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On a class of singular elliptic systems

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Abstract

We study the semilinear elliptic system

$$\begin{cases} -\Delta u = u^{-p} + v^{-q}, & u > 0 & \text{in } \Omega, \\ -\Delta v = u^{-r} + v^{-s}, & v > 0 & \text{in } \Omega, \\ u = v = 0 & & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a smooth and bounded domain, $p, q, r, s > 0$. Under suitable ranges of exponents we obtain various results regarding the well posedness of our system.

Key words: Singular elliptic system, negative exponents, boundary behavior

1991 MSC: 35J55, 35B40, 35B50

1 Introduction and the main results

We are concerned in this paper with qualitative properties of solutions to the system

$$\begin{cases} -\Delta u = u^{-p} + v^{-q}, & u > 0 & \text{in } \Omega, \\ -\Delta v = u^{-r} + v^{-s}, & v > 0 & \text{in } \Omega, \\ u = v = 0 & & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a smooth and bounded domain, $p, q, r, s > 0$.

Solutions (u, v) to (1) are understood in the classical sense, that is, $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$.

The system (1) appears as a natural extension of the single singular problem

$$\begin{cases} -\Delta u = u^{-p}, & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{on } \partial\Omega, \end{cases} \quad (2)$$

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which was considered, among other works, in [3,9,15]. A particular feature of (2) in the case $p > 0$, and in contrast to the case $p < -1$ is that it has a unique solution. This fact will be used in dealing with (1) in order to study the existence of solutions.

Another singular elliptic system recently investigated in the literature is

$$\begin{cases} -\Delta u = u^{-p}v^{-q}, & u > 0 & \text{in } \Omega, \\ -\Delta v = u^{-r}v^{-s}, & v > 0 & \text{in } \Omega, \\ u = v = 0 & & \text{on } \partial\Omega. \end{cases} \quad (3)$$

The case $p, q, r, s > 0$ represents the singular counterpart of the standard Lane-Emden system and was discussed in [5,6,10,14,17].

The case $q, s > 0$ and $p, r < 0$ corresponds to the Gierer-Meinhardt system [11,12] with homogeneous Dirichlet boundary conditions (see [1,2,4,7–10]). Such a system describes the pattern formation of spatial tissue structures in morphogenesis, a biological phenomenon discovered by Trembley [16] in 1744.

Coming back to our system (1), we first state a nonexistence result.

Theorem 1.1 (Nonexistence) *Assume that one of the following two conditions hold.*

(i) $2r < 1 + \max\{1, s\}$ and

$$\min \left\{ \frac{1}{1 + \max\{1, s\}}, 1 - \max \left\{ \frac{1}{2}, \frac{r}{1 + \max\{1, p\}} \right\} \right\} > \frac{1}{q}, \quad (4)$$

or

(ii) $2q < 1 + \max\{1, p\}$ and

$$\min \left\{ \frac{1}{1 + \max\{1, p\}}, 1 - \max \left\{ \frac{1}{2}, \frac{q}{1 + \max\{1, s\}} \right\} \right\} > \frac{1}{r}. \quad (5)$$

Then, the systems (1) has no solutions.

Corollary 1.2 *Assume that one of the following conditions hold:*

$$q > 1 + \max\{1, s\} \quad \text{and} \quad 2r < 1 + \max\{1, p\},$$

or

$$r > 1 + \max\{1, p\} \quad \text{and} \quad 2q < 1 + \max\{1, s\}.$$

Then, the systems (1) has no solutions.

More clearly but perhaps less precise, Corollary 1.2 states that if one of the exponents q and r is too small and the other is too big, then the systems (1) has no solutions.

In particular, from Corollary 1.2 we deduce that the system (1) has no solutions if

$$q > 2 + s \quad \text{and} \quad 2r < 1 + p,$$

or

$$r > 2 + p \quad \text{and} \quad 2q < 1 + s.$$

We shall next be concerned with the existence of a solution to (1). Our main result in this case is the following.

Theorem 1.3 (Existence) *Assume $p, q, r, s > 0$ satisfy*

$$q < 1 + \max\{1, s\} \quad \text{and} \quad r < 1 + \max\{1, p\}. \quad (6)$$

Then, the system (1) has at least one classical solution.

Corollary 1.4 *Assume $0 < q, r < 2$. Then, the system (1) has at least one classical solution.*

In other words, and in contrast to Corollary 1.2, if q and r are both small, then a classical solution to system (1) always exists, regardless to the size of p and s .

We should point out that there are regions for exponents $p, q, r, s > 0$ where we do not know whether the system (1) admits solutions. For instance, if

$$q > \max\{1, s\} \quad \text{and} \quad r > 1 + \max\{1, p\}$$

then, none of the conditions (4), (5) or (6) hold. In particular, for q and r large enough, we are not able to decide the (non)existence of a solution to (1).

Theorem 1.5 (C^1 -regularity of solutions up to the boundary)

Let (u, v) be a classical solution of (1).

- (i) If $p < 1$ and $2q < 1 + \max\{1, s\}$ then $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$;*
- (ii) If $s < 1$ and $2r < 1 + \max\{1, p\}$ then $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$.*

We are next concerned with the uniqueness of a solution to system (1). For the singular systems considered in [4–6,8] the uniqueness of the solution was deduced for some ranges of exponents p, q, r and s which imply either $u \in C^1(\overline{\Omega})$ or $v \in C^1(\overline{\Omega})$.

Theorem 1.6 (Uniqueness) *Assume $p, q, r, s > 0$ satisfy (6) and that one of the following holds.*

- (i) $\frac{q}{1 + \max\{1, s\}} < \frac{\max\{1, p\}}{1 + \max\{1, p\}}$ and $qr < 1$;*
- (ii) $\frac{r}{1 + \max\{1, p\}} < \frac{\max\{1, s\}}{1 + \max\{1, s\}}$ and $qr < 1$.*

Then, the system (1) has a unique classical solution.

Corollary 1.7 *Assume $p, q, r, s > 0$ satisfy*

$$2q < 1 + \max\{1, s\}, \quad 2r < 1 + \max\{1, p\} \quad \text{and} \quad qr < 1. \quad (7)$$

Then, the system (1) has a unique classical solution.

Corollary 1.8 *Assume $p, s > 0$ and $0 < q, r < 1$. Then, the system (1) has a unique classical solution.*

2 Some preliminary results

In this section we collect some basic results which will be useful in proving our main results. In the sequel Ω will be assumed to be a smooth and bounded domain of \mathbb{R}^N . We also denote by $\delta(x)$ the distance from $x \in \Omega$ to the boundary $\partial\Omega$. Given two positive functions f, g defined in Ω , we shall use $f \sim g$ to signify that $c^{-1}f \leq g \leq cf$ in Ω for some constant $c > 1$.

We start this section with the following comparison principle whose proof relies on the maximum principle.

Lemma 2.1 *Let $K \in C(\Omega)$ and assume $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

- (i) $\Delta u + u^{-p} + K(x) \leq 0 \leq \Delta v + v^{-p} + K(x)$ in Ω ,
- (ii) $u, v > 0$ in Ω and $v = 0$ on $\partial\Omega$.

Then $u \geq v$ in Ω .

The following result stems from Crandall, Rabinowitz and Tartar [3].

Lemma 2.2 *For any $p > 0$ there exists a unique solution $w \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

$$\begin{cases} -\Delta w = w^{-p} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (8)$$

Furthermore, w has the following asymptotic behavior

$$w(x) \sim \begin{cases} \delta(x) & \text{if } 0 < p < 1, \\ \delta(x) \log^{1/2} \frac{1}{\delta(x)} & \text{if } p = 1, \\ \delta(x)^{\frac{2}{1+p}} & \text{if } p > 1. \end{cases} \quad (9)$$

In particular, there exists $C > 0$ such that

$$w \geq C\delta(x)^{\frac{2}{1+\max\{1,p\}}} \quad \text{in } \Omega, \quad (10)$$

and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$w \leq C_\varepsilon \delta(x)^{\frac{2-\varepsilon}{1+\max\{1,p\}}} \quad \text{in } \Omega. \quad (11)$$

Lemma 2.3 *Let $w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$0 \leq -\Delta w \leq C\delta(x)^{-a} \quad \text{in } \Omega, \quad w = 0 \text{ in } \partial\Omega,$$

where $a \in (0, 2)$ and $C > 0$. Then, $w \in C^{0,\gamma}(\bar{\Omega})$ for some $\gamma \in (0, 1)$ and

- (i) If $a \in (0, 1)$ then $w \in C^2(\Omega) \cap C^{1-a}(\bar{\Omega})$ and $w(x) \leq c\delta(x)$ in Ω , for some $c > 0$;
- (ii) If $a = 1$ then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $w(x) \leq c_\varepsilon \delta(x)^{1-\varepsilon}$ in Ω ;
- (iii) If $a \in (1, 2)$ then $w(x) \leq c\delta(x)^{2-a}$ in Ω , for some $c > 0$.

Proof. Let \mathcal{G} be the Green's function for the negative Laplace operator. This yields

$$u(x) = - \int_{\Omega} \mathcal{G}(x, y) \Delta u(y) dy,$$

for all $x \in \Omega$. Hence

$$\begin{aligned} |u(x_1) - u(x_2)| &\leq - \int_{\Omega} |\mathcal{G}(x_1, y) - \mathcal{G}(x_2, y)| \Delta u(y) dy \\ &\leq c \int_{\Omega} |\mathcal{G}(x_1, y) - \mathcal{G}_x(x_2, y)| \delta(y)^{-a} dy, \end{aligned}$$

for all $x_1, x_2 \in \Omega$. Next, using the method in [13, Theorem 1.1] we have

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\gamma \quad \text{for some } 0 < \gamma < 1.$$

Hence $u \in C^{0,\gamma}(\bar{\Omega})$.

(i) Suppose $0 < a < 1$. Then,

$$\nabla u(x) = - \int_{\Omega} \mathcal{G}_x(x, y) \Delta u(y) dy \quad \text{for all } x \in \Omega,$$

and

$$\begin{aligned} |\nabla u(x_1) - \nabla u(x_2)| &\leq - \int_{\Omega} |\mathcal{G}_x(x_1, y) - \mathcal{G}_x(x_2, y)| \Delta u(y) dy \\ &\leq c \int_{\Omega} |\mathcal{G}_x(x_1, y) - \mathcal{G}_x(x_2, y)| \delta(y)^{-a} dy. \end{aligned}$$

The same technique as in [13, Theorem 1.1] yields

$$|\nabla u(x_1) - \nabla u(x_2)| \leq C|x_1 - x_2|^{1-a} \quad \text{for all } x_1, x_2 \in \Omega.$$

Therefore $u \in C^{1,1-a}(\bar{\Omega})$. This also implies $w(x) \leq c\delta(x)$ in Ω , for some $c > 0$.

(iii) Denote by λ_1 (resp. φ_1) the first eigenvalue (resp. eigenfunction) of $-\Delta$ in Ω . By normalization, we can assume $\varphi_1 > 0$ in Ω . Let now $\bar{w} := M\varphi_1^{2-a}$. A straightforward calculation yields

$$-\Delta \bar{w} = M(2-a)\varphi_1^{-a} \left[(a-1)|\nabla \varphi_1|^2 + \lambda_1 \varphi_1^2 \right] \quad \text{in } \Omega.$$

By Hopf boundary point lemma and the maximum principle, we have $(a-1)|\nabla\varphi_1|^2 + \lambda_1\varphi_1^2 > c > 0$ in Ω , for some positive constant $c > 0$. Thus, we can choose $M > 1$ suitably large such that

$$-\Delta\bar{w} \geq C\delta^{-a}(x) \geq -\Delta w \quad \text{in } \Omega.$$

This yields $w(x) \leq \bar{w}(x) \leq c\delta(x)^{2-a}$ in Ω .

(ii) This follows directly from part (iii) by noting that for $a = 1$ and $\varepsilon > 0$ we have $-\Delta w \leq C_\varepsilon\delta(x)^{-1-\varepsilon}$ in Ω .

Our last result in this section concerns the problem

$$\begin{cases} -\Delta w = w^{-p} + K(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (12)$$

where $p > 0$ is a constant and $K \in C(\Omega)$ is a positive function in Ω .

Lemma 2.4 (i) *If $K(x)$ satisfies*

$$\int_{\Omega} \delta(x)K(x)dx = \infty,$$

then (12) has no solutions.

(ii) *If $K(x) \geq c\delta(x)^{-a}$ in Ω , for some $a \geq 2$ then (12) has no solutions.*

(iii) *If $K(x) \leq C\delta(x)^{-a}$ in Ω , for some $a \in (0, 2)$, then (12) has a unique solution $w \in C^2(\Omega) \cap C(\bar{\Omega})$. Furthermore, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$w(x) \leq C_\varepsilon\delta(x)^{\min\left\{\frac{2-\varepsilon}{1+\max\{1,p\}}, 2-\varepsilon-\max\{1,a\}\right\}} \quad \text{in } \Omega. \quad (13)$$

Proof. Assume there exists a classical solution $w \in C^2(\Omega) \cap C(\bar{\Omega})$ of (12). For $\varepsilon > 0$ denote

$$\Omega_\varepsilon := \{x \in \Omega : \delta(x) > \varepsilon\}.$$

Thus, Ω_ε is a smooth domain provided $\varepsilon > 0$ is small enough. Denote by $\lambda_{1,\varepsilon}$ (resp. $\varphi_{1,\varepsilon}$) the first eigenvalue (resp. eigenfunction) of $-\Delta$ in Ω_ε . We can normalize φ_1 and $\varphi_{1,\varepsilon}$ and assume $\|\varphi_1\|_\infty = \|\varphi_{1,\varepsilon}\|_\infty = 1$. Consider the problem

$$\begin{cases} -\Delta w = (w + \varepsilon)^{-p} + K(x) & \text{in } \Omega_\varepsilon, \\ w > 0 & \text{in } \Omega_\varepsilon, \\ w = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (14)$$

Clearly, the solution w of (12) is a supersolution to (14) while $\underline{w} = m\varphi_{1,\varepsilon}$ is a subsolution of (14) provided $m > 0$ is small enough. By Lemma 2.1 we have $w \geq \underline{w}$ in Ω_ε .

Thus, (14) has a solution w_ε which, by elliptic regularity, satisfies $w_\varepsilon \in C^2(\overline{\Omega_\varepsilon})$. Let us next multiply by $\varphi_{1,\varepsilon}$ in (14) and integrate over Ω_ε . We obtain

$$\lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} w dx \geq \lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} w dx \geq \lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} w_\varepsilon dx \geq \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} K(x) dx.$$

A passage to the limit in the above estimate together with Fatou lemma yield

$$\begin{aligned} M := \lambda_1 \int_{\Omega} w dx &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} K(x) dx \\ &\geq \int_{\Omega} \liminf_{\varepsilon \rightarrow 0} \varphi_{1,\varepsilon} \chi_{\Omega_\varepsilon} K(x) dx \\ &= \int_{\Omega} \varphi_1 K(x) dx \\ &\geq C \int_{\Omega} \delta(x) K(x) dx = \infty, \end{aligned}$$

contradiction. Hence, (12) has no classical solution.

(ii) This follows from part (i) since

$$\int_{\Omega} \delta(x)^{-\gamma} dx = \infty, \quad \text{for all } \gamma \geq -1.$$

This can be seen by using local co-ordinates near the boundary of Ω as explained in [15].

(iii) Let w_1 be the solution of (8) and denote by \tilde{w} be solution of

$$\begin{cases} -\Delta \tilde{w} = K(x) & \text{in } \Omega, \\ \tilde{w} > 0 & \text{in } \Omega, \\ \tilde{w} = 0 & \text{on } \partial\Omega. \end{cases} \quad (15)$$

Note that such a solution \tilde{w} always exist due to the fact that $K(x) \leq C\delta(x)^{-a}$ in Ω , with $a \in (0, 2)$. It is easy to check that $M\varphi_1^{2-a}$ is a supersolution of (15) while the zero function is a subsolution. This simple observation together with the maximum principle yield the existence and uniqueness of a solution $\tilde{w} \in C^2(\Omega) \cap C(\overline{\Omega})$ of (15). By Lemma 2.3 we have

$$\tilde{w}(x) \leq \begin{cases} C\delta(x) & \text{if } 0 < a < 1, \\ C_\varepsilon \delta(x)^{1-\varepsilon} & \text{if } a = 1, \\ C\delta(x)^{2-a} & \text{if } 1 < a < 2. \end{cases}$$

We can summarize the above estimates by noting that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\tilde{w}(x) \leq C_\varepsilon \delta(x)^{2-\varepsilon-\max\{1,a\}} \quad \text{in } \Omega. \quad (16)$$

Let w_1 be the unique solution of (8). Thus, w_1 is a subsolution while $w_1 + \tilde{w}$ is a supersolution of (12). Thus, (12) admits a solution w such that

$$w_1 \leq w \leq w_1 + \tilde{w} \quad \text{in } \Omega. \quad (17)$$

The uniqueness of the solution to (12) follows from Lemma 2.1. The estimate (13) follows from (17), (11) and (16).

3 Proof of Theorem 1.1.

Assume that condition (i) in Theorem 1.1 holds and there exists a solution (u, v) of system (1). From (4) we can find $\varepsilon > 0$ small such that

$$\sigma := \min \left\{ \frac{2 - \varepsilon}{1 + \max\{1, s\}}, 2 - \varepsilon - \max \left\{ 1, \frac{2r}{1 + \max\{1, p\}} \right\} \right\} > \frac{2}{q}.$$

Let w_1 be the unique solution of (8). Then, by the comparison principle and (10), there exists a constant $C > 0$ such that

$$u \geq w_1 \geq C\delta(x)^{\frac{2}{1+\max\{1, p\}}} \quad \text{in } \Omega.$$

Using this fact in the second equation of (1) we find

$$-\Delta v \leq v^{-s} + c\delta(x)^{-\frac{2r}{1+\max\{1, p\}}} \quad \text{in } \Omega,$$

for some constant $c > 0$. Since $r < 1 + \max\{1, p\}$, we can use Lemma 2.4(iii) to deduce

$$v \leq C\delta(x)^\sigma \quad \text{in } \Omega,$$

where $C > 0$ is a constant. Using now this last estimate in the first equation of (1) we find

$$-\Delta u \geq u^{-p} + c\delta(x)^{-\sigma q} \quad \text{in } \Omega,$$

for some constant $c > 0$. But this is impossible according to Lemma 2.4(ii), since $\sigma q > 2$.

4 Proof of Theorems 1.3 and Theorem 1.5

Proof of Theorem 1.3. Let w_1, w_2 be solutions of (8) and

$$\begin{cases} -\Delta w_2 = w_2^{-s} & \text{in } \Omega, \\ w_2 > 0 & \text{in } \Omega, \\ w_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (18)$$

respectively. By Lemma 2.2 we have

$$w_1(x) \geq C\delta(x)^{\frac{2}{1+\max\{1, p\}}} \quad \text{and} \quad w_2(x) \geq C\delta(x)^{\frac{2}{1+\max\{1, s\}}} \quad \text{in } \Omega. \quad (19)$$

Hence, by (6) and Lemma 2.4, we may find $w_3, w_4 \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\left\{ \begin{array}{ll} -\Delta w_3 = w_3^{-p} + w_2^{-q} & \text{in } \Omega, \\ w_3 > 0 & \text{in } \Omega, \\ w_3 = 0 & \text{on } \partial\Omega, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} -\Delta w_4 = w_4^{-s} + w_1^{-r} & \text{in } \Omega, \\ w_4 > 0 & \text{in } \Omega, \\ w_4 = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (20)$$

Set

$$\mathcal{A} = \left\{ (u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \begin{array}{ll} w_1 \leq u \leq w_3 & \text{in } \Omega \\ w_2 \leq v \leq w_4 & \text{in } \Omega \end{array} \right\}.$$

For any $(u, v) \in \mathcal{A}$, we consider (Tu, Tv) the unique solution of the decoupled system

$$\left\{ \begin{array}{ll} -\Delta(Tu) = (Tu)^{-p} + v^{-q}, Tu > 0 & \text{in } \Omega, \\ -\Delta(Tv) = (Tv)^{-s} + u^{-r}, Tv > 0 & \text{in } \Omega, \\ Tu = Tv = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (21)$$

Since $u, v \in C(\overline{\Omega})$ and $u \geq w_1, v \geq w_2$ in Ω , by (19) we have

$$w_1^{-r}(x) \leq c\delta(x)^{-\frac{2q}{1+\max\{1,s\}}}, \quad w_2^{-q}(x) \leq c\delta(x)^{-\frac{2r}{1+\max\{1,p\}}} \quad \text{in } \Omega.$$

Thus, from (6) and Lemma 2.4, we deduce that the solution (Tu, Tv) of (21) is well posed. Define next

$$\mathcal{F} : \mathcal{A} \rightarrow C(\overline{\Omega}) \times C(\overline{\Omega}) \quad \text{by} \quad \mathcal{F}(u, v) = (Tu, Tv) \quad \text{for any } (u, v) \in \mathcal{A}. \quad (22)$$

Thus, the existence of a solution to system (1) follows once we prove that \mathcal{F} has a fixed point in \mathcal{A} . To this aim, we shall prove that \mathcal{F} satisfies the conditions:

$$\mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}, \quad \mathcal{F} \text{ is compact and continuous.}$$

Then, by Schauder's fixed point theorem we deduce that \mathcal{F} has a fixed point in \mathcal{A} , which, by standard elliptic estimates, is a classical solution to (1).

Let us show first that $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}$. Indeed, comparing (8), (21) and using Lemma 2.1 we easlily deduce $Tu \geq w_1$ in Ω . Further, since $v \geq w_2$ we have

$$\Delta w_3 + w_3^{-p} + w_2^{-q} \leq 0 \leq \Delta(Tu) + (Tu)^{-p} + w_2^{-q} \quad \text{in } \Omega$$

and $Tu, w_3 > 0$ in Ω , $Tu = w_3 = 0$ on $\partial\Omega$. By Lemma 2.1 it follows $Tu \leq w_3$ in Ω , and thus $w_1 \leq Tu \leq w_3$ in Ω . Similarly $w_2 \leq Tv \leq w_4$ in Ω which shows that $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}$.

Out next aim is to show that \mathcal{F} is compact and continuous. Let $(u, v) \in \mathcal{A}$. Since u and v are bounded, we deduce from (21) that $Tu, Tv \in C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$. Since the embedding $C^{0,\gamma}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ is compact, it follows that \mathcal{F} is also compact.

It remains to prove that \mathcal{F} is continuous. To this aim, let $\{(u_n, v_n)\} \subset \mathcal{A}$ be such that $u_n \rightarrow u$ and $v_n \rightarrow v$ in $C(\overline{\Omega})$ as $n \rightarrow \infty$. Using the fact that \mathcal{F} is compact, there exists

$(U, V) \in \mathcal{A}$ such that up to a subsequence we have

$$Tu_n \rightarrow U, \quad Tv_n \rightarrow V \quad \text{in } C(\overline{\Omega}) \quad \text{as } n \rightarrow \infty.$$

On the other hand, by standard elliptic estimates, the sequences $\{Tu_n\}$ and $\{Tv_n\}$ are bounded in $C^{2,\beta}(\overline{\omega})$ ($0 < \beta < 1$) for any smooth open set $\omega \subset\subset \Omega$. Therefore, up to a diagonally subsequence, we have

$$Tu_n \rightarrow U, \quad Tv_n \rightarrow V \quad \text{in } C^2(\overline{\omega}) \quad \text{as } n \rightarrow \infty,$$

for any smooth open set $\omega \subset\subset \Omega$. Passing to the limit in the definition of Tu_n and Tv_n we find that (U, V) satisfies

$$\begin{cases} -\Delta U = U^{-p} + v^{-q}, & U > 0 & \text{in } \Omega, \\ -\Delta V = V^{-s} + u^{-r}, & V > 0 & \text{in } \Omega, \\ U = V = 0 & & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of (21), we have that $Tu = U$ and $Tv = V$. Hence

$$Tu_n \rightarrow Tu, \quad Tv_n \rightarrow Tv \quad \text{in } C(\overline{\Omega}) \quad \text{as } n \rightarrow \infty.$$

This proves that \mathcal{F} is continuous.

We are now in a position to apply the Schauder's fixed point theorem. Thus, there exists $(u, v) \in \mathcal{A}$ such that $\mathcal{F}(u, v) = (u, v)$, that is, $Tu = u$ and $Tv = v$. By standard elliptic estimates, it follows that (u, v) is a classical solution of system (1).

Proof of Theorem 1.5. (i) Assume $p < 1$, $2q < 1 + \max\{1, s\}$ and let (u, v) be a solution to (1). By Lemma 2.1 we have $u \geq w_1$, $v \geq w_2$ in Ω , where w_1, w_2 are solutions of (8) and (18) respectively. Using the asymptotic behavior described in (10) it follows that

$$u(x) \geq C\delta(x), \quad v(x) \geq C\delta(x)^{\frac{2}{1+\max\{1, s\}}} \quad \text{in } \Omega,$$

where $C > 0$ is a constant. We next use these estimates for u and v in the first equation of our system (1). We find

$$-\Delta u \leq C \left[\delta(x)^{-p} + \delta(x)^{-\frac{2q}{1+\max\{1, s\}}} \right] \quad \text{in } \Omega.$$

By our assumption on p, q, r, s and Lemma 2.3(i) it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

The proof of (ii) is similar.

5 Proof of Theorem 1.6

We shall prove Theorem 1.6 under the assumption (i). The case where (ii) holds can be treated similarly. Our arguments are divided into two steps.

Step 1: For any solution (u, v) of (1) we have $u \sim w_1$, where w_1 satisfies (8).

With similar arguments to those used in the proof of Theorem 1.3 we have

$$w_1 \leq u \leq w_3 \quad \text{in } \Omega, \quad w_2 \leq v \leq w_4 \quad \text{in } \Omega,$$

where w_2, w_3 and w_4 are solutions of (18) and (20). Using the above estimates, Lemma 2.2 and (19) we have

$$-\Delta u \leq w_1^{-p} + \delta(x)^{-\frac{2q}{1+\max\{1,s\}}} \quad \text{in } \Omega. \quad (23)$$

If $0 < p \leq 1$ then condition (i) reads $\frac{2q}{1+\max\{1,s\}} < 1$ so by (17) we have

$$u \sim w_1 \sim \begin{cases} \delta(x) & \text{if } 0 < p < 1, \\ \delta(x) \log^{1/2} \frac{1}{\delta(x)} & \text{if } p = 1. \end{cases}$$

If $p > 1$ then from (i) and (23) we deduce

$$-\Delta u \leq c\delta(x)^{-\frac{2p}{1+p}} + \delta(x)^{-\frac{2q}{1+\max\{1,s\}}} \leq C\delta(x)^{-\frac{2p}{1+p}} \quad \text{in } \Omega,$$

where $c, C > 0$ are constants. Thus, by Lemma ... we find

$$u(x) \leq C\delta(x)^{\frac{2}{1+p}} \leq C_0 w_1(x) \quad \text{in } \Omega.$$

Hence, in both the above cases $u \sim w_1$.

Step 2: System (1) has a unique classical solution.

Let (u_1, v_1) and (u_2, v_2) be two solutions of system (1). From Step 1 we have $u_1 \sim w_1 \sim u_2$. This means that we can find a constant $C > 1$ such that $Cu_1 \geq u_2$ and $Cu_2 \geq u_1$ in Ω .

We claim that $u_1 \leq u_2$ in Ω . Supposing the contrary, let

$$M := \inf\{A > 1 : u_1 \leq Au_2 \text{ in } \Omega\}.$$

By our assumption, we have $M > 1$. From $u_1 \leq Mu_2$ in Ω , it follows that

$$-\Delta v_1 = u_1^{-r} + v_1^{-s} \geq M^{-r} u_2^{-r} + v_1^{-s} \quad \text{in } \Omega.$$

Therefore v_2 is a solution and $M^r v_1$ is a supersolution of

$$\begin{cases} -\Delta w = w^{-s} + u_2^{-r}, & w > 0 & \text{in } \Omega, \\ w = 0 & & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1 we obtain $M^r v_1 \geq v_2$ in Ω which reads

$$v_1 \geq M^{-r} v_2 \quad \text{in } \Omega.$$

The above estimate yields

$$-\Delta u_1 = u_1^{-p} + v_1^{-q} \leq u_1^{-p} + M^{rq} v_2^{-q} \quad \text{in } \Omega.$$

It follows that u_2 is a solution and $M^{-rq} u_1$ is a subsolution of

$$\begin{cases} -\Delta w = w^{-p} + v_2^{-q}, & w > 0 & \text{in } \Omega, \\ w = 0 & & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1 we now deduce

$$u_1 \leq M^{rq} u_2 \quad \text{in } \Omega.$$

Since $M > 1$ and $qr < 1$, the above inequality contradicts the minimality of M . Hence, $u_1 \geq u_2$ in Ω . Similarly we deduce $u_1 \leq u_2$ in Ω , so $u_1 \equiv u_2$ which also yields $v_1 \equiv v_2$. Therefore, the system (1) has a unique solution. This completes the proof of Theorem 1.6.

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