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On real-analytic recurrence relations for cardinal exponential B-splines

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Abstract

Let L_{N+1} be a linear differential operator of order $N + 1$ with constant coefficients and real eigenvalues $\lambda_1, \dots, \lambda_{N+1}$, let $E(\Lambda_{N+1})$ be the space of all C^∞ -solutions of L_{N+1} on the real line. We show that for $N \geq 2$ and $n = 2, \dots, N$, there is a recurrence relation from suitable subspaces \mathcal{E}_n to \mathcal{E}_{n+1} involving real-analytic functions, and with $\mathcal{E}_{N+1} = E(\Lambda_{N+1})$ if and only if contiguous eigenvalues are equally spaced.

Key words: L-splines, cardinal splines, basic spline, recurrence relation.
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1 Introduction

Recurrence relations for generalized splines have been discussed by several authors since the appearance of the pioneering work of C. de Boor and M.G. Cox in [4] and [5] respectively, cf. also [6], [7], [11], [12], [13], [18]. In order to motivate our results, let us consider briefly the case of *cardinal polynomial* splines. It is well known that the cardinal B-splines M_{N+1} and M_N (of order $N+1$ and N , and support in $[0, N+1]$ and $[0, N]$ respectively) are related by the identity

$$M_{N+1}(x) = \frac{x}{N}M_N(x) + \frac{N+1-x}{N}M_N(x-1) \quad (1)$$

for all $x \in \mathbb{R}$ (see e.g. [3, p. 86]). Analogous recurrence relations were proved for trigonometric and hyperbolic B-splines in [12] and [18] respectively, cf. [11] for a unified proof. On the other hand, L. Schumaker identified the classes of generalized splines which have B-splines bases computable by recursion relations analogous to those for polynomial, trigonometric, and hyperbolic splines. He proved in [17] that, in addition to the preceding spaces, essentially the only other space of splines admitting such a basis is a certain space of Tchebycheffian splines.

Our objective is to investigate whether there exists a recurrence relation generalizing (1) to the larger class of cardinal L-splines. This question was asked independently in the Conclusion of [19], p. 1436. Cardinal L -splines also arise in a natural way in the study of the so-called cardinal polysplines, see [8], [9], [10] and [1].

Polynomial and hyperbolic cardinal splines are special cases of cardinal L-splines, also known as *cardinal exponential splines*; here it is assumed that L is a linear differential operator of the form

$$L = \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j \right). \quad (2)$$

Throughout the paper we shall assume that the *eigen-values* $\lambda_1, \dots, \lambda_{N+1}$ are real numbers and we shall often use the notation

$$\Lambda_{N+1} := (\lambda_1, \dots, \lambda_{N+1}). \quad (3)$$

The functions in

$$E(\Lambda_{N+1}) := E(\lambda_1, \dots, \lambda_{N+1}) := \{f \in C^\infty(\mathbb{R}) : Lf = 0\} \quad (4)$$

are called *exponential polynomials*. A vector space \mathcal{E} is called an *exponential space of dimension $N + 1$* if there exists $(\lambda_1, \dots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ such that

$$\mathcal{E} = E(\lambda_1, \dots, \lambda_{N+1}). \quad (5)$$

A function $u : \mathbb{R} \rightarrow \mathbb{R}$ is a *cardinal L -spline* of order $N + 1$ if u is $(N - 1)$ -times continuously differentiable and for every $l \in \mathbb{Z}$ there exists an $f_l \in E(\Lambda_{N+1})$ such that $u(t) = f_l(t)$ whenever $t \in (l, l + 1)$. There exists (up to a scalar factor) a unique cardinal L -spline Q_{N+1} of order $N + 1$ and support (equal to) $[0, N + 1]$, called the B-spline of order $N + 1$, see [14]. We shall also write $Q_{\Lambda_{N+1}}$ or $Q_{(\lambda_1, \dots, \lambda_{N+1})}$ instead of Q_{N+1} .

We will study whether for a given fixed natural number N there exist “good” functions $a_N, b_N : \mathbb{R} \rightarrow \mathbb{R}$ such that the recurrence relation

$$Q_{N+1}(x) = a_N(x) Q_N(x) + b_N(x) Q_N(x - 1) \quad (6)$$

holds for all $x \in \mathbb{R}$. Note that a_N necessarily coincides with Q_{N+1}/Q_N on $(0, 1)$ and b_N with $Q_{N+1}/Q_N(\cdot - 1)$ on $(N, N + 1)$. Moreover, if a_N is known for $x \in [1, N]$, then the function b_N must be of the form

$$b_N(x) = \frac{Q_{N+1}(x) - a_N(x) Q_N(x)}{Q_N(x - 1)} \quad (7)$$

for $x \in (1, N)$. These arguments show that there exist many possibilities for a_N and b_N . However, if we require a_N to be real-analytic on \mathbb{R} then it is uniquely determined by its values on $(0, 1)$, and then b_N is also uniquely determined on $(0, N + 1)$. If in addition b_N is real-analytic on $(-\infty, 2)$ and (N, ∞) then b_N is completely determined on \mathbb{R} . An analogous statement can be made by interchanging the roles of a_N and b_N . On the other hand, it is not enough to require that a_N and b_N be C^∞ to obtain uniqueness, as Example 6 shows.

The main purpose of the paper is to find out under which conditions both a_N and b_N can be chosen to be real-analytic on \mathbb{R} . Let us introduce the following terminology: we say that there exists a *real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$* if there exist real-analytic functions a_N, b_N defined on \mathbb{R} such that (6) holds for all $x \in \mathbb{R}$. The following is our main result:

Theorem 1 *Let $(\lambda_1, \dots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. Then there exists a sequence of exponential spaces \mathcal{E}_j of dimension j , $j = 1, \dots, N + 1$,*

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_N \subset \mathcal{E}_{N+1} = E(\lambda_1, \dots, \lambda_{N+1}) \quad (8)$$

with real-analytic recurrence relations from \mathcal{E}_n to \mathcal{E}_{n+1} for $n = 2, \dots, N$, if and only if there exist $\alpha, \beta \in \mathbb{R}$ and a permutation σ of $\{1, 2, \dots, N+1\}$ such that $\lambda_{\sigma(k)} = \alpha + (k-1)\beta$, $1 \leq k \leq N+1$.

Let us note that the sufficiency part follows from [18] or [11] in the setting of L -splines with arbitrary knots; but it is also an easy byproduct of our methods of proof. This makes the paper self-contained.

Finally we mention that recurrence relations of a different nature were obtained by N. Dyn and A. Ron in [6] and [7]. When specialized to cardinal L -splines, and under the assumption $\lambda_1 \neq \lambda_{N+1}$, their results yield the following four-term recurrence relation (see e.g. [8]):

$$Q_{(\lambda_1, \lambda_2, \dots, \lambda_{N+1})}(x) = \frac{e^{-\lambda_{N+1}} Q_{(\lambda_2, \dots, \lambda_{N+1})}(x)}{\lambda_1 - \lambda_{N+1}} - \frac{e^{-\lambda_1} Q_{(\lambda_1, \dots, \lambda_N)}(x)}{\lambda_1 - \lambda_{N+1}} - \frac{Q_{(\lambda_2, \dots, \lambda_{N+1})}(x-1)}{\lambda_1 - \lambda_{N+1}} + \frac{Q_{(\lambda_1, \dots, \lambda_N)}(x-1)}{\lambda_1 - \lambda_{N+1}}.$$

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2 Preliminaries

The general theory of cardinal L -splines was developed by Micchelli [14], cf. also [8, Ch. 13]. Let $(\lambda_1, \dots, \lambda_{N+1}) \in \mathbb{C}^{N+1}$. We define the function φ_{N+1} for the operator L given in (2) as the unique function in the space $E(\Lambda_{N+1})$ such that

$$\frac{d^m}{dx^m} \varphi_{N+1}(0) = 0 \text{ for } m = 0, \dots, N-1 \text{ and } \frac{d^N}{dx^N} \varphi_{N+1}(0) = 1. \quad (9)$$

We shall also write $\varphi_{(\lambda_1, \dots, \lambda_{N+1})}$ instead of φ_{N+1} . Another useful way to explain properties of φ_{N+1} is the identity

$$\varphi_{(\lambda_1, \dots, \lambda_{N+1})}(x) := [\lambda_1, \dots, \lambda_{N+1}] h_x \quad (10)$$

where h_x is the function defined by $h_x(t) = e^{xt}$ and $[\lambda_1, \dots, \lambda_{N+1}]$ is the divided difference operator with respect to the variable t , see [16]. Recall that for

pairwise distinct $\lambda_1, \dots, \lambda_{N+1}$ and for any suitable function f

$$[\lambda_1, \dots, \lambda_{N+1}] f = \sum_{j=1}^{N+1} d_j f(\lambda_j), \quad d_j := \prod_{k=1, k \neq j}^{N+1} (\lambda_j - \lambda_k)^{-1}. \quad (11)$$

Note that $\varphi_{(\lambda_1)}(x) = e^{\lambda_1 x}$; furthermore $\varphi_{(\lambda_1, \lambda_1)}(x) = x e^{\lambda_1 x}$ for $\lambda_1 = \lambda_2$ and

$$\varphi_{(\lambda_1, \lambda_2)}(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \text{ for } \lambda_1 \neq \lambda_2. \quad (12)$$

From identity (11) one obtains the following simple consequence:

Lemma 2 *Let $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$ be pairwise distinct complex numbers and let $N \geq 1$. Then there exist nonzero constants c_j , $j = 2, \dots, N+1$, such that*

$$\varphi_{N+1}(x) = \sum_{j=2}^{N+1} c_j \varphi_{(\lambda_1, \lambda_j)}(x). \quad (13)$$

The last lemma can be generalized to the case of arbitrary $\lambda_1, \dots, \lambda_{N+1}$: For $0 \leq k \leq N$ one has the identity

$$[\lambda_1, \dots, \lambda_{N+1}] f = [\lambda_{k+1}, \dots, \lambda_{N+1}] F_k \text{ with } F_k(y) := [\lambda_1, \dots, \lambda_k, y] f. \quad (14)$$

This is easy to check for pairwise distinct $\lambda_{k+1}, \dots, \lambda_{N+1}$, using the classical recurrence relation for divided differences. The continuity of divided differences gives then the general case. Using this the following is easily established:

Lemma 3 *Let $\lambda_1, \dots, \lambda_{N+1}$ be complex numbers, define $F_x(\lambda) = \varphi_{(\lambda_1, \lambda)}(x)$ and denote by $F_x^{(l)}$ its l -th derivative with respect to the variable λ . Suppose that, up to a permutation, $(\lambda_2, \dots, \lambda_{N+1})$ is equal to $(\mu_1, \dots, \mu_1, \dots, \mu_r, \dots, \mu_r)$ where μ_1, \dots, μ_r are pairwise distinct and μ_j has multiplicity $\alpha_j > 0$ for $j = 1, \dots, r$. Then there exist nonzero constants $c_{j,l}$, $j = 1, \dots, r$; $l = 1, \dots, \alpha_j - 1$, such that*

$$\varphi_{N+1}(x) = \sum_{j=1}^r \sum_{l=0}^{\alpha_j-1} c_{j,l} F_x^{(l)}(\mu_j). \quad (15)$$

Set $\varphi_{N+1}^+(x) := \varphi_{N+1}(x)$ for $x \geq 0$ and $\varphi_{N+1}^+(x) := 0$ for $x < 0$. The basic cardinal L -spline Q_{N+1} is defined (up to a factor) as the unique cardinal L -spline of order $N+1$ with support in $[0, N+1]$. The basic spline Q_{N+1} can

be introduced via divided differences, see [11] and [14]. We use the formula

$$Q_{N+1}(x) = \sum_{j=0}^{N+1} s_{N+1,j} \varphi_{N+1}^+(x-j) \quad (16)$$

where the coefficients $s_{N+1,j}$ are defined by the equation

$$\prod_{j=1}^{N+1} (e^{-\lambda_j} - z) = \sum_{j=0}^{N+1} s_{N+1,j} z^j. \quad (17)$$

Later we shall use the identity

$$\sum_{j=0}^{N+1} s_{N+1,j} \varphi_{N+1}(x-j) = 0 \quad (18)$$

which implies that $Q_{N+1}(x) = 0$ for all $x \geq N+1$. Further we need the formulas $s_{N+1,N+1} = (-1)^{N+1}$, and

$$s_{N+1,0} = e^{-(\lambda_1 + \dots + \lambda_{N+1})}, \quad s_{N+1,1} = -e^{-(\lambda_1 + \dots + \lambda_{N+1})} \sum_{i=1}^{N+1} e^{\lambda_i}. \quad (19)$$

3 Real-analytic recurrence relations: necessary conditions

First, note that for $N = 1$ there exists always a real-analytic recurrence relation from $E(\lambda_1)$ to $E(\lambda_1, \lambda_2)$. Indeed, $Q_{(\lambda_1)}$ is given by $Q_{(\lambda_1)}(x) = e^{\lambda_1 x} 1_{[0,1]}$, where $1_{[0,1]}$ denotes the characteristic function of the interval $[0, 1]$. Then

$$Q_{(\lambda_1, \lambda_2)}(x) = a_1(x) Q_{(\lambda_1)}(x) + b_1(x) Q_{(\lambda_1)}(x-1) \quad (20)$$

where a_1 and b_1 are defined by real-analytic continuation of the functions $Q_{(\lambda_1, \lambda_2)}/Q_{(\lambda_1)}$ on $(0, 1)$ and $Q_{(\lambda_1, \lambda_2)}/Q_{(\lambda_1)}(\cdot - 1)$ on $(1, 2)$ respectively.

3.1 Uniqueness

We shall assume that L is of the form (2), where all λ_j are real if not otherwise stated. Then $\varphi_{N+1}(x) \neq 0$ for all $x \in \mathbb{R} \setminus \{0\}$ since φ_{N+1} has at most N real zeros on \mathbb{R} . Further we know that $Q_{N+1}(x) > 0$ for all $x \in (0, N+1)$.

Proposition 4 For any $N \geq 2$, uniqueness of the functions a_N and b_N satisfying (6) is guaranteed by requiring either a_N to be real-analytic on \mathbb{R} and b_N to be real-analytic on $(-\infty, 2)$ and (N, ∞) , or b_N to be real-analytic on \mathbb{R} and a_N to be real-analytic on $(-\infty, 1)$ and $(N - 1, \infty)$.

PROOF. By (6), (16), (19), we have for all $x \in (0, 1)$

$$a_N(x) = \frac{Q_{N+1}(x)}{Q_N(x)} = \frac{s_{N+1,0}\varphi_{N+1}(x)}{s_{N,0}\varphi_N(x)} = e^{-\lambda_{N+1}} \frac{\varphi_{N+1}(x)}{\varphi_N(x)}, \quad (21)$$

and for all $x \in (N, N + 1)$ using (18) and $s_{N+1,N+1} = (-1)^{N+1}$

$$b_N(x) = \frac{Q_{N+1}(x)}{Q_N(x-1)} = -\frac{\varphi_{N+1}(x-N-1)}{\varphi_N(x-N-1)}. \quad (22)$$

Since φ_n vanishes only at 0, with multiplicity n , the function φ_{N+1}/φ_N has a real-analytic extension to all \mathbb{R} . Thus, if we require a_N to be real-analytic on \mathbb{R} , then a_N is uniquely defined by (21) on \mathbb{R} . Since (6) implies (7) for all $x \in (1, N + 1)$, the function b_N is uniquely defined on $(1, N + 1)$. If we want b_N to be real-analytic on (N, ∞) we have to define $b_N(x)$ on (N, ∞) by (22). If we want it to be real-analytic on $(-\infty, 2)$, we have to define b_N as the real-analytic extension of b_N restricted to $(1, 2)$. Using (7), (16) for $x \in (1, 2)$, and (21) it is simple to see that for $x \in (1, 2)$

$$b_N(x) = \frac{s_{N+1,1}\varphi_{N+1}(x-1)}{s_{N,0}\varphi_N(x-1)} - \frac{s_{N,1}s_{N+1,0}\varphi_{N+1}(x)}{s_{N,0}s_{N,0}\varphi_N(x)}. \quad (23)$$

An entirely analogous argument works in the second case of the proposition.

3.2 Non-analytic recurrence relations

The preceding proof also yields the following result.

Theorem 5 Let $N \geq 2$, be a natural number. Then there exist a real-analytic function $a_N : \mathbb{R} \rightarrow \mathbb{R}$ and a function $b_N \in C^{N-2}(\mathbb{R})$, real-analytic on $\mathbb{R} \setminus \{2, \dots, N\}$, such that for all $x \in \mathbb{R}$

$$Q_{N+1}(x) = a_N(x)Q_N(x) + b_N(x)Q_N(x-1). \quad (24)$$

Positivity over the interval $(0, N + 1)$ of the functions a_N and b_N appearing in the recurrence relation is always desirable from the viewpoint of stability,

cf. also the polynomial case in (1). From (21) it is clear that a_N in Theorem 5 is always positive on the half line $(0, \infty)$. Moreover (23) implies that $b_N(1) = -\frac{s_{N,1}}{s_{N,0}}a_N(1) > 0$ since $a_N(1) > 0$, $s_{N,0} > 0$, and $s_{N,1} < 0$, cf. (19). However, in general b_N is not positive on $(1, N+1)$, cf. Example 12.

The following example shows that the functions a_N and b_N are not unique if they are only required to be C^∞ , even in the polynomial case.

Example 6 Let $\Lambda = (0, 0, 0)$ and take $N = 2$ in (1), i.e. $M_3(x) = \frac{x}{2}M_2(x) + \frac{3-x}{2}M_2(x-1)$. Then there exist $c, d \in C^\infty(\mathbb{R})$, $c \not\equiv 0, d \not\equiv 0$ such that $0 = c(x)M_2(x) + d(x)M_2(x-1)$. Thus, $M_3(x) = \left(\frac{x}{2} + c(x)\right)M_2(x) + \left(\frac{3-x}{2} + d(x)\right)M_2(x-1)$ is a different decomposition with C^∞ -coefficients.

To see why such c and d exist, one may simply take $c \neq 0$ to be a C^∞ -function with support contained in the open interval $(1, 2)$. Define d just by the equation $d(x) = -c(x)M_2(x)/M_2(x-1)$ for $x \in (1, 2)$ and 0 otherwise. Then d is a C^∞ -function.

3.3 Necessary conditions

Lemma 7 Let $L_{\Lambda_{N+1}} = \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j\right)$ and $(\lambda_1, \dots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. If $\varphi \neq 0$ is a solution of $L_{\Lambda_{N+1}}\varphi = 0$, then there exists an $M > 0$ such that φ has zeros only in a strip $|\operatorname{Re} z| \leq M$.

PROOF. This follows from the asymptotics of φ , since it is a sum of exponentials and all λ_j are real.

Theorem 8 Let $N \geq 2$ and $F_N := \varphi_{N+1}/\varphi_N$. Then each property below implies the next one:

(i) There exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$;

(ii) there exist nonzero constants A_N, B_N such that for all $x \in \mathbb{R}$

$$A_N F_N(x) + B_N F_N(x-1) + F_N(x-N-1) = 0; \quad (25)$$

(iii) the function F_N has an entire extension.

PROOF. For (i) \Rightarrow (ii) suppose that there exist real-analytic functions a_N and b_N on the real line satisfying the recurrence relation (6). Comparing (22)

with (23) one obtains (25) where

$$A_N = -\frac{s_{N+1,0}s_{N,1}}{s_{N,0}^2}, \quad B_N := \frac{s_{N+1,1}}{s_{N,0}}. \quad (26)$$

It is clear from (19) that A_N and B_N are nonzero.

Let us prove now (ii) \Rightarrow (iii). Clearly φ_{N+1}/φ_N is a meromorphic function. Hence we can write $\varphi_{N+1}/\varphi_N = \psi_{N+1}/\psi_N$, where ψ_{N+1} and ψ_N are entire functions without any common zero, and for $j = N, N+1$, each zero of ψ_j is a zero of φ_j . Now (25) implies that for each $z \in \mathbb{C}$

$$\begin{aligned} 0 &= A_N \psi_{N+1}(z) \psi_N(z-1) \psi_N(z-N-1) \\ &\quad + B_N \psi_{N+1}(z-1) \psi_N(z) \psi_N(z-N-1) \\ &\quad + \psi_{N+1}(z-N-1) \psi_N(z) \psi_N(z-1). \end{aligned}$$

We show that ψ_N has no zero in the complex plane, so ψ_{N+1}/ψ_N is entire. Suppose there exists a zero of ψ_N . By Lemma 7 there exists an $K \in \mathbb{R}$ such that all zeros of φ_N (and hence of ψ_N) satisfy $\operatorname{Re} z \geq K$. Let K_0 be the infimum of $\{\operatorname{Re} z : \psi_N(z) = 0\}$. Then there exists a zero z_0 of ψ_N with $\operatorname{Re} z_0 < K_0 + \frac{1}{2}$. It follows that $\psi_N(z_0-1) \neq 0$ and $\psi_N(z_0-N-1) \neq 0$. Then the equation above shows that $0 = A_N \psi_{N+1}(z_0) \psi_N(z_0-1) \psi_N(z_0-N-1)$. By (ii), $A_N \neq 0$, so we conclude that $\psi_{N+1}(z_0) = 0$. This contradicts the fact that ψ_{N+1} and ψ_N have no common zeros.

Theorem 9 *Assume that $\lambda_1, \dots, \lambda_{N+1}$ are given with $\lambda_1 \neq \lambda_2$. Suppose that for each $n = 2, \dots, N$ there exists a real-analytic recurrence relation from $E(\lambda_1, \dots, \lambda_n)$ to $E(\lambda_1, \dots, \lambda_{n+1})$. Then there exist pairwise distinct nonzero integers m_3, \dots, m_{N+1} such that*

$$\lambda_j - \lambda_1 = m_j (\lambda_2 - \lambda_1) \text{ for } j = 3, \dots, N+1. \quad (27)$$

We first prove the following two lemmas:

Lemma 10 *With the notations of Lemma 3, given $(\lambda_1, \dots, \lambda_{N+1})$, the following holds: All functions φ_n/φ_2 , $2 \leq n \leq N+1$, have entire extensions if and only if so do all functions $F_x^{(l)}(\mu_j)/\varphi_2$ for $0 \leq l \leq \alpha_j - 1$ and $j = 1, \dots, r$.*

PROOF. Sufficiency is clear since by Lemma 3, φ_n is a linear combination of the functions $F_x^{(l)}(\mu_j)$. For the necessity, use induction over N . For $N = 1$ the statement is trivial. Suppose now that φ_n/φ_2 , $2 \leq n \leq N+1$, have entire extensions, so they have entire extensions for $2 \leq n \leq N$. By the induction hypothesis each summand (necessarily non-zero) of φ_N/φ_2 in the

corresponding sum arising from (15) has an entire extension. By Lemma 3, φ_{N+1}/φ_2 is a linear combination of multiples of the same summands and one more term with a *nonzero* coefficient, either the value $F_x(\mu_j)/\varphi_2$ for a new μ_j , or of the type $F_x^{(l)}(\mu_j)/\varphi_2$ at an old one. Since the other summands and φ_{N+1}/φ_2 have entire extensions it follows that the new term also has an entire extension.

Lemma 11 *Suppose that $\lambda_1 \neq \lambda_2$. Given $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x)$ has an entire extension if and only if there exists a nonzero $m \in \mathbb{Z}$ such that*

$$\lambda - \lambda_1 = m(\lambda_2 - \lambda_1). \quad (28)$$

Moreover, if $x \mapsto \varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x)$ has an entire extension, it cannot be so for $x \mapsto \frac{d}{d\lambda} \varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x)$.

PROOF. Suppose that $x \mapsto \varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x)$ has an entire extension. Then by (12), any nonzero complex zero z_0 of $e^{\lambda_1 z} - e^{\lambda_2 z}$ must be a zero of $z \mapsto \varphi_{(\lambda_1, \lambda)}(z)$. Since $z_0 := 2\pi i / (\lambda_2 - \lambda_1)$ is a zero of $e^{\lambda_1 z} - e^{\lambda_2 z}$ we conclude that $0 = \varphi_{(\lambda_1, \lambda)}(z_0)$. This implies that $\lambda \neq \lambda_1$, and $e^{\lambda z_0} - e^{\lambda_1 z_0} = 0$. The existence of some nonzero integer m satisfying (28) follows immediately.

Conversely, from (28) and (12), one may derive that

$$\varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x) = \frac{1}{m} \frac{e^{m(\lambda_2 - \lambda_1)x} - 1}{e^{(\lambda_2 - \lambda_1)x} - 1}. \quad (29)$$

Since $\frac{X^m - 1}{X - 1} = 1 + X + \dots + X^{m-1}$ we conclude that $x \mapsto \varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x)$ has an entire extension.

Finally, suppose that (28) holds for some nonzero m . Then, with z_0 as above, we get

$$\frac{d}{d\lambda} \varphi_{(\lambda_1, \lambda)}(z_0) \neq 0. \quad (30)$$

Since $\varphi_{(\lambda_1, \lambda_2)}(z_0) = 0$ it follows that $\frac{d}{d\lambda} \varphi_{(\lambda_1, \lambda)}(x) / \varphi_{(\lambda_1, \lambda_2)}(x)$ is not entire.

PROOF OF THEOREM 9. Suppose that for each $n = 2, \dots, N$ there exists a real-analytic recurrence relation from $E(\lambda_1, \dots, \lambda_n)$ to $E(\lambda_1, \dots, \lambda_{n+1})$. By Theorem 8 all functions φ_{n+1}/φ_n , $2 \leq n \leq N$ have entire extensions. Thus, so do all functions φ_n/φ_2 , $2 \leq n \leq N$. The previous two lemmas prove that, if

$\lambda_1 \neq \lambda_2$, (27) holds for nonzero integers. Furthermore $\lambda_1, \dots, \lambda_{N+1}$ are pairwise distinct by the second statement of Lemma 11.

We have already seen that the coefficient function a_N in (24) is positive on $(0, \infty)$. It is a natural question whether the coefficient function b_N is also positive on $[1, N+1]$. By example we show that $b_N(x)$ can be negative on the interval $(1, 2)$.

Example 12 Let $\Lambda = (0, 1, \lambda_3)$ with $\lambda_3 > 1$, and set

$$C_{\lambda_3}(x) := \lambda_3 (\lambda_3 - 1) b_2(x) \varphi_2(x) \varphi_2(x-1). \quad (31)$$

Then C_{λ_3} and b_2 have the same sign on $(1, 2)$, and a computation shows that

$$\begin{aligned} C_{\lambda_3}(x) &= (1+e)(e^x - e^{x-1}) + (1-e^x)(1+e)e^{(x-1)\lambda_3} \\ &\quad + (e^{x-1} - e)e^{x\lambda_3} - (1-e^x)e^{\lambda_3} + \lambda_3(1-e^{x-1})(1-e^x)e^{\lambda_3}. \end{aligned}$$

Take $x = 3/2$. Then, since $e^{1.5\lambda_3}$ is the dominating term and the coefficient $(e^{0.5} - e)$ is negative, $C_{\lambda_3}(3/2) < 0$ whenever λ_3 is large enough. So $b_2(3/2)$ is also negative.

4 Existence of real-analytic recurrence relations: a characterization

At first we notice the following simple observation:

Proposition 13 *If there is a real-analytic recurrence relation from the exponential space $E(\lambda_1, \dots, \lambda_N)$ to $E(\lambda_1, \dots, \lambda_{N+1})$, then there is also one from $E(c + \lambda_1, \dots, c + \lambda_N)$ to $E(c + \lambda_1, \dots, c + \lambda_{N+1})$ for any $c \in \mathbb{R}$.*

PROOF. For simplicity sake put $c + \Lambda_N = (c + \lambda_1, \dots, c + \lambda_N)$. Using the fact that $\varphi_{c+\Lambda_N}(x) = e^{cx}\varphi_{\Lambda_N}(x)$ it is not difficult to see that $Q_{c+\Lambda_N}(x) = c_N e^{cx} Q_{\Lambda_N}(x)$ for some nonzero constant c_N . Assuming the recurrence relation $Q_{\Lambda_{N+1}}(x) = a_N(x) Q_{\Lambda_N}(x) + b_N(x) Q_{\Lambda_N}(x-1)$, it is obvious that

$$c_{N+1}^{-1} c_N Q_{c+\Lambda_{N+1}}(x) = a_N(x) Q_{c+\Lambda_N}(x) + e^c b_N(x) Q_{c+\Lambda_N}(x-1). \quad (32)$$

In the following we shall make use of a general remark: Let U_{N+1} be the linear space of functions over an open interval I , spanned by the functions $1, X, \dots, X^{N-1}$ and a real-analytic function $u(X)$ over I . Then, given $a \in I$, one

can define an element Φ_u in U_{N+1} which satisfies $\Phi_u(a) = \dots = \Phi_u^{(N-1)}(a) = 0$ by

$$\Phi_u(X) = u(X) - \sum_{k=0}^{N-1} \frac{u^{(k)}(a)}{k!} (X-a)^k. \quad (33)$$

By expanding $u(X)$ in a Taylor series about a this implies

$$\Phi_u(X) = (X-a)^N \sum_{k=0}^{\infty} \frac{u^{(k+N)}(a)}{(k+N)!} (X-a)^k. \quad (34)$$

Lemma 14 *Suppose $\Lambda_N = (0, \lambda_2, \dots, (N-1)\lambda_2)$ and $\Lambda_{N+1} = (\Lambda_N, M\lambda_2)$, with a natural number $M \geq N \geq 1$, and let φ_N, φ_{N+1} be defined by (9). Then φ_{N+1}/φ_N is an entire function of the form*

$$\frac{\varphi_{N+1}(x)}{\varphi_N(x)} = cR(e^{\lambda_2 x}) \quad (35)$$

for some non-zero constant c and a polynomial R defined by

$$R(X) = (X-1) \sum_{k=0}^{M-N} \binom{M}{k+N} (X-1)^k. \quad (36)$$

PROOF. By the assumptions of the lemma, the space $E(\Lambda_{N+1})$ is generated by $1, e^{\lambda_2 x}, \dots, e^{\lambda_2(N-1)x}$ and $e^{M\lambda_2 x}$. So we are working, up to a change of variable $X := e^{\lambda_2 x}$, in the space $1, X, \dots, X^{N-1}, X^M$ over the interval $I = (0, \infty)$. Use now the above notations Φ_u for $u(X) = X^M$ and $a = 1$ in (33). Then there exists a nonzero constant d_{N+1} with

$$\varphi_{N+1}(x) = d_{N+1} \Phi_u(e^{\lambda_2 x}). \quad (37)$$

Similarly, for the system $1, X, \dots, X^{N-2}, v(X)$ with $v(X) = X^{N-1}$, one has that $\Phi_v(X) = (X-1)^{N-1}$ and $\varphi_N(x) = d_N \Phi_v(e^{\lambda_2 x})$ for some $d_N \neq 0$. An immediate consequence of (34) is that $\frac{\Phi_u(X)}{\Phi_v(X)}$ is equal to $R(X)$ defined in (36). This completes the proof of the lemma.

The following proposition provides the central step in the proof of our main theorem. In particular it shows that for the exponential space $E(0, \dots, N)$ there exist two different exponential spaces \mathcal{E}_1 and \mathcal{E}_2 admitting a real-analytic recurrence relation from $E(0, \dots, N)$ to \mathcal{E}_j for $j = 1, 2$, namely $\mathcal{E}_1 = E(0, \dots, N+1)$ and $\mathcal{E}_2 = E(-1, 0, \dots, N)$.

Proposition 15 *Given two real numbers α, β with $\beta \neq 0$, an integer $N \geq 1$, and an integer M , $M \notin \{0, \dots, N-1\}$, let us set*

$$\begin{aligned}\Lambda_N &:= (\alpha, \alpha + \beta, \dots, \alpha + (N-1)\beta) \\ \Lambda_{N+1} &:= (\alpha, \alpha + \beta, \dots, \alpha + (N-1)\beta, \alpha + \beta M).\end{aligned}$$

Then, the following assertions are equivalent:

- (i) *There exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$;*
- (ii) *$M = N$ or $M = -1$.*

PROOF. Assume that (i) holds. Due to Proposition 13, we may assume that $\alpha = 0$. We will show that that $M = N$ if $M > 0$ and $M = -1$ if $M < 0$.

First, assume that $M > 0$. Then $M > N$ by our assumptions and we can use the last Lemma: If $M > N$, then the polynomial R defined in (36) has degree $M - N + 1 \geq 2$. Now (35) and Theorem 8 yield

$$A_N R(e^{\lambda_2 x}) + B_N R(e^{\lambda_2(x-1)}) + C_N R(e^{\lambda_2(x-N-1)}) = 0. \quad (38)$$

Putting $\gamma = e^{-\lambda_2}$ and $X = e^{\lambda_2 x}$ one arrives at

$$A_N R(X) + B_N R(\gamma X) + C_N R(\gamma^{N+1} X) = 0 \quad (39)$$

for all $X > 0$, hence for all $X \in \mathbb{R}$. Then $(A_N + B_N + C_N)R(0) = 0$, and differentiation gives the following two relations:

$$(A_N + \gamma B_N + C_N \gamma^{N+1})R'(0) = (A_N + \gamma^2 B_N + C_N \gamma^{2N+2})R''(0) = 0. \quad (40)$$

Since $R(0)$, $R'(0)$ and $R''(0)$ are nonzero and $\lambda_2 \neq 0$, this implies $A_N = B_N = C_N = 0$, a contradiction. Hence $M = N$.

Now assume that $M < 0$. We will see that this is reduced to the previous case. We apply Proposition 13 with $c := -(N-1)\beta$: so assumption (i) with $\alpha = 0$ implies that there exists a real-analytic recurrence relation from $E(c + \Lambda_N)$ to $E(c + \Lambda_{N+1})$. Now $c + \Lambda_N$ consists of the values

$$-(N-1)\beta + j\beta = (-\beta)(N-1-j) \quad (41)$$

for $j = 0, \dots, N-1$ and

$$c + \lambda_{N+1} = -(N-1)\beta + M\beta = (-\beta)(N-1-M). \quad (42)$$

Since $M < 0$ we know that $\tilde{M} := N - 1 - M > 0$. By the first case applied to $c + \Lambda_N$ and $c + \Lambda_{N+1}$ we conclude that $\tilde{M} = N$ which clearly implies that $M = -1$.

For (ii) \Rightarrow (i) we assume at first that $M = N$. Then the real change of variable $X = e^{\lambda_2 x}$ transforms the cardinal spline spaces based on $E(\Lambda_N)$ and $E(\Lambda_{N+1})$ into the polynomial splines of degree N and $N + 1$ on $(0, \infty)$ relative to the simple knots $t_j := e^{\lambda_2 j}$. Recurrence relations are known in such spaces, and their coefficients are real-analytic. This implies the statement by taking the inverse transform $x = \lambda_2^{-1} \ln X$. The case $M = -1$ is handled in a similar way.

Proposition 16 *Let α be a real number. Suppose $\Lambda_N = (\alpha, \dots, \alpha)$ and $\Lambda_{N+1} = (\Lambda_N, \lambda)$ for $\lambda \in \mathbb{R}$. Then the following assertions are equivalent:*

(i) *There exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$;*

(ii) $\lambda = \alpha$.

PROOF. Due to Proposition 13, we may assume that $\alpha = 0$. By Theorem 8 and assumption (i) there exist nonzero constants A_N, B_N, C_N such that for all $x \in \mathbb{R}$

$$A_N \frac{\varphi_{N+1}}{\varphi_N}(x) + B_N \frac{\varphi_{N+1}}{\varphi_N}(x-1) + C_N \frac{\varphi_{N+1}}{\varphi_N}(x-N-1) = 0. \quad (43)$$

Suppose that $\lambda \neq 0$. Note that $\varphi_N(x) = x^{N-1}/(N-1)!$ and, according to (33) and Lemma 14, there exists a nonzero constant d_{N+1} such that

$$\varphi_{N+1}(x) = d_{N+1}(e^{\lambda x} - R(x)), \quad R(x) = \sum_{k=0}^{N-1} \frac{(\lambda x)^k}{k!}. \quad (44)$$

Multiply (43) with $[x(x-1)(x-N-1)]^{N-1}$. It follows that there exists a polynomial Q such that

$$e^{\lambda x} P(x) - Q(x) = 0 \quad \text{for all } x \in \mathbb{R}, \quad (45)$$

where the polynomial P is defined by

$$P(x) = A_N [(x-1)(x-N-1)]^{N-1} + B_N e^{-\lambda} [x(x-N-1)]^{N-1} + C_N e^{-\lambda(N+1)} [x(x-1)]^{N-1}.$$

This is impossible unless $P = Q = 0$. But $P = 0$ implies $A_N = B_N = C_N = 0$. Thus we cannot have $\lambda \neq 0$.

For (ii) \Rightarrow (i) note that $E(\Lambda_{N+1})$ is the classical polynomial spline space.

Now we are going to prove our main result stated as Theorem 17.

PROOF OF THEOREM 1. Proof of the necessity by induction. For $n = 2$ there is nothing to prove. Suppose that there exists a sequence of exponential spaces $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_N \subset \mathcal{E}_{N+1} = E(\Lambda_{N+1})$ with real-analytic recurrence relations from \mathcal{E}_n to \mathcal{E}_{n+1} for $n = 2, \dots, N$. The recursive assumption enables us to assume, without loss of generality, that $\lambda_j = \alpha + (j - 1)\beta$ for $1 \leq j \leq N$.

Suppose that $\beta \neq 0$. From Theorem 9 we can deduce that $\lambda_{N+1} = \alpha + \beta M$ for some integer M different from $0, \dots, N - 1$. Proposition 15 ensures that either $M = N$ or $M = -1$. If $M = N$, then the equality $\lambda_j = \alpha + (j - 1)\beta$ is valid for $j = N + 1$ too. If $M = -1$, then $\lambda_{\sigma(j)} = \tilde{\alpha} + (j - 1)\beta$ for $1 \leq j \leq N + 1$, with $\tilde{\alpha} := \alpha - \beta$, and with $\sigma(1) = N + 1$, $\sigma(j) := j - 1$ for $j = 2, \dots, N + 1$.

The case $\beta = 0$ follows from Proposition 16.

Sufficiency follows from Proposition 15 and 16.

Consider the exponential space $E(\lambda_1, \lambda_2)$. For simplicity assume that $\lambda_1 = 0$, and put $\beta = \lambda_2 - \lambda_1$. Then the proof of our main theorem shows how to construct all increasing sequences of exponential spaces admitting analytic relations, starting from $E(\lambda_1, \lambda_2) = E(0, \beta)$ in the following (uncomplete) scheme:

$$\begin{array}{ccccc}
 & & E(0, \beta) & & \\
 & & \swarrow & & \searrow \\
 & E(-\beta, 0, \beta) & & E(0, \beta, 2\beta) & \\
 & \swarrow & & \searrow & \\
 E(-2\beta, -\beta, 0, \beta) & & E(-\beta, 0, \beta, 2\beta) & & E(0, \beta, 2\beta, 3\beta)
 \end{array} \tag{46}$$

Let us look at the particular case that $\Lambda_{N+1} = (\lambda_1, \dots, \lambda_{N+1})$ is ordered, so $\lambda_1 \leq \dots \leq \lambda_{N+1}$. Then there exists a real-analytic recurrence relation from $E(\lambda_1, \dots, \lambda_n)$ to $E(\lambda_1, \dots, \lambda_{n+1})$ for $n = 2, \dots, N$, if and only if

$$\lambda_n = \lambda_1 + (n - 1)(\lambda_2 - \lambda_1). \tag{47}$$

The following description is obvious from the above scheme:

Theorem 17 *Let $(\lambda_1, \dots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. Then there exist real-analytic recurrence relations from $E(\lambda_1, \dots, \lambda_n)$ to $E(\lambda_1, \dots, \lambda_{n+1})$ for $n = 1, 2, \dots, N$ if and only if for $3 \leq j \leq N + 1$*

$$\lambda_j = \lambda_1 + m_j(\lambda_2 - \lambda_1) \quad (48)$$

with either $m_{j+1} = \min\{m_1, \dots, m_j\} - 1$ or $m_{j+1} = \max\{m_1, \dots, m_j\} + 1$, and with $m_1 = 0, m_2 = 1$.

It follows from our results that the only exponential spaces admitting real-analytic recurrence relations are either the classical polynomial spaces, or transformations of polynomial spaces via an exponential map, cf. the discussion in section 6 in [17].

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