DYNAMICS OF BIHOLOMORPHIC SELF-MAPS ON BOUNDED SYMMETRIC DOMAINS

P. MELLON

Abstract

Let g be a fixed-point free biholomorphic self-map of a bounded symmetric domain B. The sequence of iterates (g^n) is known not to converge locally uniformly on B. However, $g = g_a \circ T$, for a linear isometry T, $a = g(0)$ and a transvection g_a, and it is possible to determine the dynamics of g_a. We prove that the sequence of iterates (g_a^n) converges locally uniformly on B if, and only if, a is regular, in which case, the limit is a holomorphic map of B onto a boundary component (surprisingly though, generally not the boundary component of $\frac{a}{\|a\|}$). We prove (g_a^n) converges to a constant for all non-zero a if, and only if, B is a complex Hilbert ball. The results are new even in finite dimensions, where every element is regular.

Introduction

In 1926 Wolff [20] and Denjoy [4] proved that if g is a holomorphic fixed-point free self-map of Δ, then its iterates (g^n) converge locally uniformly on Δ to a unimodular constant. This was first generalised to the finite dimensional Hilbert ball by Hervé [6] in 1963, and then again, by others, two decades later [14], [16]. Shortly afterwards, the result was shown to fail for the infinite dimensional Hilbert ball [19], while it is easy to see that it fails for other, even finite dimensional, bounded symmetric domains, as shown for the bidisc $\Delta \times \Delta$ in Example 1 of [3]. It may therefore appear hopeless to consider the iterates of such maps on arbitrary bounded symmetric domains, which is, nonetheless, the purpose of this paper.

Let Z be a JB^*-triple with open unit ball B. As is long known [8] B is a bounded symmetric domain and every bounded symmetric domain can be realised in this way. Let g be a biholomorphic self-map of B which has no fixed point in B. Then g has a unique decomposition into linear and non-linear parts, and the non-linear part is tractable, namely, we can trace its iterates. We recall that the group, G, of biholomorphic automorphisms of B, has the form [8] $G = PK = KP$, where K is all linear elements of G (equivalently, all linear isometries of Z [12]), and $P = \exp(\mathfrak{p})$, where $\mathfrak{p}$ is the Lie algebra generated by the quadratic vector fields, $X_\alpha(z) = (\alpha - \{z, \alpha, z\})\frac{\partial}{\partial z}$ on Z. In other words, each $g \in G$ may be written $g = g_a \circ T$, where $a = g(0), T \in K$, and $g_a \in P$ is called a transvection.

We show that the local uniform convergence properties of the sequence of iterates (g_a^n), unlike those of (g^n), are good, and it is our aim here to establish exactly the dynamics of (g_a^n). We use results on the boundary properties of bounded symmetric domains [13] to reduce the local uniform convergence properties of (g_a^n) to the norm convergence properties of the sequence $(g_a^n(0))$ in Z and we can thereby locate all accumulation points of (g_a^n) (with respect to the topology of local uniform convergence on B) as holomorphic maps of B onto certain boundary components. We now present our main result, noting first, that if Z is finite rank, in particular if it is finite dimensional, then every element is (what is known as) regular, giving a much simpler statement than below.

THEOREM 0.1. *Let Z be a JB^*-triple with open unit ball B and $a \in B$.*

The sequence of iterates (g_a^n) has an accumulation point, with respect to the topology of local uniform convergence on B, if, and only if, a is regular. Moreover, if a is regular, then the iterates (g_a^n) converge locally uniformly on B to a holomorphic map $g_e : B \to K_e$, where K_e is the (holomorphic) boundary component of e and e is the support tripotent of a.

We note that the limit point g_e is not in general, even in finite dimensions, a constant map, and more crucially, its image, the boundary component K_e, may not contain the point $\frac{a}{\|a\|}$, for $a \neq 0$. In fact, the following result shows that while, as one might expect, the above simplifies greatly in the case of the Hilbert ball, such simplification actually characterises the Hilbert ball within the class of all bounded symmetric domains.

THEOREM 0.2. *Let Z be a JB^*-triple with open unit ball B. The following are equivalent.*

(i) (g_a^n) converges locally uniformly on B to a constant map, for all non-zero $a \in B$.

(ii) Z is (isometrically J^*-isomorphic to) a complex Hilbert space.

We note that the results are new even in finite dimensions. For a survey of the classical case $B = \Delta$ we refer to [2].

1. Notation and Background

Throughout $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, H will denote a complex Hilbert space, $L(X, Y)$ the space of all continuous linear maps from a complex Banach space X to a complex Banach space Y, $L(X) = L(X, X)$ and $GL(X)$ is all invertible elements in $L(X)$.

1.1. *JB^*-triples*

Definition 1.1. A JB^*-triple is a complex Banach space Z with real tri-linear mapping $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x, y, z\}$ is complex linear and symmetric in x and z, and is complex anti-linear in y;
(ii) the map $z \mapsto \{x, x, z\}$, denoted $x \Box x$, is Hermitian, $\sigma(x \Box x) \geq 0$ and $\|x \Box x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum;
(iii) for all $a, b, x, y, x \in Z$ the Jordan triple identity holds, namely,

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

The triple product is continuous [5], namely, $\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|$. JB^*-triples that are also Banach dual spaces are known as JBW^*-triples, and have been much studied. It is known [8] that every bounded symmetric domain is biholomorphically equivalent to the open ball of a JB^*-triple and vice versa.

Example 1.2. (i) H is a JB^*-triple for product

$$\{x, y, z\} = \frac{\langle x, y\rangle z + \langle z, y\rangle x}{2}$$

(ii) If X is a locally compact Hausdorff space, then $C_0(X)$, the space of all continuous $\mathbb{C}$-valued functions which vanish at infinity, is a JB^*-triple for $\{x, y, z\} = x\overline{y}z$.

All C^*-algebras, JB^*-algebras and J^*-algebras are JB^*-triples, so the class of triples is large and interesting. Since the triple product encodes the holomorphic structure of B (for example, the Kobayashi metric on B [17] and [18]), it is a key tool in the study of holomorphic maps on all of these spaces.

Let Z be a JB^*-triple with open unit ball B. The most important linear maps on Z are the Bergman operators

$$B(x,y) = I - 2x\Box y + Q(x)Q(y) \in L(Z)$$

as they play a central role in the geometry of B. Here $x\Box y \in L(Z)$ is the map $z \mapsto \{x,y,z\}$ and $Q(x)$ maps $z \mapsto \{x,z,x\}$ so that $Q(x)Q(y) \in L(Z)$.

We note that for all $x \in B$, $\sigma(B(x,x)) > 0$ and $B_x := B(x,x)^{\frac{1}{2}}$ exists in the sense of the holomorphic functional calculus on $L(Z)$ [9].

1.2. *Tripotents and Peirce decompositions*

A concept of orthogonality exists in Z and we say $x, y \in Z$ are orthogonal, $x \perp y$, if $x\Box y = 0$ (or equivalently [15] if $y\Box x = 0$). Analogues of idempotents for an algebra also exist in the form of tripotents, where $e \in Z$ is a tripotent if $\{e,e,e\} = e$.

Every tripotent e induces a Peirce splitting

$$Z = Z_1(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_0(e)$$

where $Z_k(e)$ is the k eigenspace of $e\Box e$.

A tripotent e is said to be maximal if $Z_0(e) = 0$ and said to be minimal if $Z_1(e) = \mathbb{C}e$. It is known that for JB^*-triples real and complex extreme points coincide and are precisely the set of all maximal tripotents.

Z is said to have rank r if the cardinality of every set of non-zero pairwise orthogonal tripotents is $\leq r$ and there is at least one set of cardinality r. We say Z is finite rank if $r < \infty$. Of course, if Z is finite dimensional then it is finite rank.

We say $a \in Z$ is algebraic if there exists a finite family of pairwise orthogonal minimal tripotents $e_1, ..., e_r$ and $\lambda_1 = \|a\| \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ such that $a = \lambda_1 e_1 + \cdots + \lambda_r e_r$. For $a \neq 0$ algebraic, the decomposition is unique if each e_k is non-zero and $\lambda_1 = \|a\| \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. If Z is finite rank then every $a \in Z$ is algebraic. We refer to [15] for additional details.

1.3. *Spectral Theory*

There is a well developed spectral theory for JB^*-triples, as follows. Given $a \in Z$, let Z_a denote the smallest closed subtriple of Z containing a. Then there is a compact set $S = -S \subset \mathsf{R}$ such that Z_a is triple isomorphic to

$$C^-(S) = \{f \in C(S) : f(-s) = -f(s) \ \forall s \in S\}.$$

We note in particular that 0 can never be an isolated point of S and also that $\mathrm{Sp}(0) = \emptyset$. Letting $S^+ = \{s \in S : s > 0\}$, the following are triple isomorphisms $Z_a \cong C^-(S) \cong C_0(S^+)$, where we identify a with the map $a(s) = s \ \forall s \in S$ (or S^+) and elements of $C_0(S^+)$ are identified with maps in $C^-(S)$ by extension in the obvious way. Proposition 3.5 of [11] gives all necessary details. The set S, denoted $\mathrm{Sp}(a)$, is called the (triple) spectrum of a and we write $\mathrm{rank}(a) :=\dim(Z_a)$. Using this spectral theory, an odd functional calculus exists on Z, [9] or [10].

1.4. *Automorphism Group*

The structure of the group, G, of all biholomorphic automorphisms of B is long known. We refer to section 3 of [10] for details. G is a Lie group whose Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ consists of all complete holomorphic vector fields on Z, with $\mathfrak{k} = \mathrm{aut}(Z)$ being all triple derivations of Z and

$$\mathfrak{p} = \{X_\alpha : \alpha \in Z\}, \text{ where } X_\alpha(z) = (\alpha - \{z, \alpha, z\})\frac{\partial}{\partial z}.$$

In particular, for each $X \in \mathfrak{g}$, the map $t \mapsto \exp(tX)$ is a 1-parameter subgroup of G.

At the group level we have the decomposition $G = KP = PK$, where $K = \mathrm{Aut}(Z)$ is the subgroup of all surjective linear isometries (or equivalently, all triple isomorphisms) of Z and $P = \exp(\mathfrak{p})$ is a real submanifold, though not a subgroup, of G.

Each $g \in G$ therefore has a unique representation $g = g_a \circ T$, where $T \in K, a = g(0)$, and $g_a \in P$, called a transvection, is given by

$$g_a(z) = a + B_a(I + z\Box a)^{-1}z, \quad z \in B$$

where $B_a := B(a, a)^{\frac{1}{2}} \in \mathrm{GL}(Z)$. Clearly $g_0 = I$.

Moreover, $g_a = \exp(X_\alpha)$, where $\alpha = \tanh^{-1}(a)$ is defined in terms of the odd functional calculus on Z (and α and a generate the same subtriple of Z).

EXAMPLE. (i) If $B = \Delta$ then $g_a(z) = \frac{z+a}{1+\bar{a}z}$, $z \in \Delta$.

(ii) If $B = B_H$ is a complex Hilbert ball then

$$g_a(z) = \left(\sqrt{1 - \|a\|^2} P_a + Q_a\right) \left(\frac{z + a}{a + \langle z, a\rangle}\right), \quad z \in B$$

where P_a is projection onto $\frac{a}{\|a\|}$ and $Q_a = I - P_a$, $a \neq 0$.

1.5. *Boundary Components*

The boundary components of a bounded symmetric domain B are classified [13] in terms of holomorphic maps called boundary transvections.

We recall that $A \subset \overline{B}$, $A \neq \emptyset$ is a (holomorphic) boundary component of B if A is minimal with respect to the fact that

$$\text{either } f(\Delta) \subset A \text{ or } f(\Delta) \subset \overline{B} \setminus A, \ \forall f : \Delta \to Z \text{ holomorphic with } f(\Delta) \subset \overline{B}.$$

We denote the boundary component of B containing a as K_a.

For $c \in \partial B$, the local uniform limit of g_a as $a \in B$ approaches c, namely $\lim_{a\to c} g_a$, exists as a holomorphic map : $B \to Z$, is denoted g_c and called a boundary transvection. Such maps classify the boundary components of B containing tripotents, namely, if $e \in Z$ is a tripotent then

$$K_e = g_e(B) = e + B_0(e), \text{ where } B_0(e) = B \cap Z_0(e)$$

and also $K_e = g_a(B)$, for all $a \in K_e$. Of course, if $e = 0$ then $K_0 = B$ and this is the unique open boundary component of B. We note that for $c \in \partial B$, the boundary transvection g_c, unlike g_a for $a \in B$, is neither biholomorphic nor injective in general. The map $(z, a) \mapsto g_a(z)$ is, however, a continuous map on $\overline{B} \times \overline{B} \setminus (\partial B \times \partial B)$. We refer to [13], in particular Theorem 2.1 and Proposition 4.3, for proofs and details of all results in this subsection.

2. Results: Algebraic Elements

Let Z be an arbitrary JB^*-triple with open unit ball B. For holomorphic functions on B, convergence is understood throughout to mean local uniform convergence on B. Let $a \in B$. We begin by examining the iterates, g_a^n, of g_a. Fix $n \in N$.

As $g_a = \exp(X_\alpha)$, where $\alpha = \tanh^{-1}(a)$ and since the map $t \mapsto \exp(tX_\alpha)$ is a 1-parameter subgroup of G then (recall that P is not a subgroup of G)

$$g_a^n = (\exp(X_\alpha))^n = \exp(nX_\alpha) = \exp(X_{n\alpha}) \in P$$

so that $g_a^n = g_{c_n}$, for $c_n \in B$, and evaluating at 0 gives $g_a^n(0) = c_n$.

In other words, for all $a \in B$ and $n \in \mathbb{N}$

$$g_a^n = g_{g_a^n(0)}. \tag{2.1}$$

This simple identity is crucial, since in light of section 1.5 above, it immediately simplifies the process of finding accumulation points of (g_a^n) with respect to the topology of local uniform convergence on B, by allowing us instead to focus on finding accumulation points of the sequence $(g_a^n(0))$ in Z with respect to the norm topology. To this end, it is important to notice that the sequence $(g_a^n(0))$ lies entirely in the JB^*-subtriple, Z_a, generated by a. If now Z_a is just $\mathbb{C}a$, then we are already almost done. Although this is generally not the case, it is true for Hilbert spaces, where $Z_a = \mathbb{C}a$, for all a in H. For Hilbert space enthusiasts therefore, who may wish to forgo Jordan theory, we present this separately.

THEOREM 2.1. *Let H be a complex Hilbert space with open unit ball $B, a \in B\backslash\{0\}$. The sequence of iterates (g_a^n) converges locally uniformly on B to the constant map $\frac{a}{\|a\|}$.*

PROOF. For $e \in \partial B$ and $\lambda, \mu \in \Delta$ then $Z_e = \mathbb{C}e$ gives $g_{\lambda e}(\mu e) = \left(\frac{\lambda+\mu}{1+\bar{\lambda}\mu}\right) e = t_\lambda(\mu)e$, where t_λ is the classical Möbius transformation on the disc, $t_\lambda(x) = \frac{\lambda+x}{1+\bar{\lambda}x}$, for $x \in \Delta$. It follows easily by induction that $g_{\lambda e}^n(0) = t_\lambda^n(0)e,$ for $n \in \mathbb{N}$. Now fix $a \in B\backslash\{0\}$. Then $g_a^n(0) = g_{\|a\|e}^n(0) = t_{\|a\|}^n(0)e,$ for $e = \frac{a}{\|a\|} \in \partial B$. From the Denjoy-Wolff theorem on Δ, or directly, one sees that the real sequence $(t_{\|a\|}^n(0))$ converges to 1 and therefore $(g_a^n(0))$ converges in norm to $\frac{a}{\|a\|}$. But then, as described in section 1.5, $(g_{g_a^n(0)})$ converges locally uniformly on B to the boundary transvection $g_{\frac{a}{\|a\|}}$. Since $g_{\frac{a}{\|a\|}}(B) = K_{\frac{a}{\|a\|}}$, and every point in ∂B is complex extreme, then $K_{\frac{a}{\|a\|}} = \{\frac{a}{\|a\|}\}$, so that $g_{\frac{a}{\|a\|}}$ is the constant map $\frac{a}{\|a\|}$. The result then follows from (2.1).

Comment. The following two properties of Hilbert spaces are key to the above proof.

1. $g_a^n(0) \in \mathsf{C}a$, for all $n \in \mathsf{N}$ (this ensures that for $a \neq 0$ a limit, $g_{\frac{a}{\|a\|}}$, exists).
2. Every point on ∂B is extreme, namely, B is strictly convex (this ensures that the limit, $g_{\frac{a}{\|a\|}}$, is constant).

While property 2 does not hold for triples of rank > 1, we can ask if property 1 generalises to some such triples. The answer is negative, as we see below.

Proposition 2.2. *Let Z be a JB^*-triple with open unit ball B. The following are equivalent.*
(i) Z is (isometrically J^*-isomorphic to) a complex Hilbert space.
(ii) B is strictly convex.
(iii) For all $e \in \partial B$, e is a maximal tripotent.
(iv) For all $e \in \partial B$, e is a minimal tripotent.
(v) For all $a \in B$, $Z_a \cong \mathsf{C}a$.
(vi) For all $a \in B$, $g_a^n(0) \in \mathsf{C}a$. [Note that for $a = 0$, $g_a = I$.]

Proof. Though (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v)$\Rightarrow$(vi) are straightforward, we sketch them for completeness.

(i) $\Leftrightarrow$ (ii): is well known.

(ii) $\Leftrightarrow$ (iii): For a JB^*-triple real and complex extreme points of $\overline{B}$ coincide with the set of maximal tripotents in Z.

(iii) $\Rightarrow$ (iv): Assume (iii) holds. Let $e \in \partial B$. By (iii) e is a maximal tripotent. Suppose e is not minimal. Then there is a decomposition, $e = e_1 + e_2$, where e_1, e_2 are non-zero tripotents and $e_1 \perp e_2$. By (iii) e_1 must be a maximal tripotent which contradicts $e_2 \perp e_1$. Therefore e is minimal and (iv) holds.

(iv) $\Rightarrow$ (iii): Assume (iv) holds. Let $e \in \partial B$. By (iv) e is a tripotent. Suppose e is not maximal, so that there exists a tripotent $f \neq 0$ with $f \perp e$. Then $g = e + f$ is a norm 1 tripotent and by (iv) is minimal, contradicting $g = e + f$. So e is maximal and (iii) holds.

(iv) $\Rightarrow$ (v): Given $a \neq 0$ in B then (iv) implies that $Z_{\frac{a}{\|a\|}} = \mathsf{C}\frac{a}{\|a\|}$ and hence $Z_a = \mathsf{C}a$. The result is trivially true if $a = 0$. Hence (v) holds.

(v) $\Rightarrow$ (iv): Assume (v) holds. Then, for $a \neq 0$, rank(a)=dim(Z_a) $= 1$ so $a = \|a\|e$ for some rank 1 tripotent e. In other words if $a \neq 0$ then $\frac{a}{\|a\|}$ is a minimal tripotent.

(v) $\Rightarrow$ (vi) is obvious.

It remains to prove (vi) $\Rightarrow$ (v). We assume therefore that $g_a^n(0) \in \mathsf{C}a$, for all a in B, $n \in \mathsf{N}$. As the condition in (v) is trivially true for $a = 0$, we fix a in $B\backslash\{0\}$. Identifying Z_a with $C^-(S)$, $S =$Sp(a), and a with the real map $s \mapsto s$, we have $g_a(z) = \frac{z+a}{1+za}$, for $z \in Z_a$. In particular then

$$g_a^3(0) = g_a\left(g_a^2(0)\right) = g_a\left(\frac{2a}{1+a^2}\right) = \frac{3a+a^3}{1+2a^2} \in \mathsf{C}a.$$

Writing $\frac{3a+a^3}{1+2a^2} = \lambda a$, $\lambda \in \mathsf{C}$, gives $(1-2\lambda)a^3 = (\lambda - 3)a$. Since $a \neq 0$ then $\lambda \neq \frac{1}{2}$ and hence $a^3 = \left(\frac{\lambda-3}{1-2\lambda}\right) a \in \mathsf{C}a$. Now if $a^{2n-1} \in \mathsf{C}a$, say μa, then $a^{2n+1} = \{a, a^{2n-1}, a\} = \{a, \mu a, a\} = \mu a^3 \in \mathsf{C}a$ from above. By induction therefore $a^{2n+1} \in \mathsf{C}a$, for all $n \in \mathsf{N}$, and hence $p_n(a) \in \mathsf{C}a$, for all odd polynomials p_n. Since every $f \in C^-(S)$ can be written as the limit of a sequence of odd polynomials, it follows that $f(a) \in \mathsf{C}a$ for all $f \in C^-(S)$. In other words, $Z_a \subset \mathsf{C}a$ and hence $Z_a = \mathsf{C}a$. This completes (vi) $\Rightarrow$ (v).

We return now to arbitrary JB^*-triples. For the space of holomorphic functions on B, we note that nets, rather than sequences, are required to determine the topology. In particular, the set of all accumulation points of (g_a^n) with repect to the topology of local uniform convergence on B is precisely the set of limit points of all its locally uniformly convergent subnets. The following result shows that for (g_a^n) this is conveniently the same as the set of all its (locally uniform) subsequential limits.

THEOREM 2.3. *Let Z be a JB^*-triple with open unit ball B and $a \in B$. The set of accumulation points of (g_a^n) with respect to the toplogy of local uniform convergence on B is*

$$\{g_c : c \in \Gamma_a\},$$

where Γ_a is the set of all subsequential limits of $(g_a^n(0))$ in Z with respect to the norm topology. In particular, for (g_a^n) the set of accumulation points is exactly the set of its subsequential limits.

Proof. Let h be an accumulation point of (g_a^n), namely, there is a subnet $(n_\alpha)_\alpha$ of N such that $g_a^{n_\alpha} \underset{\alpha}{\to} h$ locally uniformly on B. In particular then, $g_a^{n_\alpha}(0) \underset{\alpha}{\to} h(0)$ in Z. From (2.1) $g_a^{n_\alpha} = g_{g_a^{n_\alpha}(0)}$ and therefore (section 1.5) $g_a^{n_\alpha} = g_{g_a^{n_\alpha}(0)} \underset{\alpha}{\to} g_{h(0)}$ locally uniformly on B. Uniqueness of limits gives $h = g_{h(0)}$. Since the topology on Z is determined by sequences, the set of limits points of all (convergent) subnets of $(g_a^n(0))$ is the same as the set of all its subsequential limits. In other words, $h(0) \in \Gamma_a$ and $h = g_{h(0)} \in \{g_c : c \in \Gamma_a\}$. On the other hand, let $c \in \Gamma_a$, that is, $c = \lim_k g_a^{n_k}(0)$. As above $g_a^{n_k} = g_{g_a^{n_k}(0)} \underset{k}{\to} g_c$ locally uniformly on B, completing the proof.

We note now that if e is a minimal tripotent in Z then $Z_1(e) = \mathsf{C}e$ so $Z_e = \mathsf{C}e$ and, exactly as in the Hilbert ball, $g_{\lambda e}^n$ is then easy to compute, for $\lambda \in \Delta$.

Lemma 2.4. *Let e be a minimal tripotent. Then*

$$g_{\lambda e}^n = g_{t_\lambda^n(0)e} \quad \textit{for } n \in \mathsf{N}, \lambda \in \Delta,$$

where $t_\lambda(x) = \frac{\lambda + x}{1+\bar{\lambda}x}$ is the usual Möbius map on Δ. In particular, for $\lambda \neq 0$, $(g_{\lambda e}^n)$ converges locally uniformly on B to the boundary transvection $g_{\frac{\lambda}{|\lambda|}e}$.

Proof. Let $\lambda \in \Delta$. Since e is minimal, $g_{\lambda e}^n(0) \in Z_e = \mathsf{C}e$ and we compute, for $\mu \in \overline{\Delta}$, $g_{\lambda e}(\mu e) = \left(\frac{\lambda+\mu}{1+\mu\bar{\lambda}}\right) e = t_\lambda(\mu)e$. By induction then $g_{\lambda e}^n(0) = t_\lambda^n(0)e$. If $\lambda \neq 0$ then $t_\lambda^n(0)$ converges to $\frac{\lambda}{|\lambda|}$ and the result follows, as before, from (2.1) and section 1.5.

Our next motivation comes from the fact that if $a \perp b$ then g_a is "orthogonal" to g_b in the sense that

$$g_{a+b} = g_a \circ g_b.$$

This will allow us to extend Lemma 2.4 above to finite linear combinations of tripotents, namely, to algebraic elements of Z. In order to do this we present the following simple lemmata. Proofs are given for completeness as ready references are elusive. In the finite rank case, proof of the following follows from [15].

LEMMA 2.5. *Let $a, b \in Z$ with $a \perp b$. Then $\|a+b\| = \max\{\|a\|, \|b\|\}$. In particular, if $a, b \in B$, then $a + b \in B$ whenever $a \perp b$.*

PROOF. As $a \perp b$, $a \Box b = b \Box a = 0$, so that $\|a + b\|^2 = \|(a + b) \Box (a + b)\| = \|a \Box a + b \Box b\|$. Now $\|(a \Box a + b \Box b) \frac{a}{\|a\|}\| = \|a\|^2$ as $\{b, b, a\} = 0$ and $\|(a \Box a + b \Box b) \frac{b}{\|b\|}\| = \|b\|^2$ as $\{a, a, b\} = 0$, giving $\|a + b\| \geq \max\{\|a\|, \|b\|\}$. In the opposite direction, calculating in $Z_x \cong C^-(S)$, it is easy to see that $\|x\|^{2n+1} = \|x^{2n+1}\|$, for all $n \in \mathsf{N}$. Induction arguments, together with use of the Jordan triple identity and the fact that $a \perp b$, then easily give $(a + b)^{2n+1} = a^{2n+1} + b^{2n+1}$. Therefore $\|a + b\|^{2n+1} = \|(a + b)^{2n+1}\| = \|a^{2n+1} + b^{2n+1}\| \leq \|a\|^{2n+1} + \|b\|^{2n+1}$, so $\|a + b\| \leq \lim_{n\to\infty} \left(\|a\|^{2n+1} + \|b\|^{2n+1}\right)^{\frac{1}{2n+1}} = \max\{\|a\|, \|b\|\}$.

The following result, for $n = 1$ and elements $e, v \in B$ with e a tripotent and $v \perp e$, is used in the proof of Proposition 4.3 of [13], though a proof is not given there.

LEMMA 2.6. *Let $a, b \in B$ be orthogonal. Then*

$$g^n_{a+b} = g^n_a \circ g^n_b, \text{ for all } n \in \mathsf{N}.$$

PROOF. As in Lemma 2.5, $a \perp b$ implies $(a+b)^{2n+1} = a^{2n+1} + b^{2n+1}$. If p then is any odd polynomial, it follows that $p(a + b) = p(a) + p(b)$ and, again by induction, $p(a) \perp p(b)$. This extends to any odd continuous function f, such that $f(a+b)$, $f(a)$, $f(b)$ are defined on Z in terms of the odd functional calculus, namely, $a \perp b$ implies $f(a + b) = f(a) + f(b)$ and $f(a) \perp f(b)$. Apply this now to

$$f(t) = \tanh^{-1}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n + 1}.$$

As in section 1, $g_a = \exp(X_\alpha)$, $g_b = \exp(X_\beta)$, for $\alpha = \tanh^{-1}(a)$, $\beta = \tanh^{-1}(b)$ and X_α is the vector field, $X_\alpha(z) = (\alpha - \{z, \alpha, z\}) \frac{\partial}{\partial z}$. From above, $a \perp b$ implies $\tanh^{-1}(a + b) = \tanh^{-1}(a) + \tanh^{-1}(b) = \alpha + \beta$ and, in addition, $\alpha \perp \beta$. Therefore $g_{a+b} = \exp(X_{\tanh^{-1}(a+b)}) = \exp(X_{\alpha+\beta}) = \exp(X_\alpha + X_\beta)$. As $[X_\alpha, X_\beta] = X_{\alpha\Box\beta-\beta\Box\alpha}$ then $\alpha \perp \beta$ implies $[X_\alpha, X_\beta] = 0$. This gives $\exp(X_\alpha + X_\beta) = (\exp X_\alpha) \circ$

$(\exp X_\beta)$, and $g_{a+b} = g_a \circ g_b$. Clearly $g_{a+b} = g_{b+a}$ so that $g_a \circ g_b = g_b \circ g_a$ and therefore $g^n_{a+b} = g^n_a \circ g^n_b$ by induction, for all $n \in \mathsf{N}$.

The above result extends by continuity to include boundary transvections.

LEMMA 2.7. *Let $a, b \in \overline{B}$ be orthogonal. Then $g^n_{a+b} = g^n_a \circ g^n_b$, for all $n \in \mathsf{N}$.*

We are now in a position to deal with the iterates (g^n_a), where $a \in B$ is algebraic. We recall a is algebraic if $\dim(Z_a) < \infty$. If $a \neq 0$, this means $a = \lambda_1 e_1 + ... + \lambda_r e_r$, where $\|a\| = \lambda_1 \geq ... \geq \lambda_r > 0$, $e_1, ..., e_r$ are mutually orthogonal minimal tripotents, $r = \dim(Z_a)$ and we refer to $e := e_1 + ... + e_r = \mathrm{supp}(a)$ as the support tripotent of a. Note that $\mathrm{supp}(0) = 0$. It turns out that the dynamics of g_a on B are determined entirely by g_e, giving our second main result.

THEOREM 2.8. *Let Z be a JB^*-triple and $a \in B$ be algebraic. Then (g^n_a) converges locally uniformly on B to the holomorphic map g_e, where $e = supp(a)$.*

PROOF. The case $a = 0$ is trivial, as $g_a = g_e = I$. Take $a \in B\backslash\{0\}$. We write $a = \lambda_1 e_1 + ... + \lambda_r e_r$, where $\|a\| = \lambda_1 \geq ... \geq \lambda_r > 0$, $e_1, ..., e_r$ are mutually orthogonal minimal tripotents, and $r = \mathrm{rank}\ a$. Fix $n \in \mathsf{N}$. Then

$$\begin{aligned} g^n_a &= g^n_{\lambda_1 e_1} \circ \cdots \circ g^n_{\lambda_r e_r} && \text{by Lemma 2.6} \\ &= g_{t^n_{\lambda_1}(0) e_1} \circ \cdots \circ g_{t^n_{\lambda_r}(0) e_r} && \text{by Lemma 2.4} \\ &= g_{t^n_{\lambda_1}(0) e_1 + \cdots + t^n_{\lambda_r}(0) e_r} && \text{by Lemma 2.6 again.} \end{aligned}$$

Since for $1 \leq i \leq r$, $\lambda_i \in (0, 1)$, it is clear that $t^n_{\lambda_i}(0) \underset{n}{\to} 1$. Therefore $\lim_n t^n_{\lambda_1}(0) e_1 + \cdots + t^n_{\lambda_r}(0) e_r = e_1 + \cdots + e_r = e$ and therefore, as before, g^n_a converges locally uniformly on B to g_e and we are done.

COMMENTS. 1. If Z is finite rank then all elements in Z are algebraic and, of course, every finite dimensional JB^*-triple is finite rank.

2. If Z is a JBW^*-triple then the algebraic elements are dense, cf. [11] section 2.

We now take a closer look at Theorem 2.8. The transvection g_e maps B onto the holomorphic boundary component, K_e, of e. This yields our first major surprise for, as the following examples show, the boundary components of e and $\frac{a}{\|a\|}$ are generally different, so that Theorem 2.8 diverges from the Hilbert space result (Theorem 2.1) in several distinct ways.

(i) (g_a^n) does not necessarily converge to a constant map. See Example 2.9 below.

(ii) Even if (g_a^n) does converge to a constant, that constant is not generally $\frac{a}{\|a\|}$. In fact, that constant is not generally in $K_{\frac{a}{\|a\|}}$. See Example 2.10 below.

(iii) Where it exists, the limit of (g_a^n) does not generally map into the boundary component $K_{\frac{a}{\|a\|}}$. See Example 2.9 below.

As before, convergence here refers to local uniform convergence on B.

Example 2.9. Let Z be $\mathbb{C}^3$ with ℓ_∞ norm, $\|(z_1, z_2, z_3)\| = \max_{1\leq i\leq 3} |z_i|$, so $B = \Delta^3$. Consider the rank 2 element $a = (\frac{1}{2}, \frac{1}{4}, 0) \in B$. Then

$$a = \frac{1}{2}e_1 + \frac{1}{4}e_2,\ e_1 = (1, 0, 0), e_2 = (0, 1, 0) \text{ and } e = \operatorname{supp}(a) = e_1 + e_2 = (1, 1, 0)$$

so that $g_e(B) = K_e = 1 \times 1 \times \Delta$. On the other hand,

$$\frac{a}{\|a\|} = e_1 + \frac{1}{2}e_2 \text{ and } K_{\frac{a}{\|a\|}} = K_{e_1} = 1 \times \Delta \times \Delta.$$

Clearly $K_e \cap K_{\frac{a}{\|a\|}} = \emptyset$. Note that $g_a^n(z) = (t_{\frac{1}{2}}^n(z_1), t_{\frac{1}{4}}^n(z_2), z_3)$ with $z = (z_1, z_2, z_3)$, where $t_{\frac{1}{2}}^n, t_{\frac{1}{4}}^n$ are Möbius maps on Δ that converge locally uniformly on Δ to 1. So (g_a^n) converges locally uniformly on B to g_e where $g_e(z) = (1, 1, z_3)$.

Example 2.10. Let Z be $\mathbb{C}^2$ with ℓ_∞ norm, so $B = \Delta^2$. Take

$$a = \left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}e_1 + \frac{1}{4}e_2, \text{where } e_1 = (1, 0), e_2 = (0, 1) \text{ and } e = \operatorname{supp}(a) = e_1 + e_2 = (1, 1).$$

Here e is a complex extreme point, so $K_e = \{e\}$ and g_e is the constant map e. On the other hand

$$\frac{a}{\|a\|} = \left(1, \frac{1}{2}\right) = e_1 + \frac{1}{2}e_2 \text{ and } K_{\frac{a}{\|a\|}} = K_{(1,0)} = 1 \times \Delta.$$

Of course, (g_a^n) converges locally uniformly on Δ^2 to the constant map e.

The following proposition clarifies the situation.

Proposition 2.11. *Let $a \in B$ be algebraic and $e = supp(a)$. If $a \neq 0$ then*

$$K_{\frac{a}{\|a\|}} = K_e \text{ if and only if } \frac{a}{\|a\|} \text{ is a tripotent.}$$

Proof. Let $a \in B\backslash\{0\}$ be algebraic and $e = \mathrm{supp}(a)$. From section 1.5, $K_{\frac{a}{\|a\|}} = K_e$ if and only if

$$\frac{a}{\|a\|} \in K_e = g_e(B) = e + B_0(e), \; B_0(e) = Z_0(e) \cap B.$$

In other words, $K_{\frac{a}{\|a\|}} = K_e$ if and only if $v := e - \frac{a}{\|a\|} \in B_0(e)$. Write $a = \lambda_1 e_1 + \cdots + \lambda_r e_r$, $\|a\| = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0$ and $e = e_1 + \cdots + e_r = \mathrm{supp}(a)$. Then $\frac{a}{\|a\|} = \frac{a}{\lambda_1} = e_1 + \frac{\lambda_2}{\lambda_1} e_2 + \cdots + \frac{\lambda_r}{\lambda_1} e_r$ and

$$v = \left(1 - \frac{\lambda_2}{\lambda_1}\right) e_2 + \cdots + \left(1 - \frac{\lambda_r}{\lambda_1}\right) e_r.$$

By Lemma 2.5, $v \in B$, so $v \in B_0(e)$ if, and only if, $v \square e = 0$. Since

$$v \square e = \left(1 - \frac{\lambda_2}{\lambda_1}\right) e_2 \square e_2 + \cdots + \left(1 - \frac{\lambda_r}{\lambda_1}\right) e_r \square e_r, \text{ we have}$$

$$\begin{aligned} v \in B_0(e) \Leftrightarrow & \quad \frac{\lambda_i}{\lambda_1} = 1, \quad i = 2, ..., r \\ \Leftrightarrow & \quad \|a\| = \lambda_1 = \lambda_2 = ... = \lambda_r \\ \Leftrightarrow & \; a = \|a\| e \quad \Leftrightarrow \frac{a}{\|a\|} = e \Leftrightarrow \quad \frac{a}{\|a\|} \text{ is a tripotent.} \end{aligned}$$

3. Results: Regular Elements

We now use spectral theory to extend Theorem 2.8 above to those elements a of Z, for which the spectrum, $\mathrm{Sp}(a)$, does not contain 0. In [11, Lemma 4.1] $0 \notin \mathrm{Sp}(a)$ is shown to be equivalent to several previously studied concepts of regularity, namely, $0 \notin \mathrm{Sp}(a) \Leftrightarrow a$ is regular $\Leftrightarrow a$ is strongly regular $\Leftrightarrow a$ has a generalized inverse. For this reason, if $0 \notin \mathrm{Sp}(a)$ we simply refer to a as being regular. Since 0 is never an isolated point of the spectrum, it follows that every algebraic element is regular. In particular, 0 is regular as $\mathrm{Sp}(0) = \emptyset$. As noted in section 1, we have, for $S = \mathrm{Sp}(a)$

and $S^+ = \{s \in S : s > 0\}$, triple isomorphisms $Z_a \cong C^-(S) \cong C_0(S^+)$, where a is identified with the the map $a(s) = s, \forall s \in S^+$.

We now extend the concept of support tripotent, used earlier for algebraic elements, to regular elements of Z. Let $a \in Z$ be regular and $S =$Sp(a). Since $0 \notin S$ the map $e(s) = 1, \forall s \in S^+$ is continuous, so that e defines a tripotent in $Z_a \cong C_0(S^+)$, which we refer to as the support tripotent of a, written $e = \text{supp}(a)$. Theorem 2.8 can now be generalised.

THEOREM 3.1. *Let Z be a JB^*-triple and $a \in B$ be regular. Then (g_a^n) converges locally uniformly on B to the holomorphic map g_e, where $e = supp(a)$.*

PROOF. The case $a = 0$ is trivial. Let $a \in B \setminus \{0\}$ be regular and $e = \text{supp}(a)$. We prove that $e = \lim_n g_a^n(0)$ in Z by identifying Z_a with $C_0(S^+)$, where $a(s) = s,\ \forall s \in S^+$. We note that $S^+ \subset [0, \|a\|]$. As $g_a(z) = \frac{z+a}{1+\bar{a}z},\ z \in C_0(S^+)$, we have $g_a(0)(s) = a(s) = s = t_s(0),\ s \in S^+,$ where $t_s(x) = \frac{s+x}{1+sx}$ is the usual Möbius map on Δ. By induction then

$$g_a^n(0)(s) = t_s^n(0),\ s \in S^+,\ n \in \mathsf{N}.$$

Therefore

$$\begin{aligned} \|g_a^n(0) - e\| &= \|g_a^n(0) - e\|_{Z_a} = \|g_a^n(0) - e\|_{C_0(S^+)} \\ &= \sup_{s \in S^+} |g_a^n(0)(s) - e(s)| = \sup_{s \in S^+} |t_s^n(0) - e(s)| = \sup_{s \in S^+} |t_s^n(0) - 1|. \end{aligned}$$

It is easy to see by induction that, for all $n \in \mathsf{N}$, the real map $s \mapsto t_s^n(0)$ is strictly increasing on any interval $[0, \alpha]$, $\alpha < 1$. Since S is compact and $0 \notin S$, then $s_0 := \inf S^+ > 0$, so that $S^+ \subset [s_0, \|a\|]$ and therefore, for all $n \in \mathsf{N}$, the map $s \mapsto t_s^n(0)$ is strictly increasing on S^+. In particular, $0 < t_{s_0}^n(0) \leq t_s^n(0) < 1\ \forall s \in S^+, n \in \mathsf{N}$. Therefore, for all $n \in \mathsf{N}$, $|t_s^n(0) - 1| \leq |t_{s_0}^n(0) - 1|\ \ \forall s \in S^+$ and hence

$$\|g_a^n(0) - e\| = \sup_{s \in S^+} |t_s^n(0) - 1| \leq |t_{s_0}^n(0) - 1|.$$

Since $s_0 > 0$, we know that $\lim_n t_{s_0}^n(0) = 1$ and therefore $\lim_{n\to\infty} g_a^n(0) = e$. As before, $(g_a^n) = (g_{g_a^n(0)})$ then converges locally uniformly on B to g_e and we are done.

A further look at the above proof however reveals that g_e is the only possible accumulation point of (g_a^n).

THEOREM 3.2. *Let Z be a JB^*-triple and $a \in B$. The set of all (local uniform) accumulation points of the sequence of iterates (g_a^n) is non-empty if, and only if, a is regular. In particular, (g_a^n) converges locally uniformly on B if, and only if, a is regular.*

PROOF. Theorem 3.1 gives one direction. In the opposite direction, suppose that an accumulation point of (g_a^n) exists. In other words, (g_a^n) has some subnet, $(g_a^{n_\alpha})_\alpha$ say, which converges locally uniformly on B to the map h. Then $h(0) = \lim_\alpha g_a^{n_\alpha}(0)$ exists in $Z_a \cong C^-(S)$, where $S = \mathrm{Sp}(a)$. In particular,

$$\|h(0) - g_a^{n_\alpha}(0)\|_Z = \|h(0) - g_a^{n_\alpha}(0)\|_{Z_a} = \sup_{s \in S} |h(0)(s) - g_a^{n_\alpha}(0)(s)| \longrightarrow_\alpha 0.$$

Moreover, as in the proof of Theorem 3.1 above, $g_a^{n_\alpha}(0)(s) = t_s^{n_\alpha}(0),\ \forall s \in S,\ \forall \alpha$, so that $h(0)(s) = \lim_\alpha g_a^{n_\alpha}(0)(s) = \lim_\alpha t_s^{n_\alpha}(0),\ \forall s \in S$. On the other hand, since for $s \in S$, $\lim_{n\to\infty} t_s^n(0)$ exists and, in fact,

$$\lim_{n\to\infty} t_s^n(0) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$$

it follows that all subnets of $(t_s^n(0))$ must also converge to this same limit.

Therefore, for $s \in S$,

$$h(0)(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}.$$

Since $h(0) \in C^-(S)$, it is continuous on S, which means that 0 cannot be an accumulation point of S. On the other hand, 0 is never an isolated point of S (cf. section 1.3 above) and hence $0 \notin S = \mathrm{Sp}(a)$. In other words, a is regular (and $h(0) = e$).

Theorems 3.1 and 3.2 together therefore tell us that the only possible (local uniform) accumulation point of (g_a^n) is the holomorphic map g_e, where $e = \mathrm{supp}(a)$, and if a is not regular, then such a support tripotent does not exist in Z. Therefore our results above are somehow best possible.

The proofs of Theorems 3.1 and 3.2 also contain the proof of the following alternative characterisation of regularity.

COROLLARY 3.3. *Let Z be a JB^*-triple with open unit ball B and $a \in B$. Then a is regular if, and only if, $\lim_n g_a^n(0)$ exists in Z. In particular, a has a support tripotent in Z if, and only if, $\lim_n g_a^n(0)$ exists in Z.*

COMMENT. Corollary 3.3 suggests a way to extend the concept of a support tripotent to arbitrary elements of Z, by seeking such a tripotent in Z^{**}, if one is not available in Z. Namely, if $\lim_n g_a^n(0)$ does not exist in Z, then we may seek a limit in Z^{**} using the weak*-topology. Since Z^{**} is a JBW^*-triple, its closed unit ball is weak*-compact and therefore $(g_a^n(0))$ considered in $\overline{B}_{Z^{**}}$ must have a weak*-convergent subnet. Such a subnet, by arguments similiar to those in Theorem 3.2, will converge weak* to a tripotent $\tilde{e}$, say, in Z^{**}. In other words, for arbitrary a in Z, a concept of support tripotent in Z^{**} can be defined in terms of the weak*-topology. This has already been done in [1], though from quite a different perspective. We note however that, since transvections g_a are, in general, not weak*-weak* continuous [7], this in no way conflicts with what we have done above.

The final result of our paper characterizes Hilbert spaces as being the only JB^*-triples where the iterates (g_a^n) converge locally uniformly to a constant map, for $a \in B\backslash\{0\}$. This is further evidence, if any is required, that the study of the dynamics of a holomorphic map on the Hilbert ball does not generalise in any useful way to the other bounded symmetric domains, as the strict convexity of the Hilbert ball makes it a natural outlier in this class.

THEOREM 3.4. *Let Z be a JB^*-triple. The following are equivalent.*

(i) *For all $a \in B\backslash\{0\}$, the iterates (g_a^n) converge locally uniformly on B to a constant map.*
(ii) *Z is (triple isomorphic to) a complex Hilbert space.*

Proof. (ii)⇒(i) is Theorem 2.1.

Assume (i) holds and let $a \in B\backslash\{0\}$. By Theorem 3.2, a is regular and by Theorem 3.1, (g_a^n) then converges to g_e, e =supp(a). Since (i) holds, g_e must be a constant map, c say. Then $K_e = g_e(B) = \{c\}$ which happens if, and only if, $e = c$ is complex extreme, that is, e is a maximal tripotent.

Let d now be any non-zero tripotent in Z. Since $d = \text{supp}(\lambda d)$, for $\lambda \in \Delta$, the above implies that d must be maximal. However, every non-zero tripotent in Z is maximal only if every non-zero tripotent is also minimal. In other words, all non-zero tripotents are rank 1. Returning to a, this means that $e = \text{supp}(a)$, and hence also a, is rank 1. So Z is a rank 1 JB^*-triple and is therefore triple isomorphic to a complex Hilbert space.

References

1. J. Arazy and W. Kaup, *On continuous Peirce decompositions, Schur multipliers and the perturbation of triple functional calculus*, Math. Ann. **320** (2001), 431-461.
2. R.B. Burkel, *Iterating analytic self-maps of the disk*, Monthly **88**(6) (1981), 396-407.
3. C.-H. Chu and P. Mellon, *Iteration of compact holomorphic maps on a Hilbert ball*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1771-1777.
4. A. Denjoy, *Sur l'iteration des fonctions analytiques*, C. R. Acad. Sci. Paris, **182** (1926), 255-257.
5. Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for JB^*-triples*, Duke Math. J. **53** (1986), 139-148.
6. M. Hervé, *Quelques propriétés des application analytiques d'une boule à m dimensions dans elle-meme*, J. Math. Pures et Appl. **42** (1963), 117-147.
7. J. M. Isidro and W. Kaup, *Weak continuity of holomorphic automorphisms in JB^*-triples*, Math. Z. **210** (1992), 277-288.
8. W. Kaup, *Algebraic characterization of symmetric complex Banach manifolds*, Math. Ann. **228** (1977), 39-64.
9. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **138** (1983), 503–529.
10. W. Kaup, *Hermitian Jordan Triple Systems and the Automorphisms of Bounded Symmetric Domains*, Non-Associative Algebra and Applications, Oviedo 1993 (Santos Gonzales ed.), Kluver Academic Publishers 1994, pp. 204-214.
11. W. Kaup, *On spectral and singular values in JB^*-triples*, Proc. Roy. Irish Acad. Sect. A 96, no. 1 (1996), 95-103.

12. W. Kaup and H. Upmeier, *Banach spaces with biholomorphically equivalent unit balls are isomorphic*, Proc. Amer. Math. Soc. **58** (1976), 129-133.

13. W. Kaup and J. Sauter, *Boundary structure of bounded symmetric domains*, Manuscripta Math. **101** (2000), 351-360.

14. Y. Kubota, *Iteration of holomorphic maps of the unit ball into itself*, Proc. Amer. Math. Soc. **88** (1983), 476-480.

15. O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, University of California at Irvine, 1977.

16. B. MacCluer, *Iterates of holomorphic self maps of the unit ball in* C^{N}, Michigan Math. J. **30** (1983), 97-106.

17. P. Mellon, *Holomorphic invariance on bounded symmetric domains*, J. reine angew. Math.**523** (2000), 199-223.

18. P. Mellon, *A general Wolff theorem for arbitrary Banach spaces*, Math. Proc. of Roy. Ir. Acad.**104A (2)** (2004), 127-142.

19. A Stachura, *Iterations of holomorphic self-maps of the unit ball in Hilbert space*, Proc. Amer. Math. Soc. **93** (1985), 88-90.

20. J. Wolff, *Sur l'iteration des fonctions*, C. R. Acad. Sci. Paris, **182** (1926), 42-43, 200-201.

School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland