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ON THE MIXED CAUCHY PROBLEM WITH DATA ON SINGULAR CONICS

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1. Introduction

In this paper, we shall consider the mixed Cauchy problem for holomorphic partial differential operators of the type

$$Lu = Q_k\left(D\right)u + \sum_{|\alpha|\leq k_0} a_\alpha\left(z\right)D^\alpha u \tag{1}$$

where $Q_k(D)$ is a non-trivial homogeneous, constant coefficient partial differential operator of order k, the $a_\alpha(z)$ are holomorphic functions in a domain $\Omega \subset \mathbb{C}^n$ containing 0, and k_0 is a natural number $< k$. We use standard multi-index notation with $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and D^α denoting the differential operator

$$D^\alpha := \frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}...\partial z_n^{\alpha_n}}.$$

The principal symbol of the partial differential operator L in (1) is the homogeneous polynomial $Q_k\left(\zeta\right)$ of degree k.

Let $\mathcal{O}(U)$ denote the space of holomorphic functions in $U \subset \mathbb{C}^n$. Clearly, L defines a continuous linear operator $L\colon \mathcal{O}(U) \to \mathcal{O}(U)$ for every $U \subset \Omega$. In general, this linear operator is not injective. Indeed, if L has, e.g., constant coefficients, then it is well known (and easy to prove using the idea of Fischer duality as in [31]; see also [14]) that there are non-trivial entire solutions of $Lu = 0$ as long as $k \geq 1$ and, hence, $L\colon \mathcal{O}(U) \to \mathcal{O}(U)$ is never injective in this case. The surjectivity of $L\colon \mathcal{O}(U) \to \mathcal{O}(U)$ is more subtle, even when L has constant coefficients, and depends on the geometry of the domain U. The reader is referred to [20] for further information about this question (see also [3]). However, it is an immediate consequence of the classical Cauchy-Kowalevsky theorem that, for any domain $0 \in U \subset \Omega$, there exists a subdomain $0 \in U' \subset U$ such that the equation $Lu = f$, for $f \in \mathcal{O}(U)$, has solutions $u \in \mathcal{O}(U')$.

The *mixed Cauchy problem for L (at* 0*)* that we shall consider in this paper consists of finding large classes of irreducible algebraic hypersurfaces (i.e. codimension one algebraic

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subvarieties of $\mathbb{C}^n$)

$$\Gamma_1, \dots, \Gamma_p$$

containing $0 \in \mathbb{C}^n$, and multiplicities $\mu_1, \dots, \mu_p$ such that, for every domain $0 \in U \subset \Omega$, there exists a subdomain $0 \in U' \subset U$ with the property that the boundary value problem

$$\begin{cases} Lu = f \\ (D^\beta(u-g))|_{\Gamma_j} = 0, \quad j = 1, \dots, p,\ 0 \leq |\beta| < \mu_j \end{cases} \tag{2}$$

has a unique solution $u \in \mathcal{O}(U')$ for every $f, g \in \mathcal{O}(U)$. In this case, we shall say that the mixed Cauchy problem (2) is *well posed.* The classical Cauchy-Kowalevsky theorem corresponds to the case of a hyperplane

$$\Gamma = \Gamma_1 := \{(z_1, ..., z_n) \in \mathbb{C}^n \colon z_1 = 0\}$$

(so $p = 1$ and the multiplicity is k) and $Q_k(D) = D^{(k,0,\dots,0)}$, see e.g. [28], p. 15. By an analytic change of variables the Cauchy-Kowalewsky theorem can be generalized to the case of initial conditions on a hypersurface

$$\Gamma = \Gamma_1 := \{z \colon R(z) = 0\}$$

which is non-singular at 0 (i.e. the conormal vector $\zeta := (\partial R/\partial z)(0)$ is not 0), and non-characteristic with respect to L at 0 (i.e. $Q_k(\zeta) \neq 0$), see [28], p. 22. So in this case, the Cauchy problem is well posed. However, we shall be interested in the more difficult case of *singular* hypersurfaces which is not covered by the Cauchy-Kowalevsky theorem. Our methods of proof depend on arguments using homogeneous power series in combination with new decompositions of homogeneous polynomials, known as Fischer decompositions; for more details we refer the reader to Section 5.

Before stating our main results and discussing previous results along these lines, let us remark that it suffices to consider only the mixed Cauchy problem with nulldata, i.e.

$$\begin{cases} Lu = f \\ (D^\beta u)|_{\Gamma_j} = 0, \quad j = 1, \dots, p,\ 0 \leq |\beta| < \mu_j. \end{cases} \tag{3}$$

since the equation $Lu = f$ has a solution in $\mathcal{O}(U')$ for every $f \in \mathcal{O}(U)$. For the remainder of this paper, we shall consider only the problem (3) with nulldata.

In what follows, we shall give some equivalent reformulations of the mixed Cauchy problem that will be more convenient from a technical point of view. Since each Γ_j is an irreducible algebraic hypersurface in $\mathbb{C}^n$ with $0 \in \Gamma_j$, there is an irreducible polynomial $R_j(z)$, uniquely determined up to a multiplicative constant and with $R_j(0) = 0$, such that $\Gamma_j := \{z \colon R_j(z) = 0\}$. The condition that

$$D^\beta u|_{\Gamma_j} = 0, \quad 0 \leq |\beta| < \mu_j,$$

for $u \in \mathcal{O}(\Omega')$ is equivalent to $R_j^{\mu_j}$ dividing u in the ring $\mathcal{O}(\Omega')$, henceforth denoted by $R_j^{\mu_j}|u$. Thus, if we set $P := R_1^{\mu_1} \dots R_p^{\mu_p}$, then the mixed Cauchy problem (3) can be equivalently formulated as follows,

$$\begin{cases} Lu = f \\ P|u. \end{cases} \tag{4}$$

We shall refer to the polynomial P in (4) as the *divisor* in the mixed Cauchy problem. Now, given $f \in \mathcal{O}(U)$, a function $u \in \mathcal{O}(U')$ is a solution to (3) or, equivalently, to (4) if and only if $u = Pq$, for some $q \in \mathcal{O}(U')$, and $L(Pq) = f$ in U'. In particular, *the mixed Cauchy problem for the operator L and divisor P is well posed if and only if, for every domain $0 \in U \subset \Omega$, there is a subdomain $0 \in U' \subset U$ such that there exists a unique solution $q \in \mathcal{O}(U')$ to the equation*

$$L(Pq) = f, \tag{5}$$

for every $f \in \mathcal{O}(U)$. We shall use this formulation of the mixed Cauchy problem in our main results, Theorem 2, Theorem 3 and Theorem 17.

Previous results on the mixed Cauchy problem (at 0) includes a theorem by Hörmander ([19], Theorem 9.4.2) in the case where the divisor $P(z)$ is a monomial z^γ of degree $|\gamma| = k$, $Q_k(D) = D^\gamma$, and certain sufficiently small perturbations are allowed even in the principal part of L (i.e. k_0 in (1) is allowed to be k, but the coefficient $a_\gamma(z)$ must be identically 0 and there is a "smallness" requirement for those coefficients $a_\alpha(z)$ for which $|\alpha| = k$). An early version of this theorem in two dimensions goes back to Goursat (see e.g. [31]). Another, more recent result is due to the first author, jointly with H. S. Shapiro ([14], Theorem 3.1.1): There exists a number $k_0 < k$ depending on Q_k such that the mixed Cauchy problem with divisor $P(z) = Q_k^*(z) = \overline{Q_k(\bar{z})}$ has a unique solution $u \in U'$ for every $f \in \mathcal{O}(U)$.

In this paper, we shall prove a result (see Theorem 2) for the mixed Cauchy problem for differential operators L of the type $Q_k(\zeta) = (B(\zeta))^m$ where $k = 2m$ and $B(\zeta)$ is a non-degenerate quadratic form. For the divisor $P(z)$ we only require that it is a homogeneous polynomial of degree $2m$ that is $B(\zeta)$-elliptic (see below for the definition). This result does not contain, nor is it contained in the results from [14] mentioned above. The results in [14] allow a more general class of principal symbols $Q_k(\zeta)$, but, on the other hand, for each $Q_k(\zeta)$ there is only one divisor $P(z)$ that can be used in the mixed Cauchy problem, namely $Q_k^*(z)$. The result in the present paper treats a smaller class of principal symbols, but for each such principal symbol $Q_k(\zeta)$ there is a large class of $P(z)$ that may be used as a divisor. We also give a more precise result in $\mathbb{R}^n$ for operators with the iterated Laplacian as their principal symbol. The additional precision in this theorem concerns the relation between U' and U (see Theorem 3).

The paper is organized as follows. The main results are stated in Section 2 and it is explained how Theorem 2 follows from Theorem 3. Section 3 discusses two examples as illustrations of our main results. The section after that introduces an integral that will be used throughout the paper. The Fischer norms are then introduced and some basic estimates are proved in Section 5. The next section contains further estimates and, in particular, the key estimate (Theorem 13) needed to prove Theorem 3. In the last section, Section 7, we state and prove a general result about mixed Cauchy problems in $\mathbb{R}^n$ (Theorem 17), which together with the estimate in Theorem 13 proves Theorem 3.

2. Main results

In order to state our first main result, we need the following definition. Let $B(\zeta)$ be a nondegenerate quadratic form in $\mathbb{C}^n$, i.e. $B(\zeta) = \zeta^t B\zeta$ for some invertible, symmetric $n\times n$ matrix with complex coefficients. By standard linear algebra, there exists an invertible $n \times n$ matrix A such that $B(A\tau)$ is equal to the standard nondegenerate quadratic form $\Sigma(\tau)$,

$$\Sigma(\tau) := \sum_{j=1}^{n} \tau_j^2. \tag{6}$$

Let now $P_{2p}(z)$ be a homogeneous polynomial of degree $k := 2p \geq 2$. We shall say that P_{2p} is *B-elliptic* if, for *some* invertible $n \times n$ matrix A such that $B(A\tau) = \Sigma(\tau)$, the polynomial $P_{2p}(A^{-t}x)$ is real-valued for $x \in \mathbb{R}^n$ and there is a constant $\delta > 0$ such that

$$P_{2p}(A^{-t}x) \geq \delta(B(Ax))^p = \delta|x|^{2p}, \quad x \in \mathbb{R}^n. \tag{7}$$

Here A^{-t} is the transpose of the inverse matrix A^{-1}. For instance, if $B(\zeta) = \Sigma(\zeta)$ and $P_{2p}(x)$ is *elliptic* in the usual sense, i.e. $P_{2p}(x)$ is real and satisfies $P_{2p}(x) \geq \delta|x|^{2p}$ for $x \in \mathbb{R}^n$, then of course P_{2p} is B-elliptic. However, we point out that P_{2p} can be Σ-elliptic, even if $P_{2p}(x)$ fails to be elliptic, as is illustrated by the following example.

Example 1. Let $\xi \in \mathbb{R}$ and consider the following homogeneous polynomial of degree 4,

$$\begin{aligned} P(z) = P_4(z) := (\xi^4 + (1+\xi^2)^2)z_1^4 + (\xi^4 + (1+\xi^2)^2)z_2^4 - 12\xi^2(1+\xi^2)z_1^2z_2^2 \\ + 4i\xi\sqrt{1+\xi^2}(1+2\xi^2)(z_1z_2^3 + z_1^3z_2). \end{aligned} \tag{8}$$

The polynomial $P(x)$ is not real for $x \in \mathbb{R}^n$ and, hence, is not elliptic (nor is its real part elliptic if, say, $|\xi| \geq 1$). However, if we let A be the matrix

$$A := \begin{pmatrix} i\xi & -\sqrt{1+\xi^2} \\ \sqrt{1+\xi^2} & i\xi \end{pmatrix} \tag{9}$$

then one can check that $\Sigma(A\tau) = \Sigma(\tau)$ and $P(A^{-t}x) = x_1^4 + x_2^4$. Since $P(A^{-t}x)$ is real and satisfies $P(A^{-t}x) \geq \delta|x|^4$, we conclude that P is Σ-elliptic.

We also mention that a homogeneous polynomial $P_{2p}(z)$ of degree $2p$ is B-elliptic, for a given nondegenerate quadratic form $B(\zeta)$, if and only if there exists a linear change of coordinates $z = A^{-t}w$ such that $Q_{2p}(\partial/\partial z) := (B(\partial/\partial z))^p$ in the new coordinates w

becomes $\widetilde{Q}_{2p}(\partial/\partial w) = \Delta_{\mathbb{C}}^p$, where

$$\Delta_{\mathbb{C}} := \sum_{j=1}^{n} \frac{\partial^2}{\partial w_j^2}, \tag{10}$$

and the polynomial $\tilde{P}_{2p}(w) := P_{2p}(A^{-t}w)$ is elliptic in the usual sense.

Our first main result is the following.

Theorem 2. *Let $B(\zeta)$ be a nondegenerate quadratic form in $\mathbb{C}^n$ and p an integer ≥ 1. Let $k := 2p$, $Q_k(\zeta) := (B(\zeta))^p$, and consider the holomorphic partial differential operator L given by (1) with $k_0 = p = k/2$. Suppose $P(z) = P_k(z)$ is a homogeneous polynomial of degree $k = 2p$ that is $B(\zeta)$-elliptic. Then, for any domain $0 \in U \subset \Omega$, there is a subdomain $0 \in U' \subset U$ such that the mixed Cauchy problem*

$$L(Pq) = f$$

has a unique solution $q \in \mathcal{O}(U')$ for every $f \in \mathcal{O}(U)$.

In the setting of Theorem 2, as we mentioned above, we may assume, possibly after a linear change of coordinates, that $Q_{2p}(D) = \Delta_{\mathbb{C}}^p$ and $P_{2p}(x) \geq \delta|x|^{2p}$ for $x \in \mathbb{R}^n$. Let Δ denote the usual Laplace operator in $\mathbb{R}^n$,

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{11}$$

Theorem 2 will follow from a result about a mixed Cauchy type problem in $\mathbb{R}^n$ for partial differential operators whose principal symbol is the iterated Laplace operator Δ^p. To formulate this result, we must introduce some more notation. Let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ be the open unit ball in $\mathbb{R}^n$ (where $0 < R \leq \infty$). We consider the algebra $A\left(B_R\right)$ of all infinitely differentiable functions $f : B_R \to \mathbb{C}$ such that for any compact subset $K \subset B_R$ the homogeneous Taylor series $\sum_{m=0}^{\infty} f_m\left(x\right)$ converges absolutely and uniformly to f on K; here, f_m is the homogeneous polynomial of degree m defined by the Taylor series of f

$$f_m\left(x\right) = \sum_{|\alpha|=m} \frac{1}{\alpha =!} \frac{\partial^\alpha f}{\partial x^\alpha}\left(0\right) x^\alpha.$$

Note that the functions in $A(B_R)$ are real-analytic. In fact, it is known that $A\left(B_R\right)$ is isomorphic to $\mathcal{O}(\widehat{B}_R)$, where $\widehat{B}_R \subset \mathbb{C}^n$ denotes the Lie ball of radius R

$$\widehat{B_R} := \left\{ z \in \mathbb{C}^n : |z|^2 + \sqrt{|z|^4 - |z_1^2 + \ldots + z_n^2|^2} < R^2 \right\},$$

and the isomorphism $\phi\colon \mathcal{O}(\widehat{B_R}) \to A\left(B_R\right)$ is simply given by $\phi(f) := f|_{B_R}$. (See [32] for this result; see also [30], Section 8.) We observe that the isomorphism ϕ commutes with differentiation in the following way

$$\phi\left(\frac{\partial^{|\alpha|}f}{\partial z^{\alpha}}\right)=\frac{\partial^{|\alpha|}\phi(f)}{\partial x^{\alpha}}.$$

Since any domain $0\in U$ contains a Lie ball of some radius and every Lie ball contains an open neighborhood U' of 0, we conclude, as claimed above, that Theorem 2 indeed is a consequence of the following result in $\mathbb{R}^n$.

Theorem 3. *Suppose that $P_{2p}(x)$ is homogeneous of degree $2p$ and elliptic, i.e. there exists $\delta>0$ such that $P_{2p}(x)\geq\delta|x|^{2p}$ for all $x\in\mathbb{R}^n$. Let $0\leq k_0\leq p$ be an integer, $R>0$ a positive number, $a_\alpha(x)$ functions in $A(B_R)$ for every multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\leq k_0$, and*

$$L=\Delta^p+\sum_{|\alpha|\leq k_0}a_\alpha(x)D^\alpha.$$

If $k_0<p$ then the operator $q\mapsto L(P_{2p}q)$ is a bijection from $A(B_R)$ onto $A(B_R)$. If $k_0=k$ then there exists $r>0$ such that the equation $L(P_{2p}q)=f$ has a unique solution $q\in A(B_r)$ for every $f\in A(B_R)$.

The proof of Theorem 3 hinges on new estimates for a real version of the Fischer norm (see Theorem 13) that go back to the paper [30] by the second author. Theorem 3 follows then from a general result (Theorem 17) about real mixed Cauchy type problems. The latter theorem is analogous to a similar theorem about complex Cauchy problems in [14].

We note that if $Q_{2p}(D)=\Delta_{\mathbb{C}}^p$ in Theorem 2 (as we may assume), then the homogeneous polynomial $P_{2p}(x)=Q_{2p}^*(x)=|x|^{2p}$ is B-elliptic. Thus, both Theorem 2 and Theorem 3.1.1 in [14] apply to the mixed Cauchy problem for L given by (1) with divisor $P_{2p}(z)=\sum z_j^2$. In this particular situation, the result in [14] is more general: the number k_0 in (1) can be chosen to be $3p/2$ (see [14], p. 261), whereas in the present paper only $k_0=p$ is allowed. The reason for this is that [14] utilizes the complex Fischer norm, rather than the real one used in this paper, and when $Q_{2p}(D)=\Delta_{\mathbb{C}}^p$, $P_{2p}(z)=\sum z_j^2$, a stronger estimate holds for the complex Fischer norm (see Subsection 6.1). The advantage of the real norm, of course, is that it allows a much more general class of divisors.

3. Examples and applications

In this section, we apply Theorem 3 to a couple of explicit examples. Before proceeding, we should perhaps point out that, in general, the mixed Cauchy problem for L with divisor P is *not* well posed, even if P is a homogeneous polynomial of degree k, as is illustrated by the following simple example.

Example 4. Consider the complex "Laplace operator" in two variables

$$L=\Delta_{\mathbb{C}}:=\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}$$

and the homogeneous polynomial $P(z) = z_1 z_2$. Note that $q = 1$ solves $L(Pq) = 0$ and, hence, the uniqueness fails. It is also easy to see that the equation $L(Pq) = 1$ has no solution in any neighborhood of the origin.

In [14, Section 5], it is also shown that solvability can fail even when uniqueness holds. For instance, if we take L to be the complex Laplace operator in $\mathbb{C}^2$ and

$$P(z) = z_2(z_1^2 + (z_2 - 1)^2 - 1) = z_2(z_1^2 + z_2^2 - 2z_2),$$

then uniqueness holds in the mixed Cauchy problem at 0 but $L(Pq) = f$ is in general not solvable.

The problem of deciding for which polynomials $P(z)$ the mixed Cauchy problem $L(Pq) = 0$, where L is the complex Laplace operator, has $q = 0$ as its unique solution has been addressed in e.g. [2], [1].

Example 5. Consider the holomorphic partial differential operator

$$L := \Delta_{\mathbb{C}}^2 + \sum_{j=1}^{n} a_n(z) \frac{\partial}{\partial z_j} + b(z), \tag{12}$$

where, for simplicity, the coefficients $a_j(z)$ and $b(z)$ are assumed to be entire functions in $\mathbb{C}^n$. Let $P(z) = \sum_{j=1}^{n} z_j^4$ and note that $P(x) \geq \delta |x|^4$ for $x \in \mathbb{R}^n$. Since $k_0 = 1 < 2$, it follows from Theorem 3 and the remarks preceding it that the mixed Cauchy problem

$$L(Pq) = f$$

has a unique solution $q \in \mathcal{O}(\widehat{B_R})$ for any $f \in \mathcal{O}(\widehat{B_R})$ and any $R > 0$. This illustrates the fact that if U is a Lie ball, then one can take $U' = U$ in Theorem 2 provided that $k_0 < p$ (and the coefficients are analytic in $\widehat{B_{2R}}$).

Example 6. Let $\square$ denote the wave operator in $\mathbb{R}^n \times \mathbb{R}$,

$$\square := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial t^2}$$

and consider the real partial differential operator

$$L := \square + a(x,t), \tag{13}$$

where $a(x,t)$ is, say, in $A(\mathbb{R}^{n+1})$. Let

$$P(x,t) := \sum_{j=1}^{n} x_j^2 - t^2,$$

so that $\{(x,t)\colon P(x,t)=0\}$ is the light cone. Observe that the linear change of variables $y=it$ transforms $\square$ into the Laplace operator Δ in $\mathbb{R}^{n+1}$ and $\widetilde{P}(x,y):=P(x,it)$ becomes

$$\widetilde{P}(x,y)=\sum_{j=1}^{n}x_j^2+y^2,$$

which is clearly elliptic. An application of Theorem 2 and the remark made in Example 5 above (here, $k_0=0<1=p$) yields (the probably well known result) that the real Cauchy problem

$$L(Pq)=f \tag{14}$$

has a unique solution q in $A(D_R)$ for every $f\in A(D_R)$. Here, D_R is the real domain

$$D_R:=\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}:|x|^2+|y|^2+\sqrt{\left(|x^2|+|y|^2\right)^2-\left|x_1^2+\dots x_n^2-y^2\right|^2}<R^2\right\},$$

and $A(D_R)$ denotes the restriction to D_R of functions that are holomorphic in

$$\left\{(z,w))\in\mathbb{C}^n\times\mathbb{C}:|z|^2=+|w|^2+\sqrt{\left(|z^2|+|w|^2\right)^2-\left|z_1^2+\dots z_n^2-w^2\right|^2}<R^2\right\}$$

We point out that the light cone, which carries the null data in (14), is everywhere characteristic for the wave operator $\square$.

4. A special integral

Throughout the paper we shall use frequently the following notation:

$$I_m:=\int_0^\infty e^{-r^2}r^m dr \text{ for } m\in\mathbb{N}_0.$$

This integral is well known, and for the even case (see p. 265 in [29]) we have

$$I_{2m}=\frac{\sqrt{\pi}}{2}\frac{(2m)=!}{m!}2^{-2m}=\frac{\sqrt{\pi}}{2}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2m-1)}{2^m}\le m!, \tag{15}$$

while in the odd case a simple substitution argument gives

$$I_{2m+1}=\int_0^\infty e^{-r^2}r^{2m+1}dr=\frac{1}{2}\int_0^\infty e^{-x}x^m dx=\frac{1}{2}m!. \tag{16}$$

We shall use the following identity.

Proposition 7. *For positive integers m,k,j,n,*

$$\frac{I_{2m+2jk+n-1}}{I_{2m+n-1}}=\frac{1}{2^{jk}}\left(n+2m\right)\left(n+2m+2\right)\dots\left(n+2m+2jk-2\right). \tag{17}$$

Proof. First assume that $n-1$ is even and write $n-1=2l$. Then by (15)

$$\begin{aligned}\frac{I_{2m+2jk+n-1}}{I_{2m+n-1}} &= \frac{1}{2^{jk}}\frac{1\cdot 3\cdot 5\cdot ...\cdot(2(m+jk+l)-1)}{1\cdot 3\cdot 5\cdot ...\cdot(2(m+l)-1)}\\ &= \frac{1}{2^{jk}}(2m+2l+1)(2m+2l+3)\dots(2m+2l+2jk-1)\\ &= \frac{1}{2^{jk}}(2m+n)(2m+n+2)\dots(2m+n-2+2jk).\end{aligned}$$

If $n-1$ is odd, then write $n=2l$. We obtain

$$\frac{I_{2m+2jk+n-1}}{I_{2m+n-1}}=\frac{I_{2m+2jk+2l-1}}{I_{2m+2l-1}}=\frac{(m+jk+l-1)!}{(m+l-1)!}. \tag{18}$$

On the other hand, the right hand side of (17) for $n=2l$ is equal to

$$\frac{(2l+2m)(2l+2m+2)\dots(2l+2m+2jk-2)}{2^{jk}}$$

which is equal to $(l+m)(l+m+1)\dots(l+m+jk-1)$. In view of (18) the proof is finished. □

5. Basic estimates in Fischer type spaces

Let $\mathbb{C}[x_1,...,x_n]$ be the space of all polynomials in n variables with complex coefficients. An important inner product on $\mathbb{C}[x_1,...,x_n]$ is the so-called *Fischer inner product*, or the *apolar inner product*, defined by

$$\langle P,Q\rangle_F := \sum_{\alpha\in\mathbb{N}_0^n}\alpha!c_\alpha\overline{d_\alpha}$$

for polynomials $P(x)=\sum_{|\alpha|\leq N}c_\alpha x^\alpha$ and $Q(x)=\sum_{|\alpha|\leq N}d_\alpha x^\alpha$, which has been used by several authors, see e.g. in chronological order [16], [6], [10], [23], [26], [27], [12], [24], [33], [31], [7], [11], [15](and the references given there), [36], [13], [8], [9], [34], [35], [17], [22], and [4]. This inner product has the property that the adjoint map of the differentiation operator $Q(D)$ is the multiplication operator M_{Q^*}, defined by $M_{Q^*}(f)=Q^*\cdot f$; so this means that

$$\langle Q(D)f,g\rangle_F=\langle f,Q^*\cdot g\rangle_F=\langle f,M_{Q^*}g\rangle_F \tag{19}$$

for all polynomials $f,g\in\mathbb{C}[x_1,...,x_n]$ where Q^* is the polynomial obtained by conjugating the coefficients. It was already observed by V. Bargmann in 1961 (see [6]) that

(20)

$$\langle f,g\rangle_F=\frac{1}{\pi^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x+iy)\overline{g(x+iy)}e^{-|x|^2-|y|^2}dxdy=\frac{1}{\pi^n}\int_{\mathbb{C}^n}f(z)\overline{g(z)}e^{-|z|^2}dA_z$$

where dx, dy denote the Lebesgue measure on $\mathbb{R}^n$ and dA_z the Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. In passing, we note that the *Bargmann space* $\mathcal{F}_n$ (also called *Fock* or *Fischer space*) is defined as the space of all entire functions $f : \mathbb{C}^n \to \mathbb{C}$ which satisfy

$$\|f\|_F^2 = \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dA_z < \infty.$$

In analogy with equation (20), we shall consider the following real version of the Fischer inner product:

$$\langle f, g \rangle_{rF} := \int_{\mathbb{R}^n} f(x) \overline{g(x)} e^{-|x|^2} dx, \tag{21}$$

which has been useful for solving the Hayman conjecture for uniqueness sets of polyharmonic functions and for solving the Khavinson-Shapiro conjecture for the Dirichlet problem, see [30]. Note that in (21) we consider a polynomial as a function on the space $\mathbb{R}^n$, while in (20) it is considered as a function on the space $\mathbb{C}^n$.

We should point out that the two inner products have some important differences, e.g. the adjoint map for the multiplication operator M_Q for the inner product $\langle \cdot, \cdot \rangle_{rF}$ is not the differentiation operator but just the operator M_{Q^*}. However, it is a somewhat surprising fact that the two inner product share many properties as well. As an illustrative example we begin with the following proposition, part of which will be crucial in the proof of Theorem 17 below.

Proposition 8. *Let k and n be positive integers. Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ and Σ^{2n-1} the unit sphere in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Let P_k be a homogeneous polynomial of degree k in n variables, $M_{\mathbb{R}} := \max_{\theta \in \mathbb{S}^{n-1}} |P_k(\theta)|$, and $M_{\mathbb{C}} := \max_{\eta \in \Sigma^{2n-1}} |P_k(\eta)|$. Then, for any homogeneous polynomial f_m of degree m in n variables,*

$$\|P_k f_m\|_F \leq M_{\mathbb{C}} \sqrt{\frac{I_{2m+2k+2n-1}}{I_{2m+2n-1}}} \|f_m\|_F, \ \|P_k f_m\|_{rF} \leq M_{\mathbb{R}} \sqrt{\frac{I_{2m+2k+n-1}}{I_{2m+n-1}}} \|f_m\|_{rF}. \tag{22}$$

In particular, for fixed k and n, there are constants $C_{k,n} > 0$ and $D_{k,n} > 0$ such that

$$\|P_k f_m\|_F \leq C_{k,n} M_{\mathbb{C}} \sqrt{(1+m)^k} \|f_m\|_F, \ \|P_k f_m\|_{rF} \leq D_{k,n} M_{\mathbb{R}} \sqrt{(1+m)^k} \|f_m\|_{rF}. \tag{23}$$

Proof. Let us consider first the norm $\|\cdot\|_{rF}$. By introducing polar coordinates, it is easy to see that for a homogeneous polynomial of degree m

$$\|f_m\|_{rF}^2 = \langle f_m, f_m \rangle_{rF} = I_{2m+n-1} \int_{\mathbb{S}^{n-1}} |f_m(\theta)|^2 d\theta.$$

Applied to $P_k f_m$ this gives

$$\|P_k f_m\|_{rF}^2 = I_{2m+2k+n-1} \int_{\mathbb{S}^{n-1}} |P_k(\theta) f_m(\theta)|^2 d\theta.$$

Then

$$\|P_k f_m\|_{rF}^2 \le \frac{I_{2m+2k+n-1}}{I_{2m+n-1}} M_{\mathbb{R}}^2 \|f_m\|_{rF}^2 .$$

This proves the second inequality in (22). From Proposition 7, it is easy to see that

$$\frac{I_{2m+2k+n-1}}{I_{2m+n-1}} \le (m+k+n/2-1)^k.$$

Clearly, for fixed k and n, there exists a constant $D_{k,n}$ such that the second inequality in (23) holds.

For the computation of the norm $\|\cdot\|_F$, we note that

$$\|f_m\|_F^2 = I_{2m+2n-1} \int_{\Sigma^{2n-1}} |f_m(\eta)|^2 \, d\eta.$$

Then

$$\|P_k f_m\|_F^2 \le \frac{I_{2m+2k+2n-1}}{I_{2m+2n-1}} {M_{\mathbb{C}}}^2 \|f_m\|_F^2 .$$

This proves the first inequality in (22). The first inequality in (23) follows easily from Proposition 7 as above. □

As a second example, also used in the proof of Theorem 17, we consider estimates of the derivative of homogeneous polynomials:

Proposition 9. *Let $\alpha \in \mathbb{N}_0^d$ be a multi-index and D^α be the corresponding differential operator. Then*

$$\|D^\alpha f_m\|_F \le \sqrt{m^{|\alpha|}} \|f_m\|_F \quad \text{and} \quad \|D^\alpha f_m\|_{rF} \le \sqrt{(2m)^{|\alpha|}} \|f_m\|_{rF}$$

for any homogeneous polynomial f_m of degree m.

Proof. By a simple induction argument, it is sufficient to prove the statement for the differential operator $D_j := \frac{\partial}{\partial x_j}$. In case of $\|\cdot\|_F$ we repeat (for convenience of the reader) the argument already given in [21] (or see [13, p. 256]): By Euler's formula one has

$$\sum_{j=1}^n z_j D_j f_m = m f_m.$$

Taking the Fischer inner product with f_m, and using that multiplication by z_j is adjoint to D_j one obtains

$$\sum_{j=1}^n \|D_j f_m\|_F^2 = m \|f_m\|_F^2 .$$

In particular,

$$\|D_j f_m\|_F \le \sqrt{m} \|f_m\|_F .$$

Note that the previous argument does not apply to the norm $\|\cdot\|_{rF}$ since D_j is not the adjoint of z_j. However, a simple argument using partial integration shows that for $j = 1, ..., n$ and $f, g \in \mathbb{C}[x_1, ..., x_n]$

$$\left\langle \frac{\partial}{\partial x_j} f, g \right\rangle_{rF} + \left\langle f, \frac{\partial}{\partial x_j} g \right\rangle_{rF} = 2 \langle x_j \cdot f, g \rangle_{rF} .$$

Replace f by $\frac{\partial}{\partial x_j} f$, and sum up, then

$$\langle \Delta f, g \rangle_{rF} + \sum_{j=1}^{n} \left\langle \frac{\partial}{\partial x_j} f, \frac{\partial}{\partial x_j} g \right\rangle_{rF} = 2 \sum_{j=1}^{n} \left\langle x_j \frac{\partial}{\partial x_j} f, g \right\rangle_{rF} .$$

For a homogeneous polynomial f_m of degree m Euler's formula yields

$$\langle \Delta f_m, f_m \rangle_{rF} + \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} f_m \right\|_{rF}^2 = 2m \langle f_m, f_m \rangle_{rF} . \tag{24}$$

Hence it suffices to show that $\langle \Delta f_m, f_m \rangle_{rF} \geq 0$, which will be done in the next proposition. □

Proposition 10. *For any homogeneous polynomial f_m of degree m*

$$\langle \Delta f_m, f_m \rangle_F = 0 \text{ and } \langle \Delta f_m, f_m \rangle_{rF} \geq 0$$

Proof. The identity for $\langle \cdot, \cdot \rangle_F$ is trivial since Δf is polynomial of degree $m - 2$ and homogeneous polynomials of different degree are always orthogonal for the Fischer inner product. It is an elementary fact that, for a homogeneous harmonic polynomial $h(x)$, the following formula holds

$$\Delta \left(|x|^{2s} h(x) \right) = 2s \left[2s - 2 + 2 \deg h + n \right] \cdot |x|^{2s-2} h(x) . \tag{25}$$

For the inequality for the real inner product $\langle \cdot, \cdot \rangle_{rF}$, we consider the *Gauß decomposition* of f_m: there exist homogeneous harmonic polynomials h_{m-2s} of degree $m - 2s$ such that $f_m = \sum_{s=0}^{N} |x|^{2s} h_{m-2s}$ with $N = [m/2]$, see e.g. [5], p. 76. Then, according to (25), $\Delta \left(|x|^{2s} h_{m-2s} \right) = c_s |x|^{2s-2} h_{m-2s}$, with

$$c_s := 2s \left(2s - 2 + n + 2 \deg h_{m-2s} \right) \geq 0.$$

Thus

$$\langle \Delta f_m, f_m \rangle = \sum_{s=0}^{N} \sum_{j=0}^{N} c_s \left\langle |x|^{2s-2} h_{m-2s}, |x|^{2j} h_{m-2j} \right\rangle_{rF} .$$

Furthermore,

$$\left\langle |x|^{2s-2} h_{m-2s}, |x|^{2j} h_{m-2j} \right\rangle_{rF} = I_{2m+n-3} \int_{\mathbb{S}^{n-1}} h_{m-2s}(\theta) h_{m-2j}(\theta) \, d\theta.$$

Since $\deg h_{m-2s} - \deg h_{m-2j} = 2(j-s)$, we see that $\langle h_{m-2s}, h_{m-2j}\rangle_{rF} = 0$ for $s \neq j$. Hence there is only a contribution in (5) for $s = j$, and we obtain

$$\langle \Delta f_m, f_m\rangle_{rF} = \sum_{s=0}^{N} c_s \left\langle |x|^{2s-2} h_{m-2s}, |x|^{2s} h_{m-2s}\right\rangle_{rF} \geq 0.$$

□

6. Operators acting on homogeneous polynomials

Let $\mathcal{P}_m(\mathbb{R}^n)$ denote the space of all homogeneous polynomials with complex coefficients of degree m in n variables. We consider first the operator $F_{2p} : \mathcal{P}_m(\mathbb{R}^n) \to \mathcal{P}_m(= \mathbb{R}^n)$ defined by

$$F_{2p}(q) := \Delta^p \left(|x|^{2p} q\right).$$

A simple induction argument using the formula (25) shows that, for any homogeneous harmonic polynomial h,

$$F_{2p}\left(|x|^{2s} h\right) = \Delta^p \left(|x|^{2s+2p} h\right) = d_p(s, \deg h) |x|^{2s} h \tag{26}$$

where $d_p(s,m)$ is the number

$$d_p(s,m) = 2^p (s+p) \ldots. (s+1) \cdot (2s+2p-2+n+2m) \ldots. (2s+n+2m). \tag{27}$$

From this one obtains the following well-known result; for the reader's convenience, we shall sketch the proof.

Proposition 11. *The space $\mathcal{P}_m(\mathbb{R}^n)$ has a basis consisting of eigenvectors for the operator $F_{2p} : \mathcal{P}_m(\mathbb{R}^n) \to \mathcal{P}_m(\mathbb{R}^n)$ such that the lowest eigenvalue is greater than or equal to*

$$e_{p,m} = 2^p p! (2m+n)(2m+n+2) \ldots (2m+n+2(p-1)). \tag{28}$$

Proof. Let $m \geq 1$ be fixed. Let $\mathcal{H}_{m-2s}(\mathbb{R}^n)$ be the space of all harmonic polynomials of degree $m-2s$, and let $Y_{m-2s,l}$ for $l = 1, \ldots, a_{m-2s} := \dim \mathcal{H}_{m-2s}(\mathbb{R}^n)$ be a basis of $\mathcal{H}_{m-2s}(\mathbb{R}^n)$. Then

$$|x|^{2s} Y_{m-2s,l}, \quad s = 0, \ldots, [m/2], l = 1, \ldots, a_{m-2s} \tag{29}$$

are homogeneous polynomials of degree m, and by (26) they are clearly eigenfunctions of F_{2p} with eigenvalue $d_p(s, m-2s)$ and

$$d_p(s, m-2s) := 2^p (s+p) \ldots. (s+1) \cdot (2m-2s+2p-2+n) \ldots. (2m-2s+n).$$

The minimal value for these numbers, ranging from $s = 0, \ldots, [m/2]$, is attained for $s = 0$ which gives (28). The Gauß decomposition of a polynomial (see the proof of Proposition 10) shows that (29) is indeed a basis of $\mathcal{P}_m(\mathbb{R}^n)$. □

Proposition 12. *For a homogeneous polynomial f_m of degree m, we have*

$$\left\|\Delta^p \left(|x|^{2p} f_m\right)\right\|_{rF} \geq e_{p,m} \|f_m\|_{rF}.$$

Proof. Let $f_m = \sum_{s=0}^{N} |x|^{2s} h_{m-2s}$, with $N := [m/2]$, be the Gauß decomposition with harmonic polynomials h_{m-2s} of degree $m-2s$ for $s = 0, ..., N$. We compute the inner product $\langle F_{2p} f_m, F_{2p} f_m \rangle_{rF}$ for $F_{2p} := \Delta^p \left(|x|^{2p} \cdot \right)$:

$$\langle F_{2p} f_m, F_{2p} f_m \rangle_{rF} = \sum_{s=0}^{N} \sum_{j=0}^{N} d_p(s, m-2s) d_p(j, m-2j) \left\langle |x|^{2s} h_{m-2s}, |x|^{2j} h_{m-2j} \right\rangle_{rF} .$$

Since $\deg h_{m-2s} - \deg h_{m-2j} = 2(j-s)$, we see that $\langle h_{m-2s}, h_{m-2j} \rangle_{rF} = 0$ for $s \neq j$. Hence

$$\langle F_{2p} f_m, F_{2p} f_m \rangle_{rF} = \sum_{s=0}^{N} d_p(s, m-2s)^2 \left\langle |x|^{2s} h_{m-2s}, |x|^{2s} h_{m-2s} \right\rangle_{rF} .$$

Similarly, $\langle f_m, f_m \rangle_{rF} = \sum_{s=0}^{N} \left\langle |x|^{2s} h_{m-2s}, |x|^{2s} h_{m-2s} \right\rangle_{rF}$. Hence

$$\| F_{2p} f_m \|_{rF} \geq e_{p,m} \| f_m \|_F .$$

□

We shall now give the basic $\| \cdot \|_{rF}$-estimate for the operator

$$f \longmapsto \Delta^p \left(P_{2p} \cdot f_m \right),$$

which will be used in the proof of Theorem 3. We shall show in the comments below (Subsection 6.1) that the result is sharp even if $P_{2p}(x) = |x|^{2p}$. This is in contrast with the case of the complex Fischer norm $\| \cdot \|_F$, where a better estimate than (31) holds for $P_{2p}(z) = (\sum z_j^2)^p$, see (37).

Theorem 13. *Let $P_{2p}(x)$ be a homogeneous polynomial of degree $2p$ and suppose that there is a $\delta > 0$ such that $P_{2p}(x) \geq \delta |x|^{2p}$ for all $x \in \mathbb{R}^n$. Then there exists a constant C_1 such that, for each homogeneous polynomial f_m of degree $m \in \mathbb{N}_0$,*

$$\| \Delta^p \left(P_{2p} \cdot f_m \right) \|_{rF} \geq C_1 e_{p,m} \| f_m \|_{rF} . \tag{30}$$

Moreover, there exists a constant C_2 such that, for each homogeneous polynomial f_m of degree $m \in \mathbb{N}_0$,

$$\| \Delta^p \left(P_{2p} \cdot f_m \right) \|_{rF} \geq C_2 \| P_{2p} \cdot f_m \|_{rF} . \tag{31}$$

Proof. As above, we let $F_{2p}(u) = \Delta^p \left(|x|^{2p} u \right)$. Let f_m be given and define $g_m =: \Delta^p \left(P_{2p} f_m \right)$. Since F_{2p} is a bijection there exists u_m such that $F_{2p}(u_m) = \Delta^p \left(|x|^{2p} u_m \right) = g_m$. Proposition 12 yields

$$\| \Delta^p \left(P_{2p} f_m \right) \|_{rF} = \| g_m \|_{rF} = \| F_{2p}(u_m) \|_{rF} \geq e_{p,m} \| u_m \|_{rF} . \tag{32}$$

Since obviously $\left\| |x|^{2p} u_m \right\|_{rF}^2 = \frac{I_{2m+4p+n-1}}{I_{2m+n-1}} \| u_m \|_{rF}^2$ one obtains

$$\| \Delta^p \left(P_{2p} f_m \right) \|_{rF} \geq e_{p,m} \sqrt{\frac{I_{2m+n-1}}{I_{2m+4p+n-1}}} \left\| |x|^{2p} u_m \right\|_{rF} . \tag{33}$$

Note that $\Delta^p \left(P_{2p} f_m - |x|^{2p} u_m\right) = 0$; thus there exists a homogeneous polynomial r_{m+2p} of degree $m + 2p$ such that $|x|^{2p} u_m = P_{2p} f_m + r_{m+2p}$ and $\Delta^p r_{m+2p} = 0$. A result proved in [30] (see Theorem 12 and, in particular, equation (26), loc. cit.) yields the following estimate for f_m,

$$\|f_m\|_{rF} \leq \delta^{-1} \frac{I_{2m+n-1}}{I_{2m+2p+n-1}} \left\| |x|^{2p} u_m \right\|_{rF} \tag{34}$$

where $\delta > 0$ is a constant independent of m. So we obtain from (33)

$$\|\Delta^p \left(P_{2p} f_m\right)\|_{rF} \geq \delta e_{p,m} \sqrt{\frac{I_{2m+n-1}}{I_{2m+4p+n-1}}} \frac{I_{2m+2p+n-1}}{I_{2m+n-1}} \|f_m\| .$$

It is easy to see from Proposition 7 that there is a constant C', depending only on p and n, such that

$$\sqrt{\frac{I_{2m+n-1}}{I_{2m+4p+n-1}}} \frac{I_{2m+2p+n-1}}{I_{2m+n-1}} \geq C'$$

for all natural numbers m. This proves the estimate (30).

The estimate (31) follows immediately from (30) and Proposition 8, since there is a constant $C'' > 0$ such that, for $m \geq 0$,

$$\frac{e_{p,m}}{(m+1)^p} \geq C''.$$

□

We note, by Proposition 7, that the constant $e_{p,m}$ can be expressed by means of the integrals I_m,

$$\frac{I_{2m+2p+n-1}}{I_{2m+n-1}} = \frac{1}{2^p} (2p - 2 + n + 2m) \ldots . (n + 2m) = \frac{1}{2^{2p} p!} e_{p,m}. \tag{35}$$

We also record here the following corollary of Theorem 13, which will be used to prove Theorem 3.

Corollary 14. *Suppose that $P_{2p}(x)$ is homogeneous of degree $2p$ and $P_{2p}(x) \geq \delta |x|^{2p}$ for all $x \in \mathbb{R}^n$. Then there exists a constant D such that, for each homogeneous polynomial f_m of degree $m \in \mathbb{N}_0$,*

$$\|\Delta^p \left(P_{2p} \cdot f_m\right)\|_{rF} \geq D m^p \|f_m\|_{rF} \tag{36}$$

6.1. **A comment on the difference between the real and complex Fischer norms.** The following estimates for the complex Fischer norm were proved in [21],

$$\|\Delta_{\mathbb{C}}^p(\Sigma^p \cdot q_m)\|_F \geq C\sqrt{m^p}\|\Sigma^p \cdot q_m\|, \quad \|\Delta_{\mathbb{C}}^p(\Sigma^p \cdot q_m)\|_F \geq C m^p \|q_m\|_F, \tag{37}$$

where Σ is given by (6). These estimates lead to the fact, mentioned in Section 2, that the mixed Cauchy problem, for L with principal part $\Delta_{\mathbb{C}}^p$, with divisor Σ is well posed for

$k_0 = 3p/2$ ([14]). We note that the second estimate in (37) is analogous to the estimate for the real norm in Proposition 12. The first, however, does not hold for the real norm in view of the following result.

Proposition 15. *Assume that $n > 1$. Suppose that for an integer $l \geq 0$ the following estimate holds for homogeneous polynomials f_m of degree m:*

$$\left\|\Delta^p \left(|x|^{2p} \cdot f_m\right)\right\|_{rF} \geq C\sqrt{m^l} \left\||x|^{2p} \cdot f_m\right\|_{rF} \tag{38}$$

Then $l = 0$.

Proof. Suppose that (38) holds. Since $n > 1$ we may take for f_m a homogeneous harmonic polynomial $Y_m \neq 0$ of degree m. Recall that Y_m is an eigenvector of $F_{2p} = \Delta^p \left(|x|^{2p} \cdot\right)$ and

$$\Delta^p \left(|x|^{2p} \cdot Y_m\right) = d_p(p,m) Y_m$$

where $d_p(p,m)$ is given by (27). Further

$$\left\||x|^{2p} \cdot Y_m\right\|_{rF} = \sqrt{\frac{I_{4p+2m+n-1}}{I_{2m+n-1}}} \left\|Y_m\right\|_{rF}$$

Hence (38) implies that

$$|d_p(p,m)| \geq C\sqrt{m^l} \sqrt{\frac{I_{4p+2m+n-1}}{I_{2m+n-1}}}$$

But $|d_p(p,m)| \leq Am^p$ and

$$\frac{I_{4p+2m+n-1}}{I_{2m+n-1}} \geq m^{2p}.$$

So we obtain that $C\sqrt{m^l} \leq A$, which implies that $l = 0$. □

Proposition 15 is not true for $n = 1$. Indeed, in this case, the set $\mathcal{P}_m(\mathbb{R}^n)$ consists of multiples of the polynomial x^m and one has

$$\frac{d^{2p}}{dx^{2p}}(x^{2p} \cdot x^m) = (m+2p)(m+2p-1)...(m+1)x^m.$$

7. The mixed Cauchy problem for linear partial differential operators in $\mathbb{R}^n$

As above, we let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ denote the open unit ball in $\mathbb{R}^n$ and $A(B_R)$ the algebra of all infinitely differentiable functions $f : B_R \to \mathbb{C}$ such that the homogeneous Taylor series $\sum_{m=0}^{\infty} f_m(x)$ converges absolutely and uniformly to f on compact subsets of B_R, where f_m are the homogeneous polynomials of degree m defined by the Taylor series of f. Introducing polar coordinates $x = r\theta$ with $r \geq 0$ and $\theta \in \mathbb{S}^n = \{x \in \mathbb{R}^n : |x| = 1\}$ one can write

$$f(r\theta) = \sum_{m=0}^{\infty} r^m f_m(\theta).$$

If $\sum_{m=0}^{\infty} r^m |f_m(\theta)|$ converges uniformly for all $\theta \in \mathbb{S}^{n-1}$ and $0 \leq r \leq \rho < R$ then there exists a majorant M_ρ such that

$$\rho^m |f_m(\theta)| \leq M_\rho \text{ for all } \theta \in \mathbb{S}^{n-1}. \tag{39}$$

It is easy to see that this implies

$$\limsup_m \max_{\theta \in \mathbb{S}^{n-1}} \sqrt[m]{|f_m(\theta)|} \leq R^{-1}. \tag{40}$$

Conversely, if the estimate (40) holds for a real-analytic function f in a neighborhood of 0, it is easy to see that $f \in A(B_R)$. We shall need the following lemma, which follows easily from Proposition 11 in [30].

Proposition 16. *Suppose that f_m are homogeneous polynomials of degree m for $m \in \mathbb{N}_0$. Then $\sum_{m=0}^{\infty} f_m$ converges uniformly on compact subsets of B_R if and only if, for every $0 < \rho < R$, there is a constant C_ρ such that*

$$\max_{\theta \in \mathbb{S}^{n-1}} |f_m(\theta)| \leq C_\rho \rho^{-m}, \quad \forall\, m \in \mathbb{N}_0, \tag{41}$$

if and only if, for every $0 < \rho < R$, there is a constant C_ρ such that

$$\|f_m\|_{rF} \leq C_\rho \rho^{-m} \sqrt{m!}, \quad \forall\, m \in \mathbb{N}_0. \tag{42}$$

Our main result in this section is the following. Theorem 3 follows directly from this result in view of Corollary 14.

Theorem 17. *Let P_k and Q_k be homogeneous polynomials of degree k and suppose that there exist a constant $C > 0$ and an exponent p with $0 < p \leq k$ such that, for all homogeneous polynomials q_m of degree $m \geq 0$,*

$$\|Q_k(D)(P_k q_m)\|_{rF} \geq C m^p \|q_m\|_{rF}. \tag{43}$$

Let $0 \leq k_0 < k$ be an integer, $R > 0$ a positive number, $a_\alpha(x)$ functions in $A(B_R)$ for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k_0$, and

$$L = Q_k(D) + \sum_{|\alpha| \leq k_0} a_\alpha(x) D^\alpha. \tag{44}$$

If $k_0 < p$ then the operator $q \mapsto L(P_k q)$ is a bijection from $A(B_R)$ onto $A(B_R)$. If $k_0 = p$ (in particular $p < k$) then there exists $r > 0$ such that $L(P_k q) = f$ has a unique solution $q \in A(B_r)$ for every $f \in A(B_R)$.

Proof. Let $f=\sum_{m=0}^{\infty} f_m$ be a function in $A(B_R)$ given in terms of its homogeneous Taylor series as above. Consider the equation

$$L(P_k q) = f. \tag{45}$$

We shall look for a solution q in terms of its homogeneous Taylor series $\sum_{m=0}^{\infty} q_m$. To prove Theorem 17, it suffices to show that the homogeneous polynomials q_m are uniquely determined by (45) and that the series $\sum_{m=0}^{\infty} q_m$ converges uniformly on compact subsets of B_r, for some $r>0$, and that one can take $r=R$ if $k_0<p$. Let us fix $m\geq 0$ and identify the homogeneous part of degree m in (45). To this end, we expand the coefficients a_α in terms of their Taylor series, $a_\alpha=\sum_{m=0}^{\infty} a_{\alpha,m}$, and obtain from (45)

$$Q_k(D)(P_k q_m) = f_m - \sum_{l=0}^{k_0} \sum_{|\alpha|=l} \sum_{i=0}^{m+l-k} a_{\alpha,i} D^\alpha (P_k q_{m+l-k-i}), \tag{46}$$

where of course the last sum only occurs for those l (if any) for which $m+l-k\geq 0$. Note that (43) implies, in particular, that $q_m \mapsto Q_k(P_k q_m)$ is injective and, hence, also surjective as an operator from the vector space of homogeneous polynomials (including the zero polynomial) into itself. Since $k_0<k$, we conclude from (46) that q_m is uniquely determined by q_j, with $0\leq j\leq m-1$, and f_m, and that q_0 is uniquely determined by f_0. This proves the injectivity of $q\mapsto L\left(D\right)\left(P_k q\right)$.

To prove the existence of a solution to (45), we must estimate the $\|\cdot\|_{rF}$ -norms of q_m. For the remainder of this proof, we shall only deal with the norm $\|\cdot\|_{rF}$ and, for simplicity of notation, shall denote this norm simply by $\|\cdot\|$. The inequality (43) implies that, for $m\geq 1$,

$$\begin{aligned} \|q_m\| &\leq C^{-1} m^{-p} \|Q_k(D)(P_k q_m)\| \\ &\leq C^{-1} m^{-p} \left(\|f_m\| + \sum_{l=0}^{k_0} \sum_{|\alpha|=l} \sum_{i=0}^{m+l-k} \|a_{\alpha,i} D^\alpha (P_k q_{m+l-k-i})\| \right). \end{aligned} \tag{47}$$

Let us fix a radius $r>0$ with $r\leq R$. To show that $q=\sum_{m=0}^{\infty} q_m$ converges to a function in $A(B_r)$, it suffices, in view of Proposition 16, to show that for every $0<\rho<r$ there is a constant $B=B_\rho$ such that

$$\|q_j\| \leq B\rho^{-j}\sqrt{j!} \tag{48}$$

for all $j=1,2\ldots$. Let us choose two radii ρ and σ with $0<\rho<\sigma<r$. Since $f\in A(B_R)$, there is, in view of Proposition 16, a constant $D=D_\rho$ such that

$$\|f_m\| \leq D\rho^{-m}\sqrt{m!}, \quad m=0,1,2\ldots. \tag{49}$$

Moreover, since $a_\alpha \in A(B_R)$, there is, in view of Proposition 16, a constant $E_\alpha=E_{\alpha,\sigma}$ such that

$$\max_{\theta\in\mathbb{S}^{n-1}} |a_{\alpha,i}(\theta)| \leq E_\alpha \sigma^{-i}, \quad i=0,1,2\ldots. \tag{50}$$

Proposition 8 (with P_k replaced by $a_{\alpha,i}$ and using that $|\alpha| = l$), shows that

$$\|a_{\alpha,i}D^\alpha(P_k q_{m+l-k-i})\| \le \sqrt{\frac{I_{2m+n-1}}{I_{2m-2i+n-1}}} E_\alpha \sigma^{-m} \|D^\alpha(P_k q_{m+l-k-i})\|. \tag{51}$$

Proposition 9 applied to $\|D^\alpha(P_k q_{m+l-k-i})\|$ and Proposition 8 applied to $\|P_k q_{m+l-k-i}\|$, denoting the constant $D_{k,n}M_{\mathbb{R}}$ in the latter simply by M, yield

$$\|D^\alpha(P_k q_{m+l-k-i})\| \le M 2^{l/2}(m+l-i)^{l/2} \sqrt{\frac{I_{2m+2l-2i+n-1}}{I_{2m+2l-2k-2i+n-1}}} \|q_{m+l-k-i}\|. \tag{52}$$

Let us define n^* to be the natural number $\frac{1}{2}n - 1$ for even n and $n^* = (n-1)/2$ for odd n. It is easy to see that

$$\frac{I_{2m+2l-2i+n-1}}{I_{2m+2l-2k-2i+n-1}} \le \frac{(m+l-i+n^*)!}{(m+l-k-i+n^*)!}.$$

If we set $A := 2 + n^*$, then $t + n^* \le A \cdot t$ for all natural numbers $t \ge 1$. From this we obtain the estimate

$$\frac{(m+l-i+n^*)!}{(m+l-k-i+n^*)!} \le A^k \frac{(m+l-i)!}{(m+l-k-i)!}.$$

Since $m+l-i \le m+l-i+s$ for $s = 1, ..., l$ we finally obtain

$$\frac{I_{2m+2l-2i+n-1}}{I_{2m+2l-2k-2i+n-1}}(m+l-i)^l \le A^k \frac{(m-i+2l)!}{(m+l-k-i)!}. \tag{53}$$

Now we conclude from (47) in combination with (49) and (51), (52), (53) that

$$\|q_m\| \le C^{-1} m^{-p} D \rho^{-m} \sqrt{m!} + S_m \tag{54}$$

where we define

$$\begin{aligned} S_m \ : \ &= C^{-1} m^{-p} \sum_{l=0}^{k_0} \sum_{|\alpha|=l} \sum_{i=0}^{m+l-k} E_\alpha \sigma^{-i} A^{k/2} M 2^{l/2} \cdot \\ &\sqrt{\frac{(m+n^*)!}{(m-i+n^*)!} \frac{(m-i+2l)!}{(m+l-k-i)!}} \|q_{m+l-k-i}\| \end{aligned}$$

If we denote by E_l the number

$$E_l := A^{k/2} M 2^{l/2} \binom{n+l-1}{l} \cdot \max_{|\alpha|=l} E_\alpha$$

and observe that the number of multi-indices $\alpha \in \mathbb{N}_0^n$ for which $|\alpha| = l$ is

$$\binom{n+l-1}{l},$$

then we get

$$S_m \leq \frac{C^{-1}}{m^p} \sum_{l=0}^{k_0} \sum_{i=0}^{m+l-k} E_l \sigma^{-i} \sqrt{\frac{(m+n^*)!\,(m-i+2l)!}{(m-i+n^*)!\,(m+l-k-i)!}} \|q_{m+l-k-i}\|.$$

We shall show that there exists an integer m_0 with the following property: If there exists a constant B such that (48) holds for all $j = 0, 1, \ldots, m-1$, for any m with $m \geq m_0$, then (48) holds also for $j = m$. The existence of such an integer m_0 clearly implies, by induction, that (48) holds for all $j = 0, 1, 2, \ldots$, which in turn completes the proof of the theorem.

So suppose that, for some B, (48) holds for all $j = 0, 1, \ldots, m-1$. Without loss of generality, of course, we may assume $B \geq 1$. The estimate (54) can be written as

$$\|q_m\| \leq B\rho^{-m}\sqrt{m!}(C^{-1}DB^{-1}m^{-p} + T_m), \tag{55}$$

where we have defined

$$T_m := \frac{C^{-1}}{m^p} \sum_{l=0}^{k_0} \sum_{i=0}^{m+l-k} E_l \sigma^{-i} \sqrt{\frac{(m+n^*)!\,(m-i+2l)!}{m!\,(m-i+n^*)!}} \rho^{-(l-k-i)}.$$

Recall that $B \geq 1$, so that $C^{-1}DB^{-1}m^{-p} \leq C^{-1}Dm^{-p}$. We shall show that there is m_0 such that $C^{-1}Dm^{-p} + T_m \leq 1$ for all $m \geq m_0$. For this, it suffices to show that there exists $0 < \theta < 1$ and m_0 such that

$$T_m \leq \theta \text{ for all } m \geq m_0.$$

We set

$$N(\rho) := C^{-1}(k_0+1) \max_{0 \leq l \leq k_0} E_l \cdot \max_{0 \leq l \leq k_0} \rho^{k-l}$$

and obtain

$$T_m \leq \frac{N(\rho)}{m^p} \sum_{i=0}^{m+k_0-k} \left(\frac{\rho}{\sigma}\right)^i \sqrt{\frac{(m+n^*)!\,(m-i+2k_0)!}{m!\,(m-i+n^*)!}}.$$

First suppose that $2k_0 \geq n^*$. Then

$$T_m \leq \frac{N(\rho)}{m^p} (m+n^*)^{n^*/2} (m+2k_0)^{(2k_0-n^*)/2} \sum_{i=0}^{\infty} \left(\frac{\rho}{\sigma}\right)^i.$$

If $k_0 < p$, then the right hand side clearly converges to zero as $m \to \infty$. If $k_0 = p$, then we can make $T_m \leq \theta < 1$ for all sufficiently large m by making $\rho < r$ sufficiently small. (Observe that $N(\rho) \to 0$ as $\rho \to 0$, since $k_0 < k$.)

Now suppose that $2k_0 < n^*$. Clearly we obtain

$$T_m \leq \frac{N(\rho)}{m^p} (m+n^*)^{n^*/2} \sum_{i=0}^{m+k_0-k} \left(\frac{\rho}{\sigma}\right)^i \frac{1}{\sqrt{(m-i+2k_0+1)\ldots(m-i+n^*)}}. \tag{56}$$

In order to estimate the latter sum, we fix a number δ with $0<\delta<1$. We consider only m with $m>\delta^{-1}$. The estimate $m-i\geq(1-\delta)\,m$ holds for all $i\leq\delta m$. We split the sum in (56) into two sums I_1+I_2, the first one containing only indices with $i\leq[\delta m]$ and the second one indices i with $i>[\delta m]$. We estimate

$$I_1\leq\sum_{i=0}^{[\delta m]}\left(\frac{\rho}{\sigma}\right)^i\frac{1}{\left((1-\delta)\,m\right)^{(n^*-2k_0)/2}}\leq K\frac{1}{m^{(n^*-2k_0)/2}}\frac{1}{1-\frac{\rho}{\sigma}}$$

where

$$K:=\frac{1}{(1-\delta)^{(n^*-2k_0)/2}}.$$

For the second sum I_2, we use the estimate

$$I_2\leq\sum_{i=[\delta m]}^{m+k_0-k}\left(\frac{\rho}{\sigma}\right)^i\leq\left(\frac{\rho}{\sigma}\right)^{[\delta m]}\frac{1}{1-\frac{\rho}{\sigma}}$$

and, hence, obtain

$$T_m\leq\frac{N(\rho)}{1-\frac{\rho}{\sigma}}\frac{(m+n^*)^{n^*/2}}{m^p}\left(\frac{K}{m^{(n^*-2k_0)/2}}+\left(\frac{\rho}{\sigma}\right)^{[\delta m]}\right).$$

As before it is easy to see that T_m converges to 0 for $k_0<p$, since $m^s\left(\frac{\rho}{\sigma}\right)^{[\delta m]}$ converges to 0 for any integer s (recall that $\rho/\sigma<1$). If $k_0=p$, we use, as above, the fact that $N(\rho)\to$ as $\rho\to 0$ to conclude that, if r is sufficiently small, then $T_m\leq\theta<1$ for large m. This completes the proof of Theorem 17. □

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