \; b_{n,2} = a \frac{-e_n'(1) + k e_n(1)}{\cos(\theta_2) + k\sin(\theta_2)}.$$

Now, the condition $b_{n,1} = 0$ implies $e_n'(0) + k e_n(0) = 0$ for all $k \geq k_0$. This yields $e_n(0) = e_n'(0) = 0$ and we infer by Cauchy uniqueness the contradiction $e_n = 0$. Thus $b_{n,1} \neq 0$ for all $n \geq 1$. The same reasoning shows that $b_{n,2} \neq 0$ for all $n \geq 1$. This concludes the proof.