Iterates of a compact holomorphic map on a finite rank homogeneous ball

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ABSTRACT. We study iterates, f^n, of a fixed-point free compact holomorphic map $f : B \to B$ where B is the open unit ball of any JB^*-triple of finite rank. These spaces include $L(H,K)$, H,K Hilbert, $\dim(H)$ arbitrary, $\dim(K) < \infty$, or any classical Cartan factor or C^*-algebra of finite rank. Apart from the Hilbert ball, the sequence of iterates $(f^n)_n$ does not generally converge (locally uniformly on B) and little is known of accumulation points. We present a short proof of a Wolff theorem for B and establish key properties of the resulting f-invariant subdomains. We define a concept of closed convex holomorphic hull, $Ch(x)$, for $x \in \partial B$ and prove the following.
There is a unique tripotent u in ∂B such that all constant subsequential limits of $(f^n)_n$ lie in $Ch(u)$. As consequence we also get a short proof of the classical Hilbert ball results.

Introduction

Let B be the open unit ball of a Banach space Z and $f : B \to B$ be a holomorphic map with no fixed point in B. Study of the iterates, $(f^n)_n$, for $f^1 = f, f^n = f \circ f^{n-1}, n \geq 2$, has been ongoing for nearly a century since it was shown in the 1920s [**7, 26**] for $B = \Delta$ that (f^n) converges uniformly on compact subsets to a constant $\xi \in \partial\Delta$. Strong convexity yielded generalisations of this to the Hilbert ball; in 1963 for finite dimensions [**10**] and 1997 for infinite dimensions when f is also compact (that is, $\overline{f(B)}$ is compact) [**6**]. There are more recent extensions to other domains with similar convexity properties [**1, 2, 3, 4, 5, 11**]. Without strong convexity however (or similar), the iterates do not generally converge [**6**, Example 2], [**23**, Corollary 2.10]. We seek instead accumulation points of (f^n) with respect to the topology of local uniform convergence on B. This paper gives such results for infinite dimensional balls that are not strongly convex.

Recall that an open unit ball B is homogeneous if, and only if, the underlying Banach space Z is a JB^*-triple and such balls classify bounded symmetric domains [**12**]. JB^*-triples include Hilbert spaces, C^*-algebras and classical spaces known as Cartan factors, among many others. There is a long established concept of rank for triples [**16, 19**]. An example of a finite rank JB^*-triple is $L(H,K)$, where $\dim(K) < \infty$. For $x \in \partial B$, we define a concept of closed convex hull, $Ch(x)$, in terms of the triple product. This coincides with a concept introduced in [**2**] for convex domains in $\mathbb{C}^n$. For convergence of the iterates the natural

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topology, τ, here is local uniform convergence on B, which in finite dimensions coincides with uniform convergence on compact sets. We prove the following.

THEOREM 0.1. *Let B be the open unit ball of a finite rank JB^*-triple and $f : B \to B$ be a compact holomorphic map with no fixed point in B. Then there exists $u \in \partial B$ such that all constant τ-accumulation points of (f^n) belong to $Ch(u)$.*

In particular, all constant subsequential limits of (f^n) belong to $Ch(u)$.

The results are new in infinite dimensions for triples of rank ≥ 2 and allow short proofs of the classical Hilbert ball results [**10**] and [**6**]. We also provide in Theorem 2.1 below a much shorter proof of an alternative version of a recent Wolff's theorem for all finite rank triples [**22**], which establishes central f-invariant subdomains of B. Several key properties of these f-invariant domains are established and used.

1. Notation and background

Throughout, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For X and Y complex Banach spaces, $L(X,Y)$ denotes the space of continuous linear maps from X to Y, $X' = L(X,\mathbb{C}),\ L(X) = L(X,X)$ and $GL(X)$ is all invertible elements in $L(X)$.

DEFINITION 1.1. A *JB**-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x,y,z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.
(ii) The map $z \mapsto \{x,x,z\}$, denoted $x\Box x$, is Hermitian, $\sigma(x\Box x) \geq 0$ and $\|x\Box x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum.
(iii) The product satisfies

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

The triple product is continuous and $\|\{x,y,z\}\| \leq \|x\|\|y\|\|z\|$ [**9**]. Odd powers of an element x exist, with $x^{2n+1} := \{x,x^{2n-1},x\}, n \in \mathbb{N}, n \geq 1$, allowing us to define $p(x)$, for any odd polynomial p, leading to an odd functional calculus and a triple spectrum of the element x [**14**]. There are natural linear maps $x\Box y \in L(Z) : z \mapsto \{x,y,z\}$, $Q_x \in L_{\mathbb{R}}(Z) : z \mapsto \{x,z,x\}$, and the Bergman operators $B(x,y) = I - 2x\Box y + Q_xQ_y \in L(Z)$.

EXAMPLE 1.2. $L(H,K)$, for complex Hilbert spaces H and K, is a *JB**-triple with $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$, where y^* denotes the usual adjoint of y.

A tripotent is $e \in Z$ satisfying $\{e,e,e\} = e$. Every tripotent e induces a splitting of Z, as $Z = Z_0(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_1(e)$, where $Z_k(e)$ is the k eigenspace of $e\Box e$. Mutually orthogonal projections of Z onto $Z_0(e)$, $Z_{\frac{1}{2}}(e)$, and $Z_1(e)$ are given by $P_0(e) = B(e,e), P_{\frac{1}{2}}(e) = 2(e\Box e - Q_eQ_e)$ and $P_1(e) = Q_eQ_e$, respectively . We say $x,y \in Z$ are orthogonal, $x\perp y$, if $x\Box y = 0$ (equivalently $y\Box x = 0$). In particular, if c and e are orthogonal tripotents then $c+e$ is also a tripotent. This gives a partial ordering on the set, M, of all tripotents in Z as follows.

DEFINITION 1.3. For tripotents c and e we say $c < e$ if $e - c \in M$ and $(e-c)\perp c$.

With respect to this ordering, e is maximal if, and only if, $Z_0(e) = 0$ and e is minimal if $Z_1(e) = \mathbb{C}e$. Z is said to have finite rank r if every element $z \in Z$ is contained in a subtriple of (complex) dimension $\leq r$, and r is minimal with this property. The rank 1 triples are the Hilbert spaces. If Z has finite rank r, a frame is a set $\{e_1, \ldots, e_r\}$ of non-zero pairwise orthogonal minimal tripotents and $z \in Z$ has a unique spectral decomposition, called its Peirce decomposition, as $z = \lambda_1 e_1 + \cdots + \lambda_r e_r$, for some frame $\{e_1, \ldots, e_r\}$ and scalars $0 \leq \lambda_1 \leq \ldots \leq \lambda_r = \|z\|$. Real and complex extreme points of $\overline{B}$ coincide with the set of maximal tripotents. For all details see [**19**].

DEFINITION 1.4. [**15**, 4.1] A non-empty set $A \subset \overline{B}$ is a holomorphic boundary component of B if A is minimal with respect to the fact that, for all $f \in \mathcal{F} = \{f : \Delta \to Z$ holomorphic with $f(\Delta) \subset \overline{B}\}$, either

$$f(\Delta) \subset A \text{ or } f(\Delta) \subset \overline{B} \setminus A.$$

It follows that if $g : B \to \overline{B}$ is a holomorphic map then $g(B)$ must lie in a single such boundary component. By replacing $\mathcal{F}$ in the above definition with the set of all complex affine maps $: \Delta \to \overline{B}$ we get the definition of (complex) affine boundary components.

The following shows that holomorphic and affine boundary components coincide in the finite rank case.

THEOREM 1.5. [**15**, 4.2, 4.3, 4.4] *Let Z be a finite rank JB^*-triple with open unit ball B. The following hold.*

(i) Holomorphic and affine boundary components coincide and are precisely the sets

$$K_e = e + B_0(e)$$

where e is a tripotent and $B_0(e) = B \cap Z_0(e) = P_0(e)(B)$.

(ii) The map $e \to K_e$ is a bijection between the set, M, of tripotents in Z and the set of boundary components of B, with $x \in K_e$ if, and only if, $e = \lim_{n\to\infty} x^{2n+1}$.

(iii) $\overline{K}_e = \bigcup_{d \geq e} K_d$.

Henceforth therefore, for finite rank triples, we refer simply to boundary components and write K_x for the boundary component of x in $\overline{B}$. We note that for $x \in B$, $K_x = K_0 = B$ and this is the only open boundary component.

THEOREM 1.6. [**17**] [**11**] *Let X be a Banach space with open unit ball B. If $f : B \to B$ is a compact holomorphic map then the following are equivalent.*

(i) f has a fixed point in B;

(ii) there exists $z \in B$ and subsequence $(n_i)_i$ such that $\sup_i \|f^{n_i}(z)\| < 1$;

(iii) there exists $z \in B$ such that $\sup_n \|f^n(z)\| < 1$;

(iv) for each $z \in B$ we have $\sup_n \|f^n(z)\| < 1$.

2. New Results

Throughout Z is a JB^*-triple of finite rank r, with open unit ball B and $f : B \to B$ is a compact ($\overline{f(B)}$ is compact) holomorphic map with no fixed points in B. Denote the iterates of f by $f^n := f \circ f^{n-1}, n \geq 2, f^1 = f$, and let $\Gamma(f)$ be the set of accumulation points of (f^n) with respect to the topology of local uniform convergence on B (used throughout). Since f

is compact, $\Gamma(f) \neq \emptyset$ and contains subsequential limits [**6**, Lemma 1]. Elements of $\Gamma(f)$ are holomorphic maps from B to $\overline{B}$. We start by locating the so-called Wolff point, ξ, of f as follows. Choose a sequence $(\alpha_k)_k$, $0 < \alpha_k < 1$, $\alpha_k \uparrow 1$. Let $f_k := \alpha_k f$ for all k. As f_k maps B strictly inside B, the Earle–Hamilton theorem [**8**] implies that f_k has a fixed point, z_k, in $\alpha_k B$. By compactness of f, we can assume that $z_k \to \xi \in \overline{B}$. If ξ were in B then $f(\xi) = \xi$ which is impossible, hence $\xi \in \partial B$. We note that as Z is finite rank, it is reflexive [**16**] and hence $\overline{B}$ is both weakly compact and weakly sequentially compact.

Let the Peirce decomposition of ξ be $\xi = \mu_1 e_1 + \cdots + \mu_r e_r$, for frame $\{e_1, \cdots, e_r\}$ and $0 \leq \mu_1 \leq \cdots \leq \mu_r = \|\xi\| = 1$. Let now

$$p \in \{0, \ldots, r-1\} \text{ be such that } p+1 = \min_k \{1 \leq k \leq r : \mu_k = 1\}.$$

Putting $\mu_0 := 0$ gives $\mu_p < 1$. We let $v := \mu_1 e_1 + \cdots + \mu_p e_p$ if $p \neq 0$ and $v = 0$ otherwise and let e be the tripotent $e = e_{p+1} + \cdots + e_r \in \partial B$. Then $\xi = e + v$ and Theorem 1.5 gives $K_\xi = K_e$. Of course,

$$p = 0 \Leftrightarrow v = 0 \Leftrightarrow \xi = e \text{ is a maximal tripotent } \Leftrightarrow \xi = e \text{ is extreme in } \overline{B}.$$

We also consider Peirce decompositions for each z_k, namely,

$$z_k = \gamma_{k1} e_{k1} + \cdots + \gamma_{kr} e_{kr}, \text{ for frame } \{e_{k1}, \ldots, e_{kr}\} \text{ and scalars } 0 \leq \gamma_{k1} \leq \cdots \leq \gamma_{kr} = \|z_k\|.$$

Motivated now by insights from [**20, 22**], we make several definitions, passing to a subsequence whenever necessary for convergence.

Fix $\lambda > 0$. For $1 \leq i \leq r$ we define

$$(1) \qquad a_i := \lim_k \frac{1-\gamma_{kr}^2}{1-\gamma_{ki}^2} = \lim_k \frac{1-\|z_k\|^2}{1-\gamma_{ki}^2} \text{ and } s_i (= s_i(\lambda)) := \sqrt{1 - \sqrt{\frac{\lambda}{a_i + \lambda}}}, \; 1 \leq i \leq r.$$

Note that $0 \leq a_1 \leq \cdots \leq a_r = 1$, $\left(1 - s_i^2\right)^2 = \frac{\lambda}{a_i+\lambda}$ and $s_i = 0 \Leftrightarrow a_i = 0$.

Our crucial definition is now

$$E_\lambda := c_\lambda + T_\lambda(B)$$

where

$$(2) \qquad c_\lambda := \sum_{i=p+1}^{r} \left(\frac{a_i}{a_i+\lambda}\right) e_i, \;\; v_\lambda := \sum_{i=p+1}^{r} s_i e_i, \;\text{ and }\; T_\lambda := B(v_\lambda, v_\lambda) \in L(Z).$$

Of course, E_λ depends on ξ and is often written as $E_\lambda = E_{\xi,\lambda}$ [**20**] to [**25**]. E_λ is constructed to be exactly the limit, in a certain sense, of a sequence of f_k-invariant Kobayashi balls (such balls are fully algebraically described in [**20**, Section 2]). Recent results [**22**, Theorem 2.8, Proposition 2.9] show that E_λ are then f-invariant, giving a Wolff theorem for finite rank triples, generalising earlier results in finite dimensions [**20**, Theorem 3.10] and for the Hilbert ball [**10**]. As the proofs in [**22**] are quite demanding however, we present here a much shorter proof of the following slightly weaker version which is adequate for our later purposes.

THEOREM 2.1. *Let B be the open unit ball of a finite rank JB*-triple and $f : B \to B$ be a compact holomorphic map with no fixed point in B. For all $\lambda > 0$, E_λ is a non-empty convex domain in B satisfying*

$$f^n(E_\lambda) \subset \overline{E}_\lambda \cap B.$$

In particular, $\overline{E_\lambda} \cap B$ is f-invariant.

Proof. Fix $\lambda > 0$. Clearly E_λ is non-empty and convex. By orthogonality $\|c_\lambda\| = \max_i \frac{a_i}{a_i+\lambda} = \frac{1}{1+\lambda}$ and $\|v_\lambda\| = \max_i s_i = \sqrt{1-\sqrt{\frac{\lambda}{1+\lambda}}}$, so $c_\lambda,\ v_\lambda \in B$ and hence $T_\lambda \in GL(Z)$ [**12**]. Since T_λ is invertible it follows that E_λ is open. For $k \in \mathbb{N}$ we let

$$r_k := \sqrt{1 - \frac{(1-\|z_k\|^2)}{\lambda}}.$$

By [**20**, Corollary 2.2, Proposition 2.3] and [**19**, Corollary 3.5] there is then a sequence of Kobayashi balls, $D_k \subset B$, which are f_k-invariant, namely, $D_k := c_k + T_k(B)$ for

$$c_k = \sum_{i=1}^{r} \left(\frac{1-r_k^2}{1-r_k^2\gamma_{ki}^2} \right) \gamma_{ki} e_{ki}, \quad v_k = \sum_{i=1}^{r} s_{ki} e_{ki} \ \text{ and } \ T_k = r_k B(v_k, v_k), \tag{3}$$

where

$$s_{ki} \ \text{ satisfies } (1-s_{ki}^2)^2 = \frac{1-\gamma_{ki}^2}{1-r_k^2\gamma_{ki}^2}, \ 1 \le i \le r.$$

From [**20**, Theorem 3.10]

$$\lim_k \left(\frac{1-r_k^2}{1-r_k^2\gamma_{ki}^2} \right) = \frac{a_i}{a_i+\lambda} \quad \text{and} \quad \lim_k \frac{1-\gamma_{ki}^2}{1-r_k^2\gamma_{ki}^2} = \frac{\lambda}{a_i+\lambda}, \ 1 \le i \le r. \tag{4}$$

Crucially then Theorem 2.1 of [**22**] proves that $e_i = \lim_k e_{ki}$, for $p+1 \le i \le r$ and

$$\gamma_i := \lim_k \gamma_{ki} < 1 \Leftrightarrow 1 \le i \le p \ \text{ giving } \ a_i = s_i = 0 \text{ for } 1 \le i \le p. \tag{5}$$

It follows therefore from (3) and (4) that

$$\lim_k c_k = c_\lambda, \ \lim_k v_k = v_\lambda \text{ and } \lim_k T_k = T_\lambda = B(v_\lambda, v_\lambda) \in GL(Z). \tag{6}$$

We show $E_\lambda \subset B$. Let $x \in E_\lambda$. Then $x = c_\lambda + T_\lambda(y)$, some $y \in B$. For $k \in \mathbb{N}$, let $x_k := c_k + T_k(y) \in c_k + T_k(B) = D_k \subset B$. From (6) $x = \lim_k x_k$, giving $E_\lambda \subset \overline{B}$, and as E_λ is open, it follows $E_\lambda \subset B$.
For f-invariance, fix $n \in \mathbb{N}$ arbitrary. As D_k is f_k^n-invariant, $f_k^n(x_k) \in D_k$, namely, $f_k^n(x_k) = c_k + T_k(u_k)$, for some $u_k \in B$. Since $T_\lambda \in GL(Z)$, (6) gives $\lim_k T_k^{-1} = T_\lambda^{-1} \in GL(Z)$ and hence $u := \lim_k u_k = T_\lambda^{-1}(f^n(x) - c_\lambda)$ exists in $\overline{B}$. Then $f^n(x) = c_\lambda + T_\lambda(u) \in c_\lambda + T_\lambda(\overline{B}) \subset \overline{E}_\lambda$. In other words, $f^n(E_\lambda) \subset \overline{E}_\lambda \cap B$ and we are done. □

E_λ is an affine subset of B, referred to as an operator ball. If Z is a Hilbert space then the E_λ are exactly the classical ellipsoids [**10, 21**].

REMARKS 2.2. The culmination of results in [**22**] admits the alternative description

$$E_\lambda = \{w \in B : \phi(w) < \lambda\}, \ \text{ for } \ \phi(w) = \|B(w,w)^{-1}B(w,\xi)R\|, \text{ with } R \in L(Z)$$

and $\phi : B \to \mathbb{R}$ is continuous. (We note that, since $B(w,w)$ is invertible if, and only if, $w \in B$ [**12**], ϕ cannot be continuously extended to ∂B.) In particular, for $0 < \mu < \lambda$ we have $\overline{E}_\mu \cap B \subset E_\lambda$. This immediately converts Theorem 2.1 above to full f-invariance, since for $x \in E_\lambda$ we can choose λ_x with $\phi(x) < \lambda_x < \lambda$, so that $x \in E_{\lambda_x}$ and hence $f(x) \in f(E_{\lambda_x}) \subset \overline{E}_{\lambda_x} \cap B \subset E_\lambda$. In other words, $f(E_\lambda) \subset E_\lambda$.

We return to the search for accumulation points of (f^n), namely $\Gamma(f)$. Fix $g \in \Gamma(f)$. Then there is a subnet (f^{n_α}) with $f^{n_\alpha} \to g$ locally uniformly on B. Let $\lambda > 0$ be arbitrary and take $z \in E_\lambda$. Theorem 2.1 gives $f^n(z) \in \overline{E}_\lambda$, for $n \in \mathbb{N}$ and hence $g(z) \in \overline{E}_\lambda$. Since Z is Banach, $\overline{E}_\lambda$ is sequentially determined and this means that there is a subsequence $(n_k)_{k\in\mathbb{N}}$ (possibly depending on z) with $g(z) = \lim_k f^{n_k}(z)$. By Theorem 1.6 then $g(z) \in \partial B$ as otherwise f would have a fixed point and hence

$$g(E_\lambda) \in \overline{E}_\lambda \cap \partial B, \quad \text{for all } g \in \Gamma(f), \lambda > 0. \tag{7}$$

In the rank one (Hilbert space) case, strong convexity gives $\overline{E}_\lambda \cap \partial B = \{\xi\}$, yielding results for iterates on Hilbert balls that do not generally hold for other JB^*-triples [**21**, Example 4.4]. In other spaces conditions are often imposed (on similar E_λ) to ensure $\overline{E}_\lambda \cap \partial B = \{\xi\}$ [**1, 11, 18, 4, 2**]. Proposition 2.7 below will show that this is not possible here.

LEMMA 2.3. *$\overline{E}_\lambda = c_\lambda + T_\lambda(\overline{B})$, for all $\lambda > 0$.*

Proof. This follows immediately from invertibility of T_λ. □

While boundary components $K_u = u + P_0(u)(B)$, for a tripotent u, are similarly affine, the linear projection $P_0(u)$ is not invertible so proof is required for the following.

PROPOSITION 2.4. *For $x \in \overline{B}$, then $\overline{K}_x = u + P_0(u)(\overline{B})$, where $u := \lim_n x^{2n+1}$.*

Proof. By Theorem 1.5 (ii) it suffices to show that $\overline{P_0(u)B} = P_0(u)(\overline{B})$ for all tripotents u. Boundedness of $P_0(u)$ gives $P_0(u)\overline{B} \subset \overline{P_0(u)B}$. Let $y \in \overline{P_0(u)B}$ and choose $y_n \in P_0(u)B$ with $y_n \to y$. We may write $y_n = P_0(u)x_n$, for $x_n \in B$. Since $P_0(u)$ is a contraction, it follows $y_n \in B$ and $y \in \overline{B}$. Moreover $P_0(u)$ is a projection, so $P_0(u)y_n = y_n$ and $y = P_0(u)y \in P_0(u)\overline{B}$ as required. □

Recall, (1) and (5), that $0 = a_1 = \cdots = a_p \le a_{p+1} \le \cdots \le a_r = 1$. Let now $m := min_k\{p+1 \le k \le r : a_k \ne 0\}$ and define $\widetilde{e} = \sum_{i=m}^r e_i$. Then $\widetilde{e}$ is a non-zero tripotent satisfying

$$e_r \le \widetilde{e} \le e \text{ and } \widetilde{e} = e \Leftrightarrow a_{p+1} \ne 0. \tag{8}$$

Of course, (2),

$$c_\lambda = \sum_{i=m}^r \left(\frac{a_i}{a_i+\lambda}\right) e_i, \text{ and } v_\lambda = \sum_{i=m}^r s_i e_i. \tag{9}$$

We abbreviate $\lim_{\lambda\to 0^+}$ as $\lim_\lambda$.

PROPOSITION 2.5. *$\lim_\lambda c_\lambda = \lim_\lambda v_\lambda = \widetilde{e}$ in Z and $\lim_\lambda T_\lambda = P_0(\widetilde{e})$ in $L(Z)$.*

Proof. For $a_i \ne 0$, $\lim_{\lambda\to 0^+} \frac{a_i}{a_i+\lambda} = 1$ and $\lim_\lambda s_i = \lim_\lambda \sqrt{1-\sqrt{\frac{\lambda}{a_i+\lambda}}} = 1$. The stated limits follow immediately from (9). □

We now establish key properties of the f-invariant domains E_λ. We write w-lim for weak limits.

PROPOSITION 2.6.

$$\bigcap_{\lambda>0} E_\lambda = \emptyset \ \textit{and} \ \emptyset \ne \bigcap_{\lambda>0} \overline{E_\lambda} \subset \partial B.$$

In fact, $\bigcap_{\lambda>0} \overline{E_\lambda} \subset \overline{K}_{\widetilde{e}}$.

Proof. Fix $\lambda > 0$. Direct calculation gives $e = c_\lambda + T_\lambda(e), \widetilde{e} = c_\lambda + T_\lambda(\widetilde{e})$ and hence, from Lemma 2.3, $e, \widetilde{e} \in \overline{E}_\lambda$, for all $\lambda > 0$. In particular $\bigcap_{\lambda>0} \overline{E_\lambda} \neq \emptyset$. Consider then $x \in \bigcap_{\lambda>0} \overline{E_\lambda}$. From Lemma 2.3, for $\lambda > 0$, $x = c_\lambda + T_\lambda(u_\lambda)$, for some $u_\lambda \in \overline{B}$. As $\overline{B}$ is weakly compact (without loss of generality) $u := w\text{-}\lim_\lambda u_\lambda$ exists in $\overline{B}$. Fix $\phi \in Z'$. Proposition 2.5 gives $\lim_\lambda \phi \circ T_\lambda = \phi \circ P_0(\widetilde{e})$ in Z' and hence $\lim_\lambda \phi \circ T_\lambda(u_\lambda) = \phi \circ P_0(\widetilde{e})(u)$. Therefore, using Proposition 2.5,

$$\phi(x) = \lim_\lambda (\phi(c_\lambda) + \phi \circ T_\lambda(u_\lambda)) = \phi(\widetilde{e} + P_0(\widetilde{e})(u)), \text{ for all } \phi \in Z'.$$

Hence $x = \widetilde{e} + P_0(\widetilde{e})(u) \in \widetilde{e} + P_0(\widetilde{e})(\overline{B}) = \overline{K}_{\widetilde{e}}$ (from Lemma 2.4). In other words, $\bigcap_{\lambda>0} \overline{E_\lambda} \subseteq \overline{K}_{\widetilde{e}} \subset \partial B$. As $\bigcap_{\lambda>0} E_\lambda \subset B$ and $\bigcap_{\lambda>0} E_\lambda \subset \bigcap_{\lambda>0} \overline{E_\lambda} \subset \partial B$, it follows that $\bigcap_{\lambda>0} E_\lambda = \emptyset$ and we are done. □

Our next result shows that $\bigcap_{\lambda>0} \overline{E}_\lambda$ may be quite large (see also [**25**, Example 4.4]).

PROPOSITION 2.7.

$$\overline{K}_e \subseteq \bigcap_{\lambda>0} \overline{E_\lambda} \subseteq \overline{K}_{\widetilde{e}} \quad \textit{with equality if, and only if, } a_{p+1} \neq 0.$$

Proof. Let $x \in K_e$. Theorem 1.5 (i) gives $x = e + w$, for $w \perp e$, $\|w\| < 1$. Fix $\lambda > 0$ arbitrary. As above $c_\lambda + T_\lambda(e) = e$. From [**19**, Corollary 3.15 (4)], $T_\lambda = B(v_\lambda, v_\lambda)$ is the identity map on $Z_0(e)$, so $T_\lambda(w) = w$, giving $c_\lambda + T_\lambda(x) = x$. Choose $(\beta_n)_n, 0 < \beta_n < 1, \beta_n \uparrow 1$ and let $x_n := \beta_n x \in B$. Then $x = \lim_n c_\lambda + T_\lambda(x_n) \in \overline{E_\lambda}$. Thus $K_e \subseteq \bigcap_{\lambda>0} \overline{E_\lambda}$ and hence $\overline{K}_e \subseteq \bigcap_{\lambda>0} \overline{E_\lambda}$. Proposition 2.6 and (8) complete the proof. □

Proposition 2.7 motivates the following.

DEFINITION 2.8. For $x \in \overline{B}$, define $Ch(x) := \overline{K}_x$.

PROPOSITION 2.9. *For $x \in \overline{B}$ the following hold.*

(i) $Ch(x)$ is the smallest closed convex set containing the boundary component of x.
(ii) $Ch(x)$ is affinely (or holomorphically) connected $\Leftrightarrow$ x is extreme $\Leftrightarrow$ $Ch(x) = \{x\}$.
(iii)

$$Ch(x) = \overline{K}_u = \bigcup_{d \geq u} K_d = \bigcup_{d \geq u} d + B_0(d), \textit{ for tripotent } u := \lim_n x^{2n+1}.$$

In particular, for $x \in B$, $Ch(x) = \overline{B}$.

Proof. From Theorem 1.5(ii), it suffices to consider tripotents. For a tripotent u, $K_u = u + P_0(u)(B)$ is clearly convex, so $\overline{K}_u$ is a closed convex set, giving (i). Theorem 1.5(iii) gives $Ch(u) = \bigcup_{d \geq u} K_d$ (for tripotents d). In particular, since each K_d is an affine component, $Ch(u)$ is affinely connected $\Leftrightarrow$ u is maximal $\Leftrightarrow$ $K_u = \{u\} \Leftrightarrow Ch(u) = \{u\}$. As the extreme points of $\overline{B}$ are exactly the maximal tripotents, for $x \in \overline{B}$ and $u = \lim_n x^{2n+1}$, x is extreme if, and only if, $x = u$ and thus $\{x\} = \{u\} = K_x = K_u = Ch(x) = Ch(u)$, giving (ii) and (iii). □

$Ch(x)$ is therefore a closed convex holomorphic hull of x. Proposition 3.2 of [**24**] shows that $Ch(x)$ coincides (for finite dimensional Z) with a hull, defined in terms of complex supporting hyperplanes, introduced in [**2**] for convex domains in $\mathbb{C}^n$. The following is now immediate from Proposition 2.7.

COROLLARY 2.10.

$$Ch(e) \subseteq \bigcap_{\lambda>0} \overline{E_\lambda} \subseteq Ch(\widetilde{e}) \quad \textit{with equality if, and only if, } a_{p+1} \neq 0.$$

COROLLARY 2.11. *All constant maps in $\Gamma(f)$ lie in $Ch(\widetilde{e})$. In particular, all constant subsequential limits of (f^n) lie in $Ch(\widetilde{e})$.*

Proof. Suppose $g \in \Gamma(f)$ is constant, equal to μ say. Fix $\lambda > 0$ and $z \in E_\lambda$. From (7) $\mu = g(z) \in \overline{E_\lambda} \cap \partial B$ and thus, from Corollary 2.10, $\mu \in \bigcap_{\lambda>0} \overline{E_\lambda} \subseteq Ch(\widetilde{e})$. □

An immediate consequence of Corollary 2.11 is a short proof of the classical Denjoy-Wolff theorem for the Hilbert ball in finite and infinite dimensions [**10, 6**].

COROLLARY 2.12. *Let Z be a Hilbert space with open unit ball B and $f : B \to B$ be a compact holomorphic map with no fixed point in B. Then there exists ξ in ∂B such that f^n converges locally uniformly on B to the constant mapping ξ.*

Proof. Let $g \in \Gamma(f)$. From (7) $g(B) \subset \partial B$ so $g(B) \subset K_\xi$, for some $\xi \in \partial B$. Since Z has rank 1, $K_\xi = \{\xi\}$ so g is constant and, in addition, $e_r = \widetilde{e} = e = \xi$ and $Ch(\widetilde{e}) = \{\xi\}$. From Corollary 2.11 all subsequential limits of (f^n) are ξ. Since f is compact, every subsequence of f^n has itself a convergent subsequence and therefore (f^n) itself must converge to ξ. □

Our final result gives a condition guaranteeing that $\widetilde{e}$ itself is the only possible constant map in $\Gamma(f)$.

COROLLARY 2.13. *If $\widetilde{e}$ is extreme then ξ is extreme and the only possible constant map in $\Gamma(f)$ is ξ and, in particular, the only possible constant subsequential limit of (f^n) is ξ. This holds, in particular, if ξ is extreme and $a_1 \neq 0$.*

Proof. If $\widetilde{e}$ is extreme then $Ch(\widetilde{e}) = \{\widetilde{e}\}$ and, since from (8), $\widetilde{e} \leq e$ then $\widetilde{e} = e$ and e extreme gives $\xi = e = \widetilde{e}$ and the result follows from Corollary 2.11. If $\xi = e + v$ is extreme then $\xi = e$ and $v = 0$. If also $a_1 \neq 0$ then (8) gives $\widetilde{e} = e = \xi$ so $\widetilde{e}$ is extreme and we are done. □

EXAMPLE 2.14. Define a holomorphic function on the bidisc Δ^2, which is the open unit ball of the JB*-triple $(\mathbb{C}^2, \|\cdot\|_\infty)$, by $f(z,w) = (g_{\frac{1}{2}}(z)^2, g_{\frac{1}{2}}(w)g_{\frac{2}{11}}(w))$, where g_a is the Möbius map on Δ which takes 0 to a. One can show that (f^n) converges locally uniformly to the constant $(1, \frac{1}{2})$. Note that f has no fixed point in Δ^2 and extends continuously to (and, holomorphically, beyond) $\overline{\Delta^2}$ so that the Wolff-point ξ must then be a boundary fixed point. While $(1,1)$ and $(1, \frac{1}{2})$ are both boundary fixed points, only $(1, \frac{1}{2})$ is attracting so $\xi = (1, \frac{1}{2})$. Here $e = e_r = (1,0)$ so (8) gives $\widetilde{e} = e$ and $Ch(\widetilde{e}) = 1 \times \overline{\Delta}$.

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