



Title	Meshfree Sequentially Linear Analysis of Concrete
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Publication date	2016-03
Publication information	Al-Sabah, Salam, and Debra F. Laefer. "Meshfree Sequentially Linear Analysis of Concrete." American Society of Engineers, March 2016. https://doi.org/10.1061/(ASCE)CP.1943-5487.0000474 .
Publisher	American Society of Engineers
Item record/more information	http://hdl.handle.net/10197/7456
Publisher's version (DOI)	10.1061/(ASCE)CP.1943-5487.0000474

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1 Meshfree, sequentially linear analysis of concrete

2 A. Salam Al-Sabah¹ and Debra F. Laefer²

3 **Abstract:**

4 New meshfree method employing the Node-based Smoothed Point Interpolation Method
5 (NS-PIM) is presented as an alternative to the non-linear finite element approach for concrete
6 members. The non-linear analysis is replaced by sequentially linear analyses (SLA), and
7 smeared, fixed concrete cracking model was used. A notched concrete beam was employed
8 for validation. Using a crack band width factor of 2.0 and a 10 mm nodal spacing, the peak
9 load differed by only 3.5% from experimental ones. Overall results were similar to experi-
10 mental ones, as well as to those published by researchers using finite element SLA. The ap-
11 proach provides two major advantages over finite element-based SLA: (1) nodal distortion
12 insensitivity and (2) nodal spacing insensitivity.

14 **Introduction**

15
16 The finite element method (FEM) is the most widely used numerical method to study linear
17 and non-linear behaviour (for both materials and geometric components) of structures. The
18 method, in its application to non-linear structural analysis, has matured sufficiently to be the
19 basis of many commercial software packages (ANSYS, Abacus, ATINA, etc.). Despite sig-
20 nificant progress in its theoretical and numerical aspects, some weaknesses persist. These can
21 be summarised as follows:

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- 22 • Results are mesh-dependent, with good results requiring a high quality mesh and each
23 element's geometry satisfying shape and aspect ratio limits.
- 24 • Models are stiffer than the actual structures. Hence, displacements are underestim-
25 ed.
- 26 • In analysis of geometric non-linearity, elements can become distorted sufficiently to
27 compromise output accuracy.
- 28 • Crack propagation usually requires re-meshing, and the robustness of automatic re-
29 meshers is questionable, particularly in three-dimensional problems.

30

31 Modelling of reinforced concrete is an important topic, as it is one of the most widely used
32 composite materials in construction. Predicting its behaviour is complicated by factors such
33 as reinforcement yielding, non-linear reinforcement-concrete bond behaviour, non-linear be-
34 haviour of concrete in compression, and tension cracking of the concrete. This last aspect
35 contributes most significantly to the early, non-linear behaviour of reinforced concrete beams
36 and slabs. The application of non-linear FEM in the analysis of reinforced concrete structures
37 can be traced back to the 1960s when the first reinforced concrete finite element model which
38 includes the effect of cracking was developed by Ngo and Scordelis (1967).

39

40 When loaded in tension, concrete fails suddenly after reaching its tensile limit. The heteroge-
41 neous nature of concrete results in a quasi-brittle behaviour that is greatly affected by soften-
42 ing damage (Bazant and Jirásek 2002). To represent this, several fracture models have been
43 proposed as summarized by Rots and Blaauwendraad (1989). An important component of
44 these models is the Fracture Process Zone (FPZ), defined as the zone ahead of the crack tip in
45 which concrete undergoes softening behaviour due to microcracking. Two widely used crack-
46 ing models are the Fictitious (or cohesive) Crack Model (FCM) introduced by Hillerborg et

47 al. (1976), and Crack Band Model (CBM) as proposed by Bazant and Oh (1983). In the first
48 model, the FPZ is represented as a fictitious line that can transmit normal stress. Fracture en-
49 ergy is then expressed as a function of critical crack separation (or opening width, w_c) (Bazant
50 and Jirásek 2002). In the CBM, fracturing is modelled as a band of parallel, densely distribut-
51 ed microcracks in the FPZ that has a certain width, which is referred to as the crack band
52 width (Bazant and Oh 1983). The average strain over the FPZ can be related to its defor-
53 mation through the crack band width. The fracture energy can then be represented as a func-
54 tion of a stress-strain curve and the crack band width.

55
56 Concrete fracture models, combined with non-linear models for concrete and steel are typi-
57 cally combined with the FEM to produce numerical procedures for non-linear analysis of re-
58 inforced concrete. Early efforts to overcome this encountered two main challenges. The first
59 was the numeric instability due to tensile cracking. The second related to the softening por-
60 tion of the behaviour. The first was solved by adopting the incremental-iterative solution
61 method (Crisfield 1996), where the unbalanced forces were allowed to dissipate through solu-
62 tion iterations. Since the second resulted from the negative tangent stiffness of the softening
63 part of behaviour, it generated an unstable equilibrium with associated numerical issues in
64 solving the stiffness equation. To surmount this, several methods were initially proposed to
65 control the load or the displacement (Crisfield 1996). Prominent amongst these were the arc
66 length method (Crisfield 1996; Riks 1979) and its variations, the minimum residual dis-
67 placement method (Chan 1988), and the line search method (Crisfield 1996). Yet challenges
68 remained. These non-linear solution methods required the specification of many control pa-
69 rameters, which depended upon user experience and did not guarantee convergence. Inherent
70 to this are expectations that the user is a highly knowledgeable and experienced practitioner
71 and that the results are obtained after many mesh and parameter refinement attempts. This is

72 particularly true for concrete, where the sudden release of strain energy due to tensile crack-
73 ing can cause the numerical solution to fail. As such, the aim of this paper was to implement
74 an alternative to non-linear FEM in its application to concrete members.

75

76 **Methodology**

77

78 The following paragraphs describe the background and details of the particular meshfree
79 method adopted for this analysis, as well as the sequentially linear analysis method that was
80 employed.

81

82 *Mesh free methods*

83 When the FEM was introduced in the 1950s, the most widely used numerical method for
84 solving differential equations was the finite difference method (Courant et al. 1967). This
85 strong-form method had a simple mathematical foundation and was easy to implement nu-
86 merically. The main previous limitation was the need for a regular grid of points to define the
87 analysis domain. These limitations added to the general acceptance of the FEM as a better
88 and more flexible alternative. Although further research related to finite difference overcame
89 the necessity of a regular grid (Liszka and Orkisz 1980), the FEM came to dominate popular
90 usage because of its ability to define complicated geometries, its basis on a robust mathemat-
91 ical foundation, and its ease in conducting error analyses (Thoméé 2001).

92

93 A fundamental alternative came in the form of meshfree methods. The first member of the
94 group was the Smoothed-Particle Hydrodynamics (SPH) (Gingold and Monaghan 1977; Lucy
95 1977) in 1977, which was initially applied in solving astrophysical problems. Since then,
96 multiple meshfree methods have been proposed (e.g. Li and Mulay 2013; Liu 2009). These

97 vary in their formulation procedure (strong, weak, weakened weak, or boundary integral) and
98 their local function approximations (moving least square, integral, differential, point, or parti-
99 tion of unity). Despite its name, most meshfree methods still require background cells to con-
100 duct the numerical integration of the system matrices. However, the meshfree methods that
101 are based on a strong formulation (e.g. the irregular finite difference method, the finite point
102 method, and local point collocation methods) do not usually require background cells. Unfor-
103 tunately, most of these methods suffer from reduced accuracy and instability due to node ir-
104 regularity (Atluri and Zhu 1998; Liu 2009).

105
106 In the work presented herein, a special meshfree Point Interpolation Method (PIM) called the
107 meshfree Node-based Smoothed Point Interpolation Method (meshfree NS-PIM) is used. The
108 method was first developed by Liu et al. (2005) under the name Linearly Conforming Point
109 Interpolation Method (LC-PIM). This was later changed to the Nodal Smoothing Operation
110 (2009) to distinguish it from the Edge-based Smoothed Point Interpolation Methods (ES-
111 PIM). Further details on NS-PIM are presented by Liu and Zhang (2013).

112
113 Meshfree NS-PIM was formulated using polynomial basis functions that have the Kronecker
114 delta function property, which allowed straightforward implementation of the essential
115 boundary conditions. Furthermore, the Generalized Smoothed Galerkin (GS-Galerkin) weak
116 form was used, which allowed use of incompatible assumed displacement functions. The
117 method is linearly conforming, with upper bound results that are free from volumetric locking
118 (Liu, 2009).

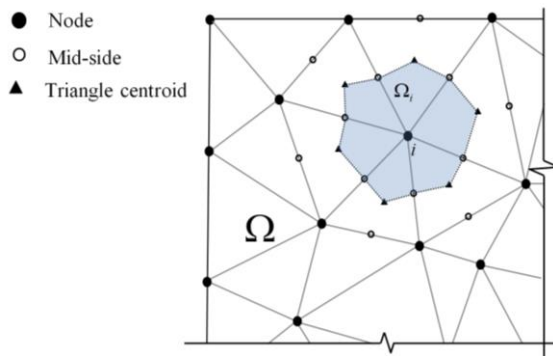
119
120 In NS-PIM, as in PIM, the displacement, u^h , of any domain point, x , is approximated using
121 a shape (interpolation) function, $\Phi_I(\mathbf{x})$. This function operates within a small local domain

122 around x (the support domain). The function interpolates the nodal displacement, u_I , of the
 123 nodes within the support domain of x (or the support nodes, S_n):

$$124 \quad u^h(\mathbf{x}) = \sum_{I \in S_n} \Phi_I(\mathbf{x}) u_I \quad (1)$$

125 The choice of the support domain size and location relative to x , explicitly influence the
 126 shape function's ability to interpolate accurately between nodal displacements. If the support
 127 domain has a poor arrangement of support nodes, inaccurate interpolation results will be pro-
 128 duced. Different schemes were introduced to resolve this issue. In the research presented
 129 herein, the T3-scheme (Liu 2009) was adopted, Fig. 1. In that scheme, the background cells
 130 that are required to conduct the meshfree weak form integration are used for support node se-
 131 lection. The cells can be generated by triangulating between the nodes. Any point inside a tri-
 132 angle is surrounded by three nodes. This allows the construction of a linear shape function
 133 that will result in a constant strain approximation. In this work Delaunay triangulation was
 134 used to generate the triangulated support domain used for the T3-scheme.

135



136

137 Fig. 1. T-3 scheme for background cells.

138

139 The triangulated problem domain (Ω) is divided into a number of smoothing domains (N_s),
 140 each centred on a node. As such, N_s equals the number of nodes. The boundaries of smooth-

141 ing domains do not overlap and have no gaps in between; hence they cover the whole do-
 142 main.

143
 144 The Generalized Smoothed Galerkin (GS-Galerkin) weak form, as shown in Eq. (2), can be
 145 converted to its discretized form, as shown in Eq. (3), where the domain integration is con-
 146 verted into a summation, thereby adding the effect over the smoothing domains (Liu 2009):

$$147 \int_{\Omega} \delta \bar{\boldsymbol{\varepsilon}}^T \mathbf{c} \bar{\boldsymbol{\varepsilon}} d\Omega - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma} \delta \mathbf{u}^T \mathbf{t} d\Gamma = 0 \quad (2)$$

$$148 \sum_{i=1}^{N_s} A_i^s (\delta \bar{\boldsymbol{\varepsilon}}_i)^T \mathbf{c} \bar{\boldsymbol{\varepsilon}}_i - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma} \delta \mathbf{u}^T \mathbf{t} d\Gamma = 0 \quad (3)$$

149

150 where:

151 $\bar{\boldsymbol{\varepsilon}}_i$: smoothed strain of domain i

152 A_i^s : area of smoothing domain i

153 \mathbf{c} : material property matrix

154 \mathbf{b} : body force vector

155 Ω : domain bounded by Γ

156 \mathbf{t} : boundary stress vector

157 N_s : number of smoothing domains

158 The smoothed strain $\bar{\boldsymbol{\varepsilon}}$ for node i , Fig. 1, can be found as the approximate strain $\tilde{\boldsymbol{\varepsilon}}(\mathbf{u}^h)$ aver-
 159 aged over the smoothing domain of the node, Ω_i^s , as follows:

$$160 \bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i) = \frac{1}{A_i^s} \int_{\Omega_i^s} \tilde{\boldsymbol{\varepsilon}}(\mathbf{u}^h) d\Omega \quad (4)$$

161

162 The assumed displacement $\mathbf{u}^h(\mathbf{x})$ can be used to find the smoothed strain matrix $\bar{\mathbf{B}}_I$, with its
 163 elements representing the smoothed shape function derivatives:

164
$$\bar{\mathbf{B}}_I(\mathbf{x}_i) = \begin{bmatrix} \bar{\phi}_{I,x}(x_i) & 0 \\ 0 & \bar{\phi}_{I,y}(x_i) \\ \bar{\phi}_{I,y}(x_i) & \bar{\phi}_{I,x}(x_i) \end{bmatrix} \quad (5)$$

165 This matrix can be used to find the approximate smoothed strain (Liu 2009):

166
$$\bar{\boldsymbol{\varepsilon}}^h(\mathbf{x}_i) = \sum_{I \in S_s} \bar{\mathbf{B}}_I(\mathbf{x}_i) \mathbf{u}_I \quad (6)$$

167 where:

168 S_s : support nodes of the smoothing domain that are used in the interpolation

169 The numeric integration required to calculate the smoothed strain matrix \bar{B}_I resulting from
 170 the linear shape function can be conducted using one integration point. The integration can be
 171 altered from an area to a line integration using Green's theorem (Thomas et al. 2004). This al-
 172 lows for a more efficient closed-form numerical implementation of the integration.

173 As in the FEM approach, the stiffness matrix is obtained from the strain matrix as follows:

174
$$\bar{\mathbf{K}} = \sum_{i=1}^{N_s} A_i^s \bar{\mathbf{B}}_i^T \mathbf{c} \bar{\mathbf{B}}_i \quad (7)$$

175 In the research presented herein, the background triangular cells are generated using a Delau-
 176 nay triangulation. Once the nodal stiffness matrices are calculated, the global stiffness matrix
 177 can be assembled. In meshfree NS-PIM, the boundary conditions can be applied explicitly, in
 178 a manner similar to the normal procedure in FEM. The overall solution steps are also similar
 179 to those applied in the FEM. In this research, meshfree NS-PIM was combined with sequen-
 180 tial linear analysis.

181

182 *Sequentially linear analysis*

183 Sequentially linear analysis (SLA) was first proposed by Rots (Rots 2001) with the aim of
 184 simplifying non-linear finite element analysis of concrete due to tension cracking. The local-
 185 ly brittle, snap-type response of many reinforced concrete structures inspired the idea to cap-

186 ture these events directly rather than trying to iterate around them with a Newton-Raphson
187 scheme. SLA was based on the finite element re-analyzing of the structure at each cycle from
188 an unloaded state, with the analysis following the secant modulus rather than the tangent. As
189 such, the numerical difficulties typically encountered in tangent non-linear analysis, particu-
190 larly in the softening part, were avoided, as the secant modulus is always positive.
191 Specifically, the non-linear analysis is substituted by a series of linear analyses, with the
192 structure at each cycle slightly modified from the previous cycle. At each cycle, the element
193 (or integration point) that is closest to cracking is identified. The following cycle analyses a
194 structure with a crack at the element (or integration point), as identified in the previous cycle.
195 The analysis is conducted according to the following steps (Rots and Invernizzi 2004; Rots et
196 al. 2008).

197

- 198 a. The structure is loaded and analyzed with a normalized unit load.
- 199 b. The critical element closest to cracking is identified.
- 200 c. The load and analysis results are scaled to produce a crack at the critical element.
- 201 d. The structure is modified by changing the properties of the cracked element.
- 202 e. The previous steps are repeated, until the desired damage level is obtained.

203

204 ***Cracking***

205 The final technical aspect relates to cracking. Cracking can be modeled discretely in both fi-
206 nite element and meshfree methods (Ngo, D. and Scordelis 1967; Rots and Blaauwendraad
207 1989), (ACI Report:446.3R-97 1997), where the crack tip stress and crack direction can be
208 identified. The ability to model geometrically actual separation produces a numerical model
209 that can accurately describe the actual cracking behavior. The main challenge (especially in
210 three-dimensional bodies) with the discrete crack method is the need to continuously update

211 the numerical model topology with the crack progression. Furthermore, a refined numerical
212 model is required, particularly around the crack tip. The cost of conducting a discrete crack-
213 ing analysis is large and, thus, mainly used for particularly detailed analysis of relatively
214 small structures.

215
216 An alternative is the smeared crack method first introduction in 1968 by Rashid (Rashid
217 1968). It is currently widely used in finite element crack analysis mainly due to its numerical
218 efficiency when compared to the discrete cracking method. This efficiency was achieved
219 through maintaining the same geometric model and assuming that the actual crack effect can
220 be distributed over the finite element width by changing the constitutive properties. This
221 smearing effect is just an approximation of the actual discrete crack. In this approximation,
222 some of the details are inevitably lost.

223
224 In addition to the above two methods, it is possible to use the extended FEM to model crack-
225 ing. In this method, special enriching functions are added to the finite element approximation
226 using the framework of partition of unity (Moës et al. 1999). In this approach, there is no
227 need to modify the model topology with the crack propagation, as strong discontinuities can
228 be modelled. However, the enrichment requires substantial numerical calculations that can
229 slow the analysis.

230
231 The approach proposed herein uses the smeared method to model cracking. For such a model,
232 it was previously observed that finite element results depend on element size (Bazant and
233 Cedolin 1979; Bazant and Oh 1983; Cedolin and Bazant 1980). To maintain mesh objectivity
234 and independence, Bazant and Cedolin (Bazant and Cedolin 1979),(Cedolin and Bazant

235 1980) proposed the concept of crack band width (h) to normalize the stress-strain curve with
236 the aim of maintaining constant fracture energy.

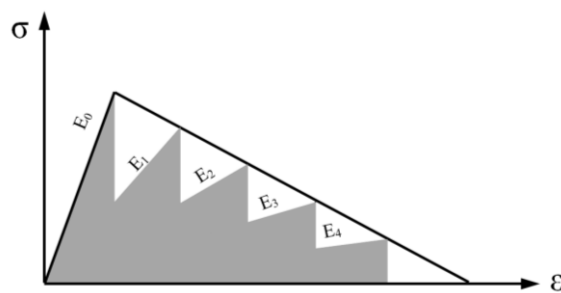
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238 Finally, two models exist to follow the crack development, the fixed and rotating crack mod-
239 els (Rots and Blaauwendraad 1989). In the first, it is assumed that the crack direction remains
240 the same after its initiation, while in the second; the crack is allowed to change its direction
241 with continuous change of stress state. In the approach proposed herein, the fixed crack mod-
242 el is employed.

243

244 *Saw-tooth approximation*

245 The application of SLA method to concrete is tightly linked to its tensile cracking. Many
246 modeling options are available. One approach is to treat concrete as an ideal brittle material
247 where the secant modulus of elasticity is instantly reduced to zero upon cracking. The results
248 based of such a model will likely be mesh-dependent, as the crack fracture energy will not
249 converge to the correct value upon mesh refinement (Bazant and Oh 1983). Alternatively, a
250 gradual reduction of the secant stiffness in the softening part of the stress-strain curve will
251 produce the saw-tooth approximation that is typically used in the SLA, Fig. 2.



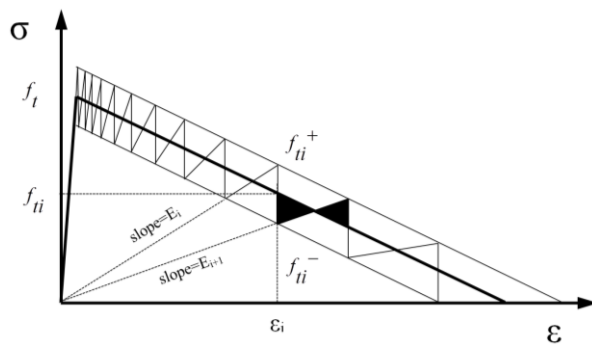
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253

Fig. 2. Typical SLA curve.

254

255 The basic stress-strain curve can be modified to maintain the same fracture energy, which is
 256 related to the area under the softening stress-strain curve. The shaded area under the saw-
 257 tooth diagram is smaller than the total area under the stress-strain curve. To maintain constant
 258 fracture energy irrespective of the tooth count, the saw-tooth diagram needs to be adjusted.
 259 To achieve this, different regularized curves have been proposed (Rots and Invernizzi 2004;
 260 Rots et al. 2008). The most elegant was called Model C (Rots et al. 2008), which is based on
 261 a linear softening behavior and obtained by modifying both the tensile strength and ultimate
 262 tensile strain. The actual softening stress-strain curve is considered to represent the base val-
 263 ue. Modified saw-tooth stresses are allowed to fluctuate around the base value within a spe-
 264 cific band. For each tensile strength (f_{ii}), there is a larger value (f_{ii}^+) defining the maximum
 265 fluctuation limit and a smaller value (f_{ii}^-) defining the minimum fluctuation limit, Fig. 3.
 266 The resulting softening part can be generated as a series of secant lines, each with a progres-
 267 sively reduced tensile strength and slope and with a progressively increased maximum strain.



268
 269 Fig. 3. Saw-tooth model C.
 270

271 To regularize the softening behavior against the mesh size, the crack band width normaliza-
 272 tion concept was introduced by Bazant and Cedolin (Bazant and Cedolin 1979). Following
 273 that concept, Rots et al. (Rots and Invernizzi 2004; Rots et al. 2008) proposed using Eq. 8 to

274 regularize the ultimate strain (ε_u) by relating it to the fracture energy (G_f), tensile strength
275 (f_t), and crack band width (h):

$$276 \quad \varepsilon_u = 2(G_f / h) / f_t \quad (8)$$

277 The saw-tooth curve was constructed such that the value of f_{ii}^- for the last saw-tooth was
278 equal to zero. This condition was set to maintain (in all practically) equal positive and nega-
279 tive areas above and below the actual stress-strain curve, to ensure constant total fracture en-
280 ergy.

281
282 In FEM, the value of the crack band width is related to the element size, element type, num-
283 ber of element integration points, and crack direction. The crack band width (h) can be ex-
284 pressed as:

$$285 \quad h = h_{fac} \cdot b \quad (9)$$

286 where

287 h_{fac} : crack band width factor

288 b : element size

289 For simple plane strain triangles of regular uniform shape, where the cracks are parallel to the
290 element side, it was found that the crack band width factor equals to 1.0 and that the element
291 size is the element side length (Rots 1988).

292
293 The meshfree NS-PIM formulation implemented herein uses a linear shape function, as well
294 as equal, regular, nodal spacing in both directions, with the cracks oriented mainly parallel to
295 the nodal grid. For these conditions, it was found in the research herein that the equivalent
296 element size is the nodal spacing and that a crack band width factor of 2.0 can produce the
297 best overall results. This crack band width factor value can be related to the formulation of

Comment [1]: Since this relates to FEM do we need it. If so, please add a sentence relating it to the NS-PIM

320

321

Fig. 4. Graphical user interface of PISLA.

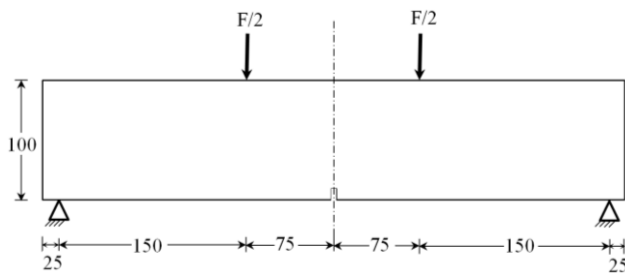
322

323 To test the application of a meshfree method on SLA, a test beam model was used. The mod-
324 el was previously used by Rots et al. (Rots and Invernizzi 2004; Rots et al. 2008) to investi-
325 gate the implementation of SLA with a finite element approach. The symmetric concrete
326 beam was 500 mm long, with a 450 mm span, a 100 mm height, a 50 mm width, and a mid-
327 span notch depth of 10 mm. Load was applied at the third-points of the free 450 mm span
328 (Fig. 5). The maximum, constant, bending moment was generated within the beam's middle
329 third.

330

331 The adopted material properties in this analysis were those used by Rots et al. (Rots and In-
332 vernizzi 2004; Rots et al. 2008): initial modulus of elasticity 38 GPa, initial tensile strength 3
333 MPa, and fracture energy 0.06 N mm/mm^2 . The beam was modeled in its entirety, without
334 any attempt to exploit its symmetry. Sensitivity of the saw-tooth model was tested by varying
335 the number of teeth: 5, 10, 20, and 40; mesh sensitivity was tested by changing the nodal
336 spacing: 20 mm, 10 mm, 5 mm, and 3.33 mm (Fig. 6); and the numerical models were re-
337 fined mainly around the notched section.

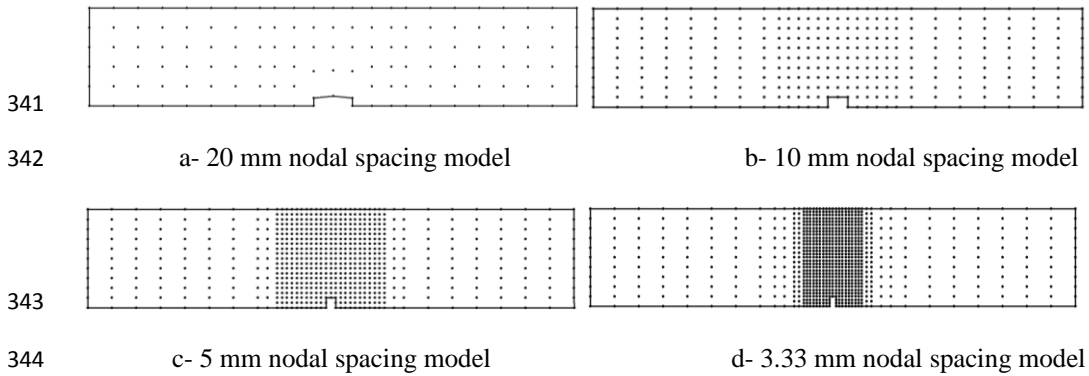
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339

Fig. 5. Experimental notched concrete beam (23, 24).

340



345 Fig. 6. The four numerical models analyzed.

346

347 **Results**

348

349 In the following sections, the analysis results from the different models are presented.

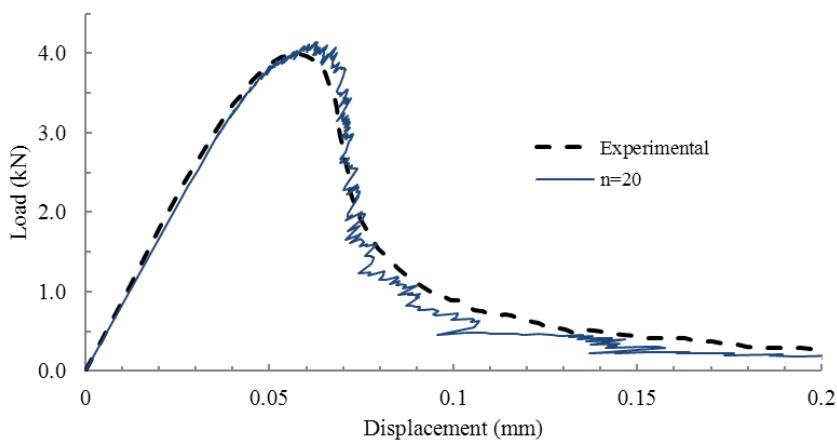
350

351 **Base model**

352 The base model had a nodal spacing of 10 mm and a saw-tooth with 20 teeth. The load-

353 deflection curve of this model was constructed by linking the points resulting from the SLA

354 (Fig. 7).



355

356 Fig. 7. Load-deflection results for the 10 mm nodal spacing model, 20 teeth.

357

358 The numerical results correlated well with the experimental behavior. The maximum numeri-
359 cal load was 4.14 kN, only 3.5% more than the maximum experimental value of 4.0 kN. The
360 results were also found to be close to the results previously obtained using finite elements
361 SLA with different saw-tooth curves and mesh densities (Rots and Invernizzi 2004; Rots et
362 al. 2008).

363 The load-deflection curve resulting from current numerical method showed irregular behav-
364 ior, particularly in the softening part. This behavior is associated with SLA method where
365 damage is traced sequentially as it progresses through the structure. The irregular load-
366 deflection behavior can be seen as a global reflection of damage represented on the local lev-
367 el by the saw-tooth approximation (Rots and Invernizzi 2004).

368

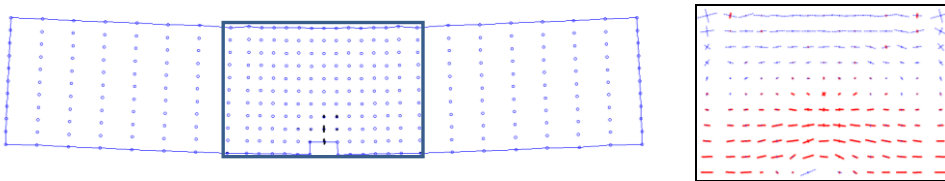
369 The presence of the notch resulted in cracking being limited to the area around the notch. The
370 cracking progression, exaggerated deflection, and stress distribution in the central part of the
371 beam are shown in Fig. 8. These results are shown at three stages of loading: 3.89 kN, 2.57
372 kN, and 0.27 kN, corresponding to cycle numbers 41, 195, and 341, respectively. All of these
373 stages are in the softening part of behavior. In Fig. 8, the crack length is related to the saw-
374 tooth number at the particular stage of cracking. The stress distribution is shown for the mid-
375 dle part of the beam, where the bending moment is constant.

376

377 At an early cracking stage, the cracks were nearly vertical and the stress distribution was
378 nearly symmetric (Fig. 8-a). Early cracks appeared at the weakest section, corresponding to
379 the notch location. With the cracking progression, the symmetry in cracking and stress distri-
380 bution was lost. Theoretically symmetric results would have been expected. However, the
381 presences of slight numerical approximations in the double precision calculations usually re-

382 result in the loss of exact symmetry. There was no need to vary the section or material proper-
383 ties from one beam location to the other to break the symmetry, as done previously by some
384 other researchers. The actual behavior of the physically tested model was similar due to slight
385 imperfections in material properties, dimensions, support conditions, and/or loading posi-
386 tions.

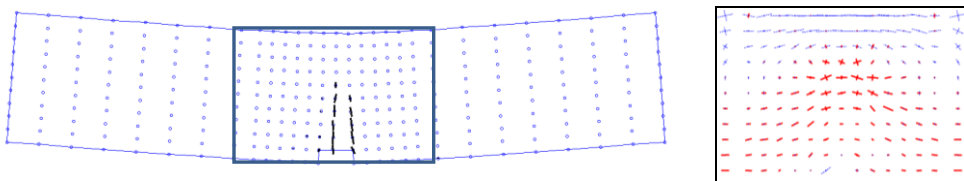
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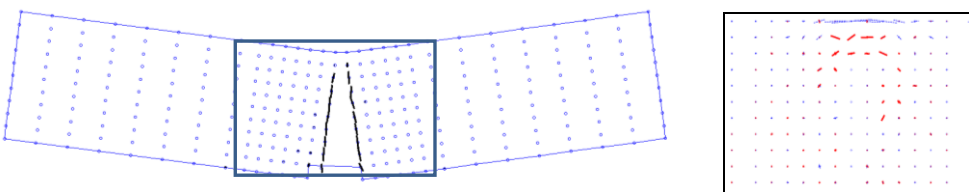
389 a- Crack pattern, exaggerated deflected shape, and principal stresses near mid-span at cycle
390 41 (load = 3.89 kN)

391



392 b- Crack pattern, exaggerated deflected shape, and principal stresses near mid-span at cycle
393 195 (load = 2.57 kN)

394



395 c- Crack pattern, exaggerated deflected shape, and principal stresses near mid-span at cycle
396 341 (load = 0.27 kN)

397

398 Fig. 8. Graphical outputs of the results of the 10 mm nodal spacing model with 20 teeth.

399

400 Fig. 8 shows clearly that the stress across the cracks reduced as the crack lengths increased.

401 This behavior is related to the saw-tooth diagram, where f_{ii}^+ reduced with the saw-tooth

402 number of the crack, i as shown in Fig. 3.

403

404 The following sections present the results obtained from studying the effects of number of

405 teeth in the saw-tooth model, nodal spacing, and node-distortion.

406

407 *Effect of tooth count*

408 Figure 9 shows the base model with 10 mm nodal spacing, the results for 5, 10, 20, and 40

409 teeth, where n indicates the number of teeth. The results were obtained for a crack band

410 width factor of 2.0. Increasing the number of teeth improved the quality of results. The load-

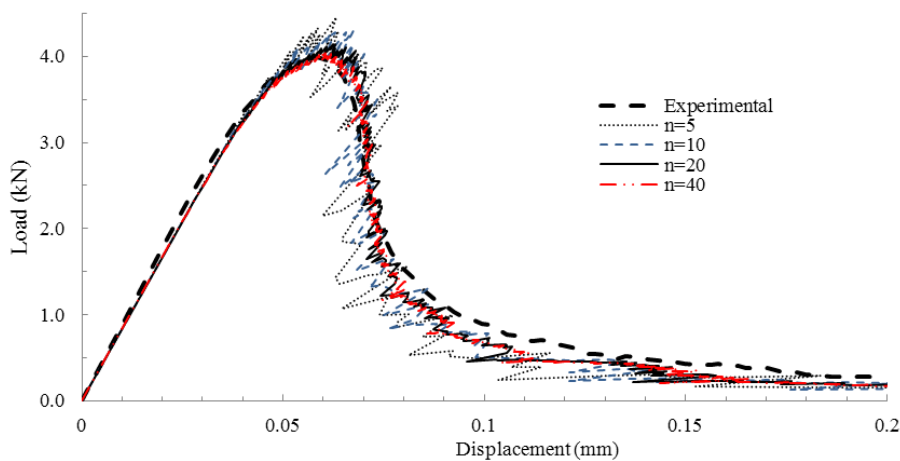
411 deflection curve became smoother and with less oscillation amplitude, as the number of teeth

412 was increased. The overall average curve location remained stable, indicating similar fracture

413 energy release. This behavior was expected and results from the more gradual release of frac-

414 ture energy, as the number of teeth increases. However, this more refined behavior comes

415 with a penalty in analysis time proportional to number of teeth.



416

417 Fig. 9. Effect of number of teeth on the load-deflection results for the 10 mm nodal spacing.

418

419 The numerical peak load ranged from a high of 4.45 kN with 5 teeth to a low of 4.06 kN for
420 40 teeth, respectively (Fig. 9). This value decreased with an increase of teeth count, although
421 at a reducing rate. This behavior reflects the saw-tooth model, in which the stress value oscil-
422 lates above and below the base curve (Fig. 3). The oscillation amplitude reduces with tooth
423 count. In this case, the model with 40 teeth was closest to the experimental value of 4.0 kN.

424

425 *Effect of model refinement*

426 Four models with nodal spacing of 20 mm, 10 mm, 5 mm, and 3.33 mm were tested (Fig. 6).

427 The results obtained from these models for a saw-tooth model with 20 teeth are shown in Fig.

428 10. The results were obtained for a crack band width factor of 2.0. The meshfree method is

429 known to produce results that are more flexible than the actual structure (Liu 2009). This

430 trend was also observed in the current analysis. The initial slope of the numerical load-

431 deflection curve was less than the experimental results. The numerical results converged to

432 the test results, as the model was refined further. The very coarse 20 mm nodal spacing model

433 showed rough and flexible behavior. There was, however, an overall similarity with the ex-

434 perimental behavior. Results of the other numerical models were closer to the experimental

435 results. The peak load was predicted accurately by the 10 mm nodal spacing model, with a

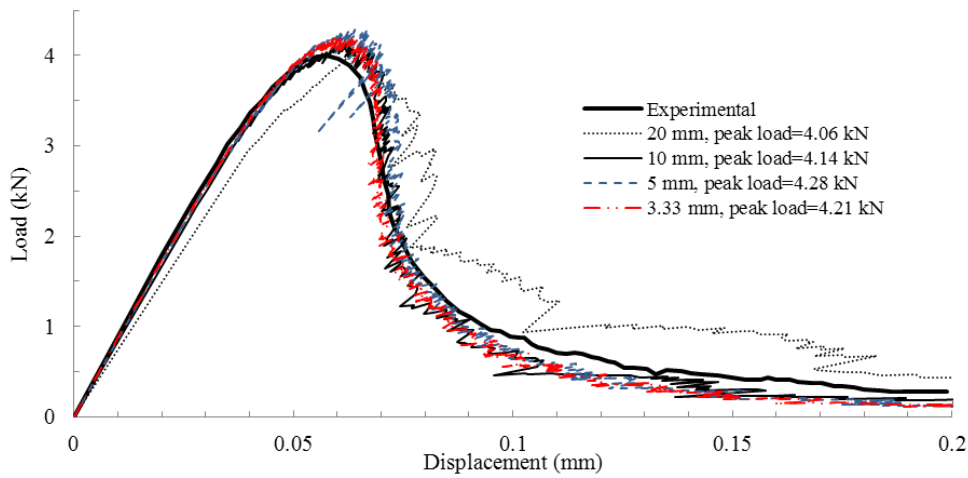
436 difference of 3.5% from the experimental results. The more refined 5 mm and 3.33 mm mod-

437 els predicted slightly higher peak values. Thus, the predicted behavior was still dependent, to

438 a small degree, on nodal spacing. The maximum load resulting from the four nodal spacing

439 was as little as 4.06 kN for the 20 mm nodal spacing to as much as 4.21 kN for the 3.33 mm

440 nodal spacing.



441

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Fig. 10. Results of mesh refinement study, all results for 20 teeth.

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This behavior relates to the tension cracking of concrete and the implications of using a smeared cracking method to model the actual cracks. In fracture mechanics, the energy released during the softening behavior per unit volume (or area) of concrete material is the fracture energy (G_f). In a meshfree (as well as finite element) analysis, the smeared crack is assumed to cover all of the smoothing domain (or element) regardless of the nodal spacing. Larger nodal spacing will result in the nodes (or elements) having larger fracture energy. To maintain constant value of fracture energy, regularization is used (Bazant and Cedolin 1979; Bazant and Oh 1983). One assumption of regularization is that the element can contain the material zone affected by the cracks, usually referred to as the FPZ (as described in section 1). The regularization scheme should work for any nodal spacing, as long as the spacing is larger than the FPZ width. One consequence of using a nodal spacing smaller than the FPZ width is that each node (or element) contained in the larger FPZ can dissipate the fracture energy. This numerical model will then overestimate the strength due to its ability to handle more fracture energy. This would explain the current results, with a slight anomaly for the 5 mm nodal spacing relative to those obtained from the 10 mm and 3.33 mm nodal spacing.

459 One possible reason is the fluctuating nature of the SLA analysis. Therefore, a single peak
 460 value is possibly not the best response measure. An average of many points around the peak
 461 result might be more appropriate. Overall, however, the softening behavior of the numerical
 462 models showed reasonable agreement with the experimental results. The generally similar
 463 overall behavior resulting from the four numerical models also indicates the accurate nodal
 464 stress values resulting from the meshfree NS-PIM analysis. This is one of the advantages of
 465 NS-PIM over the FEM approach with triangular elements (Liu 2009).

466
 467 The analysis details of the four, studied, nodal spacings are shown in Table 1. The details in-
 468 cluding number of cycles, analysis time per cycle, total analysis time, and total analysis time
 469 expressed as a ratio relative to the total analysis time of the base model. From these results, it
 470 is clear that the analysis time of the most refined model, with 3.33 mm nodal spacing, was
 471 more than 41 times that required for the base model. Although more detailed results can be
 472 obtained from more refined models, the time penalty was disproportionately high. In this anal-
 473 ysis, the 10 mm nodal spacing seems to provide a reasonable compromise between running
 474 time and level of result details. Notably the halving of the nodal spacing increases the total
 475 analysis time by more than an order of magnitude.

476

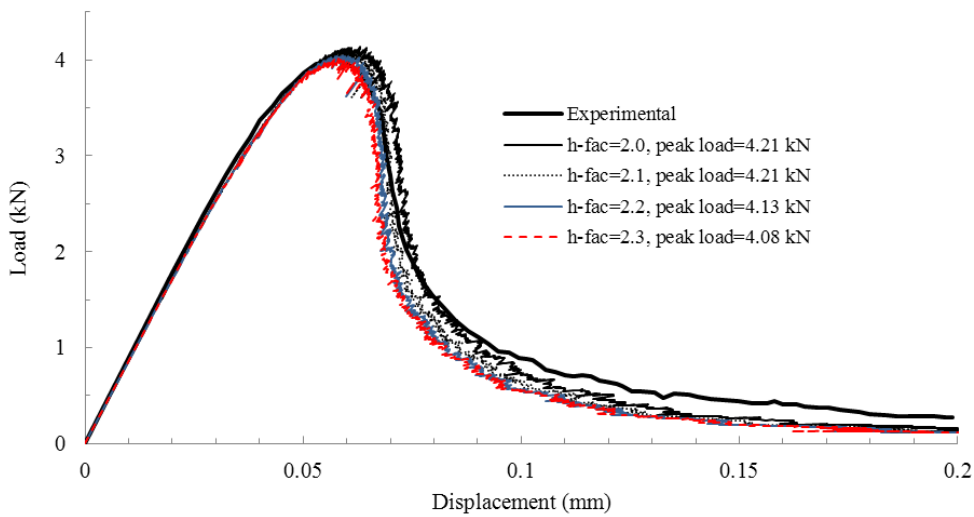
477 Table 1. Analysis details for different nodal spacing

Nodal spac- ing (mm)	No. of nodes	No. of analy- sis cycles	Analysis time/cycle (sec)	Total analysis time (sec)	Relative total analysis time
20	137	157	0.17	27	0.08
10	318	363	0.93	338	1 (reference)
5	679	823	4.81	3959	11.7
3.33	890	1213	11.55	14019	41.5

478

479 *Effect of crack band width factor*

480 To study the effect of changing the crack band width factor, the 3.33 mm nodal spacing mod-
481 el was used. The results obtained from various crack band width factors and 20 teeth saw-
482 tooth model are shown in Fig. 11. The peak load values were as much as 4.21 kN and de-
483 creased to 4.08 kN, as the crack band width factors went from 2.0 to 2.3. There was a slight
484 tendency of peak load reduction, as the crack band width factor increased. The general shape
485 of the load-deflection curve was affected also, progressively dropping below the softening
486 part of the experimental curve, as the factor value increased.



487

488 Fig. 11. Results of crack band width factor study.

489

490 For the specific problem presented in this research, with a nodal spacing of 10 mm, a crack
491 band width factor of 2.0 worked well. For smaller nodal spacing, it is still possible to use that
492 factor; however, the peak load will be overestimated slightly.

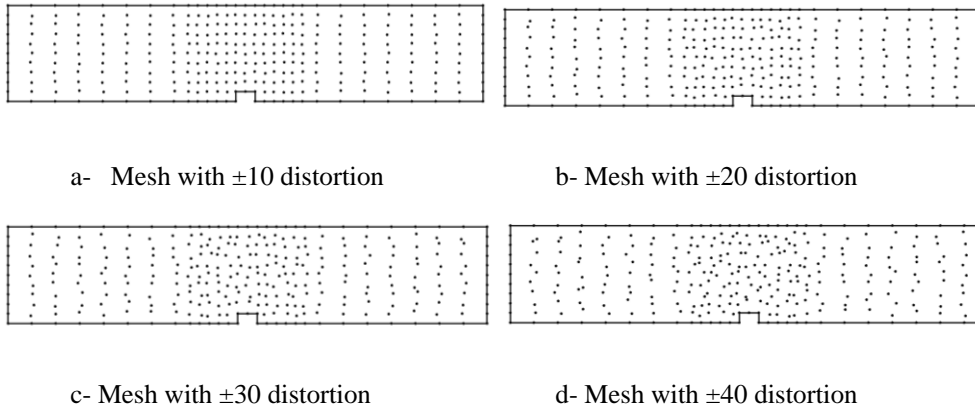
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494

495 ***Sensitivity to nodal distortion***

496 Finite element automatic mesh generators often produce irregular meshes. These can be dis-
497 torted, particularly for complex geometries. In FEM, output accuracy is sensitive to element
498 distortion. However, the meshfree method is known to be less sensitive to nodal distortion,
499 (Liu 2009). To study the effect of nodal distortion on the accuracy of the results, the base
500 model with 10 mm nodal spacing and 20 teeth was randomly distorted by $\pm 10\%$, $\pm 20\%$,
501 $\pm 30\%$, and $\pm 40\%$, as shown in Fig. 12. The distortion was introduced by adding a random
502 value within the appropriate range to the nodal coordinates. The random value was calculated
503 by multiplying a pseudo random number in the range $[-1, 1]$ by the nodal spacing and the
504 specific distortion percentage.

505

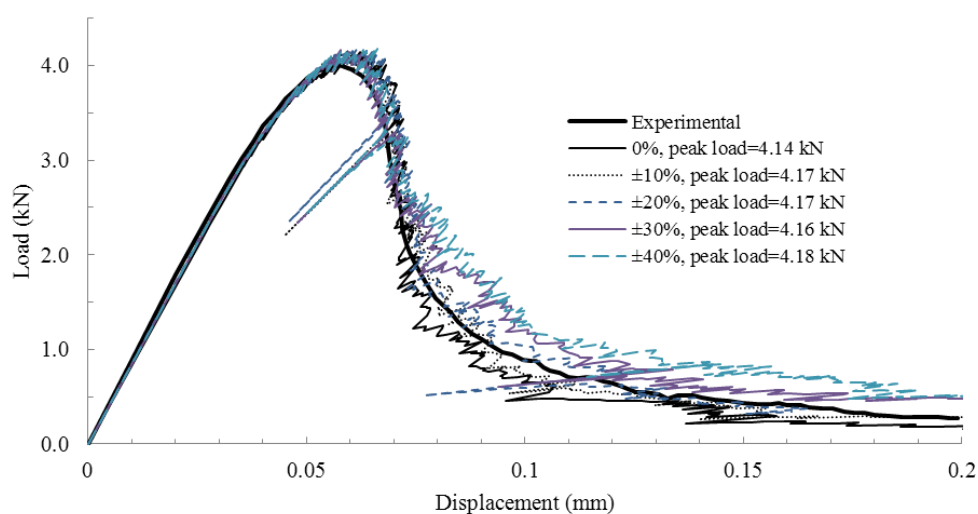


510 Fig. 12. Distorted models.

511

512 The initial stiffness for all the numerical models was similar regardless of distortion, thereby
513 proving that the meshfree NS-PIM method is quite robust and insensitive to distortion. The
514 peak loads ranged from 4.14 kN at no nodal distortion to 4.18 kN for a nodal distortion of
515 $\pm 40\%$. Furthermore, as shown in Figure 13, the overall shape of the numerical results was
516 quite close to the experimental ones for distortion values up to $\pm 20\%$. There was a trend of

517 increased strength in the descending part of the curve with increased distortion. As the initial
518 stiffness was unaffected by distortion, it is reasonable to relate the changed softening behav-
519 ior to the effect of distortion on the crack band width, rather than the mesh free analysis.
520 There are indications from the results that the peak load and the overall behavior are insensi-
521 tive to nodal distortions. However, further work is required to establish solid conclusions in
522 this regard.



523

524

Fig. 13. Distorted model results.

525

526 Discussion

527

528 Since SLA is not currently supported in commercial FEM programs. So a direct comparison
529 would be difficult to undertake. However, the general accuracy of results obtained from
530 meshfree methods, was demonstrated by the current results. The method was quite stable
531 within the range of the parameters studied. No numerical instability was encountered at any
532 analysis stage.

533 One of the main issues of non-linear FEM, as well as finite element-based SLA, is the long
534 analysis time. In both FEM and meshfree methods, output accuracy is related in part to the to-
535 tal number of degrees of freedom, hence the stiffness matrix size. As a comparison, the
536 meshfree NS-PIM using the T3 scheme can produce more accurate stress results when com-
537 pared with FEM using triangular elements (Liu 2009). The effect is directly reflected in SLA
538 with its stress-based softening behavior. The other factor affecting SLA solution time is the
539 total number of cycles. This number is the summation over all cracked nodes (integration
540 points) of the last tooth number. Thus, there is an obvious advantage in reducing the total
541 number of nodes to a level not affecting stress accuracy, as that will directly reduce the num-
542 ber of SLA analysis cycles. In this respect, meshfree NS-PIM can provide an advantage over
543 comparable FEM approaches, however the exact correlations and comparisons need to be the
544 subject of future studies.

545 **Conclusions**

546
547 Meshfree NS-PIM method was applied in the SLA analysis of cracking concrete. The method
548 was used to analyze a notched concrete beam that was previously studied by Rots et al. (Rots
549 and Invernizzi 2004; Rots et al. 2008) using finite element-based SLA. The meshfree SLA
550 managed to produce numeric results that were close to experimental ones, namely the peak
551 load and overall load-deflection behavior (including the softening part). The base model with
552 10 mm nodal spacing and 20 teeth predicted a peak load only 3.5% more than the experi-
553 mental value, and increasing the number of teeth in the saw-tooth model produced a load-
554 deflection curve that was more even and with less oscillation amplitude. The overall average
555 location of the curves remained stable. However, there was an increase in the analysis time
556 linearly related to the number of teeth in the saw-tooth mode.

557

558 What was also found was that in spite of maintaining equal fracture energy for the different
559 numerical models with different nodal spacing, the results still depended slightly on nodal
560 spacing. The normal crack band width factor value used in the analysis was 2.0. To maintain
561 the same peak load output from the different models, the crack band width factor needs to be
562 increased slightly with more refined models. More research is required to study this behavior.

563

564 Additionally, the numerical results were shown to be relatively insensitive to nodal spacing
565 and model size. By reducing the nodal spacing by a factor of 6 from 20 mm to 3.33 mm, the
566 peak load changed by less than 3.7%. This is an indication of the generally accurate stress re-
567 sults obtained from meshfree methods and the possibility of using smaller number of nodes to
568 model large structures while still obtaining accurate results at the benefit of reduced analysis
569 time. Furthermore, since the meshfree approach generates models that are less stiff than the
570 actual structures, the displacements are not underestimated. Finally, and perhaps most im-
571 portantly the predicted peak load was found to be insensitive to nodal distortions up to $\pm 40\%$,
572 and the overall behaviour insensitive to distortions of up to $\pm 20\%$. Thus, there are two areas
573 where a meshfree approach may hold strategic advantages over a FEM. The first is in the in-
574 vestigation of large deformation (in concrete, as well as other materials). The second is in the
575 auto-generation of meshes from remote sensing data (e.g. laser scanning, photogrammetry).
576 In such a case, the external geometry of an existing structure can be captured and transformed
577 directly into a solid model. This could be of tremendous value in the assessment of older
578 metal bridges and will be the subject of further study by the authors.

579

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581

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