# TAME KERNELS AND FURTHER 4-RANK DENSITIES

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Abstract. There has been recent progress on computing the 4-rank of the tame kernel $K_2(\mathcal{O}_F)$ for $F$ a quadratic number field. For certain quadratic number fields, this progress has led to "density results" concerning the 4-rank of tame kernels. These results were first mentioned in [6] and proven in [8]. In this paper, we consider some additional quadratic number fields and obtain further density results of 4-ranks of tame kernels. Additionally, we give tables which might indicate densities in some generality.

## 1. Introduction

We are interested in the structure of the 2-Sylow subgroup of $K_2(\mathcal{O}_F)$. As $K_2(\mathcal{O}_F)$ is a finite abelian group, it is a product of cyclic groups of prime power order. We say the $2^j$-rank, $j \geq 1$, of $K_2(\mathcal{O}_F)$ is the number of cyclic factors of $K_2(\mathcal{O}_F)$ of order divisible by $2^j$. In [12], the 2-rank of the tame kernel is given by Tate's 2-rank formula. In the case where F is a quadratic number field, Browkin and Schinzel in [3] simplified the 2-rank formula. In their formula, we can determine the 2-rank by counting the number of elements in $\{\pm 1, \pm 2\}$ which are norms from the given quadratic field and the number of odd primes which are ramified in the given quadratic field. Now what about 4-ranks of $K_2(\mathcal{O}_F)$?

In [6], Conner and Hurrelbrink characterize the 4-rank of $K_2(\mathcal{O})$ for certain quadratic number fields in terms of positive definite binary quadratic forms. This characterization led to a connection between densities of certain sets of primes and 4-rank values. Specifically, the author in [8] considers the 4-rank of $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$. In [6], it was shown that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank } K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$

$$\text{4-rank } K_2(\mathcal{O}_F) = 0 \text{ or } 1.$$

2000 *Mathematics Subject Classification.* Primary: 11R70, 19F99, Secondary: 11R11, 11R45.

The idea in [8] is to fix $p \equiv 7 \bmod 8$ and consider the set

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1\}.$$

In [8], the following was proved.

**Theorem 1.1.** *For the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in $\Omega$. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in $\Omega$.*

In this paper, we consider the 4-rank of $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ for primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$ and $\mathbb{Q}(\sqrt{pl})$ for primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$. We will see that for the primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$,

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \text{ or } 2,$$
$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \text{ or } 2.$$

For the primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$, we will see

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \text{ or } 1.$$

Let us fix a prime $p \equiv 1 \bmod 8$ and consider the sets

$$A = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = 1\},$$

$$B = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = -1\}.$$

The goal of this paper is to prove two theorems analogous to Theorem 1.1, namely:

**Theorem 1.2.** *For the field $\mathbb{Q}(\sqrt{pl})$, 4-rank 1 and 2 appear with natural density $\frac{3}{4}$ and $\frac{1}{4}$ in $A$. For the field $\mathbb{Q}(\sqrt{-pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in $A$.*

**Theorem 1.3.** *For the field $\mathbb{Q}(\sqrt{pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in $B$.*

Now for squarefree, odd integers d, consider the sets

$$X = \{d : d = pl\}$$

and

$$Y = \{d : d = -pl\}$$

for distinct primes $p$ and $l$.

We have computed the following: For $15 \leq d < 10^6$, there are 168331 d's in $X$. Among them, there are 35787 d's (21.26%) yielding 4-rank 0, 128468 d's (76.32%) yielding 4-rank 1, and 4076 d's (2.42%) yielding 4-rank 2.

For $-10^6 < d \leq -15$, there are 168330 d's in $Y$. Among them, there are 104056 d's (61.82%) yielding 4-rank 0, 63054 d's (37.46%) yielding 4-rank 1, and 1220 d's (.72%) yielding 4-rank 2. As a consequence of Theorems 1.2, 1.3 and Tables I and II in [9] and [10], we obtain:

**Corollary 1.4.** *For the fields $\mathbb{Q}(\sqrt{pl})$, 4-rank 0, 1, and 2 appear with natural density $\frac{13}{64}$, $\frac{97}{128}$, $\frac{5}{128}$ respectively in $X$.*

**Corollary 1.5.** *For the fields $\mathbb{Q}(\sqrt{-pl})$, 4-rank 0, 1, and 2 appear with natural density $\frac{37}{64}$, $\frac{13}{32}$, and $\frac{1}{64}$ respectively in $Y$.*

## 2. Preliminaries

Let $\mathcal{D}$ be a Galois extension of $\mathbb{Q}$, and $G = Gal(\mathcal{D}/\mathbb{Q})$. Let $Z(G)$ denote the center of G and $\mathcal{D}^{Z(G)}$ denote the fixed field of $Z(G)$. Let $p$ be a rational prime which is unramified in $\mathcal{D}$ and $\beta$ be a prime of $\mathcal{D}$ containing $p$. Let $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right)$ denote the Artin symbol of $p$ and $\{g\}$ the conjugacy class containing one element $g \in G$. In Sections 5 and 6 we use the following elementary lemma from [8].

**Lemma 2.1.** $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ *for some $g \in Z(G)$ if and only if $p$ splits completely in $\mathcal{D}^{Z(G)}$.*

Thus if we can show that rational primes split completely in the fixed field of the center of a certain Galois group G, then we know the associated Artin symbol is a conjugacy class containing one element. Note that determining the order of $Z(G)$ gives us the number of possible choices for the Artin symbol. The order of $Z(G)$ can be computed using the following setup.

Let $G_1$ and $G_2$ be finite groups and $A$ a finite abelian group. Suppose $r_1 : G_1 \to A$ and $r_2 : G_2 \to A$ are two epimorphisms and $\mathcal{G} \subset G_1 \times G_2$ is the set $\{(g_1, g_2) \in G_1 \times G_2 : r_1(g_1) = r_2(g_2)\}$. Since A is abelian, there is an epimorphism $r : G_1 \times G_2 \to A$ given by $r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$. Thus $\mathcal{G} = ker(r) \subset G_1 \times G_2$. One can check that $Z(\mathcal{G}) = \mathcal{G} \cap Z(G_1 \times G_2)$. From [8], we provide:

**Lemma 2.2.** $Z(\mathcal{G}) = Z(G_1) \times Z(G_2) \iff r_1\Big|_{Z(G_1)}$ *and* $r_2\Big|_{Z(G_2)}$ *are both trivial.*

We will use the following definition throughout this paper.

**Definition 2.3.** For primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$, $\mathcal{K} = \mathbb{Q}(\sqrt{2p})$, and $h^+(\mathcal{K})$ the narrow class number of $\mathcal{K}$, we say:

*l satisfies* $\langle 1, 32\rangle$ if and only if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$

*l satisfies* $\langle p, -2\rangle$ if and only if $l^{\frac{h^+(\mathcal{K})}{4}} = pn^2 - 2m^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$

*l satisfies* $\langle 1, -2p\rangle$ if and only if $l^{\frac{h^+(\mathcal{K})}{4}} = n^2 - 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

## 3. First Extension

Consider the fixed prime $p \equiv 1 \bmod 8$. Note $p$ splits completely in $\mathcal{L} = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ and so

$$p\mathcal{O}_{\mathcal{L}} = \mathfrak{B}\mathfrak{B}'$$

for some primes $\mathfrak{B} \neq \mathfrak{B}'$ in $\mathcal{L}$. The field $\mathcal{L}$ has narrow class number $h^+(\mathcal{L}) = 1$ as $h(\mathcal{L}) = 1$ and $N_{\mathcal{L}/\mathbb{Q}}(\epsilon) = -1$ where $\epsilon = 1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$. Similar to Lemma 2.1 in [6],

**Lemma 3.1.** *The prime $\mathfrak{B}$ which occurs in the decomposition of $p\mathcal{O}_{\mathcal{L}}$ has a generator $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$, unique up to a sign and to multiplication by the square of a unit in $\mathcal{O}_{\mathcal{L}}^*$ for which $N_{\mathcal{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = p$.*

The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p})$ as $N_{\mathcal{L}/\mathbb{Q}}(\pi) = p$. Set

$$N = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p}).$$

Then $N$ is Galois over $\mathbb{Q}$ and $[N : \mathbb{Q}] = 8$. By Corollary 24.5 in [4], 4 divides the narrow class number of $\mathbb{Q}(\sqrt{2p})$. Moreover $N$ over $\mathbb{Q}(\sqrt{2p})$ is unramified at all finite primes. Similar to Lemma 2.3 in [6], $N$ is the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{2p})$.

Consider the rational primes $l \equiv 1 \bmod 8$ for which $\left(\frac{l}{p}\right) = 1$. These primes split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ over $\mathbb{Q}$. We characterize such primes $l$ that split completely in $N$ over $\mathbb{Q}$. As $N$ is the unique unramified cyclic degree 4 extension of $\mathbb{Q}(\sqrt{2p})$, mimicing Lemma 3.3 in [6] yields

**Lemma 3.2.** *Let $l \equiv 1 \bmod 8$ be a prime such that $\left(\frac{l}{p}\right) = 1$. Then:*

*l splits completely in $N$ if and only if l satisfies $\langle 1, -2p\rangle$.*

Similar to Lemma 3.4 in [6], with 2 (respectively, $\mathfrak{D}$, the unique dyadic prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{2p})}$ ) replaced by $p$ (respectively $\mathfrak{p}$, the prime over $p$ whose class is the unique element of order 2 in the narrow ideal class group of $\mathbb{Q}(\sqrt{2p})$ ), we obtain

**Lemma 3.3.** *Let $l \equiv 1 \bmod 8$ be a prime such that $\left(\frac{l}{p}\right) = 1$. Then:*
*$l$ does not split completely in $N$ if and only if $l$ satisfies $\langle p, -2\rangle$.*

We now relate the characterizations of Lemmas 3.2 and 3.3 to the quadratic symbol $\left(\frac{\pi}{l}\right)$. From Lemma 3.1, we have a presentation $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$ with $N_{\mathcal{L}/\mathbb{Q}}(\pi) = p$. Let $\mathfrak{P}$ be a prime above $l$ in $\mathcal{O}_{\mathcal{L}}$. As $l$ splits in $\mathcal{L}$ over $\mathbb{Q}$, then the residue field $\mathcal{O}_{\mathcal{L}}/\mathfrak{P}$ is isomorphic to $\mathbb{Z}/l\mathbb{Z} = \mathbb{F}_l$, the field with $l$ elements. As 2 is a square modulo $l$, we have $2 \equiv \alpha^2 \bmod l$ for some $\alpha \in \mathbb{F}_l{}^*$. Thus we can identify $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$ with $a + b\alpha \in \mathbb{F}_l$. When we write the symbol $\left(\frac{\pi}{l}\right)$, it is understood that we mean $\left(\frac{a+b\alpha}{l}\right)$. From the discussion in Section 3 of [6], the symbol $\left(\frac{\pi}{l}\right)$ is well defined and $l$ splits completely in $N$ over $\mathbb{Q}$ if and only if $\left(\frac{\pi}{l}\right) = 1$. Combining this discussion with Lemmas 3.2 and 3.3, we have:

**Proposition 3.4.** *Let $l \equiv 1 \bmod 8$ be a prime with $\left(\frac{l}{p}\right) = 1$. Then:*

$$l \text{ satisfies } \langle 1, -2p\rangle \iff \left(\frac{\pi}{l}\right) = 1,$$
$$l \text{ satisfies } \langle p, -2\rangle \iff \left(\frac{\pi}{l}\right) = -1.$$

## 4. Matrices and Symbols

Hurrelbrink and Kolster [7] generalize Qin's approach in [9], [10] and obtain 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Let us be more specific. Let $F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$, squarefree. Let $p_1, p_2, \ldots, p_t$ denote the odd primes dividing $d$. Recall 2 is a norm from F $\iff$ all $p_i$'s are $\equiv \pm 1 \bmod 8$. If so, then d is a norm from $\mathbb{Q}(\sqrt{2})$, thus

$$d = u^2 - 2w^2$$

for $u, w \in \mathbb{Z}$. Now consider two matrices:

If $d < 0$,

$$M'_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \ldots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \ldots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \ldots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \ldots & (-d, v)_{p_t} \\ (-d, -1)_2 & (-d, -1)_{p_1} & \ldots & (-d, -1)_{p_t} \end{pmatrix}$$

If $d > 0$,

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d,p_1)_2 & (-d,p_1)_{p_1} & \dots & (-d,p_1)_{p_t} \\ (-d,p_2)_2 & (-d,p_2)_{p_1} & \dots & (-d,p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d,p_{t-1})_2 & (-d,p_{t-1})_{p_1} & \dots & (-d,p_{t-1})_{p_t} \\ (-d,v)_2 & (-d,v)_{p_1} & \dots & (-d,v)_{p_t} \\ (d,-1)_2 & (d,-1)_{p_1} & \dots & (d,-1)_{p_t} \end{pmatrix}$$

If 2 is not a norm from F, set $v = 2$. Otherwise, set $v = u + w$. Replacing the 1's by 0's and the $-1$'s by 1's, we calculate the matrix rank over $\mathbb{F}_2$. Why look at these matrices ? From [7],

**Lemma 4.1.** *Let $F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$, squarefree. Then*
*(i)If $d < 0$, then 4-rank $K_2(\mathcal{O}_F) = t\ -$ rk $(M'_{F/\mathbb{Q}})$*
*(ii) If $d > 0$, then 4-rank $K_2(\mathcal{O}_F) = t\ -$ rk $(M_{F/\mathbb{Q}})\ + a' - a$*

*where*

$$a = \begin{cases} 0 & \text{if 2 is a norm from } F \\ 1 & \text{otherwise} \end{cases}$$

*and*

$$a' = \begin{cases} 0 & \text{if both -1 and 2 are norms from } F \\ 1 & \text{if exactly one of -1 or 2 is a norm from } F \\ 2 & \text{if none of -1 or 2 are norms from } F. \end{cases}$$

Recall that our cases are:

- $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ where $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$,
- $\mathbb{Q}(\sqrt{pl})$ for $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$.

In both cases 2 is a norm from $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{-pl})$. Before we view the matrices for our cases, we characterize the symbol $(-d, v)_2$ for $d = pl, -pl$ (see Lemmas 5.3 and 5.15 in [7]).

- $(-pl, v)_2 = 1 \iff$ both $p$, $l$ satisfy $\langle 1, 32\rangle$ or neither $p$, $l$ satisfy $\langle 1, 32\rangle$,
- $(pl, v)_2 = 1$.

Also, $v$ is an $l$-adic unit and hence

$$(-pl, v)_l = (l, v)_l = \left(\frac{v}{l}\right).$$

Similarly, $(-pl, v)_p = \left(\frac{v}{p}\right)$. In the entries of the matrices below, we write $(-pl, v)_2$, $\left(\frac{v}{l}\right)$, and $\left(\frac{v}{p}\right)$ remembering to first evalute the symbols, make the substitutions 1 for 0 and -1 for 1, and then calculate the matrix rank over $\mathbb{F}_2$. Now what are the matrices in our situations?

- For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have:

$$M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ (-pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix},$$

$$M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

- For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$, we have:

$$M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 1 & 1 \\ (-pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

**Remark 4.2.** For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff$ rank $M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 1 \iff (-pl, v)_2 = 1$, $\left(\frac{v}{l}\right) = -1$ or $(-pl, v)_2 = -1 \iff$ both $p$, $l$ satisfy $\langle 1, 32\rangle$, $\left(\frac{v}{l}\right) = -1$ or neither $p$, $l$ satisfy $\langle 1, 32\rangle$, $\left(\frac{v}{l}\right) = -1$, or exactly one of $p$, $l$ satisifes $\langle 1, 32\rangle$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \iff$ rank $M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 0 \iff (-pl, v)_2 = 1$, $\left(\frac{v}{l}\right) = 1 \iff$ both $p$, $l$ satisfy $\langle 1, 32\rangle$, $\left(\frac{v}{l}\right) = 1$ or neither $p$, $l$ satisfy $\langle 1, 32\rangle$, $\left(\frac{v}{l}\right) = 1$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \iff$ rank $M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = 1 \iff \left(\frac{v}{l}\right) = -1$.
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 2 \iff$ rank $M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = 0 \iff \left(\frac{v}{l}\right) = 1$.

**Remark 4.3.** For $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff$ rank $M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 1 \iff (-pl, v)_2 = 1 \iff$ both $p$, $l$ satisfy $\langle 1, 32\rangle$ or neither $p$, $l$ satisfy $\langle 1, 32\rangle$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \iff$ rank $M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 2 \iff (-pl, v)_2 = -1 \iff$ exactly one of $p$, $l$ satisfies $\langle 1, 32\rangle$.

We can now prove Theorem 1.3.

*Proof.* Consider the sets

$$\mathcal{A}_1 = \{l \text{ prime: } l \equiv 1 \bmod 8 \text{ and } l \text{ satisfies } \langle 1, 32\rangle\ \},$$
$$\mathcal{A}_2 = \{l \text{ prime: } l \equiv 1 \bmod 8 \text{ and } l \text{ does not satisfy } \langle 1, 32\rangle\ \}.$$

By the discussion before Corollary 24.2 in [4], $\mathcal{A}_1$ and $\mathcal{A}_2$ each have density $\frac{1}{2}$ in the set of all primes $l \equiv 1 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progessions, $\mathcal{A}_1$ and $\mathcal{A}_2$ each have density $\frac{1}{8}$ in the set of all primes $l$. Note that for primes $p \equiv 1 \bmod 8$, the sets

$$\mathcal{B}_1 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and } l \text{ satisfies } \langle 1, 32\rangle\ \},$$
$$\mathcal{B}_2 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and } l \text{ does not satisfy } \langle 1, 32\rangle\ \}.$$

each have density $\frac{1}{2}$ in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ have densities $\frac{1}{16}$ in the set of all primes $l$. If $p$ satisfies $\langle 1, 32\rangle$, then by Remark 4.3:

$$\mathcal{B}_1 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and 4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1\ \},$$
$$\mathcal{B}_2 = \{l \text{ prime: } l \equiv 1 \bmod 8, \left(\tfrac{l}{p}\right) = -1, \text{ and 4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0\}.$$

For each $\mathcal{B}_i$, $i = 1, 2$, we have:

$$\left\{\begin{array}{l}\text{Density of } \mathcal{B}_i \text{ in the}\\ \text{set of all primes } l\end{array}\right\} = \left\{\begin{array}{l}\text{Density of}\\ \mathcal{B}_i \text{ in } B\end{array}\right\} \cdot \left\{\begin{array}{l}\text{Density of } B \text{ in the}\\ \text{set of all primes } l\end{array}\right\}$$

where $B$ has density $\frac{1}{8}$ in the set of all primes $l$. Thus 4-rank 0 and 4-rank 1 each appear with natural density $\frac{1}{2}$ in $B$. A similar argument works if $p$ does not satisfy $\langle 1, 32\rangle$.

□

For the primes $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$, let us relate the Legendre symbol $\left(\frac{v}{l}\right)$ to the quadratic symbol $\left(\frac{\pi}{l}\right)$. For primes $l \equiv 1 \bmod 8$, the quadratic symbol $\left(\frac{1+\sqrt{2}}{l}\right)$ is well defined and satisfies, see [1],

$$\left(\frac{1+\sqrt{2}}{l}\right) = 1 \iff l \text{ satisfies } \langle 1, 32\rangle.$$

**Proposition 4.4.** *Let $d = \pm pl$ be as above, $d = u^2 - 2w^2$ with $u,w \in \mathbb{Z}$. Then:*

$$\left(\frac{v}{l}\right) = \left(\frac{\pi}{l}\right)\left(\frac{1+\sqrt{2}}{l}\right) \text{ if } d = pl$$
$$\left(\frac{v}{l}\right) = \left(\frac{\pi}{l}\right) \text{ if } d = -pl.$$

*Proof.* From the proof of Proposition 4.6 in [6], we use the identity

$$\left(\frac{v}{l}\right) = \left(\frac{\gamma+\delta\sqrt{2}}{l}\right)\left(\frac{1+\sqrt{2}}{l}\right)$$

where $\frac{d}{l} = N_{\mathcal{L}/\mathbb{Q}}(\gamma + \delta\sqrt{2})$ for $\gamma$, $\delta \in \mathbb{Z}$. For $d = pl$, we have $\frac{d}{l} = p = N_{\mathcal{L}/\mathbb{Q}}(\pi)$ and thus $\gamma + \delta\sqrt{2} = \pi$, up to squares. For $d = -pl$, we have $\frac{d}{l} = -p = -N_{\mathcal{L}/\mathbb{Q}}(\pi)$ and so $\gamma + \delta\sqrt{2} = (1 + \sqrt{2})\pi$, up to squares.

□

In view of Proposition 3.4, Remark 4.2, and Proposition 4.4, we can determine the 4-rank of the tame kernel in terms of quadratic forms.

**Proposition 4.5.** *For* $p \equiv l \equiv 1 \bmod 8$ *with* $\left(\frac{l}{p}\right) = 1$*:*

- *4-rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff$ *both* $p$, $l$ *satisfy* $\langle 1, 32\rangle$, $l$ *satisfies* $\langle p, -2\rangle$ *or neither* $p$, $l$ *satisfy* $\langle 1, 32\rangle$, $l$ *satisfies* $\langle p, -2\rangle$ *or exactly one of* $p$, $l$ *satisfies* $\langle 1, 32\rangle$
- *4-rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \iff$ *both* $p$, $l$ *satisfy* $\langle 1, 32\rangle$, $l$ *satisfies* $\langle 1, -2p\rangle$ *or neither* $p$, $l$ *satisfy* $\langle 1, 32\rangle$, $l$ *satisfies* $\langle 1, -2p\rangle$
- *4-rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \iff l$ *satisfies* $\langle p, -2\rangle$
- *4-rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 2 \iff l$ *satisfies* $\langle 1, -2p\rangle$.

It should be noted that Qin Yue has obtained characterizations of 4-rank values, similar to Proposition 4.5, by additionally assuming that the fundamental unit of $\mathbb{Q}(\sqrt{2p})$, $p \equiv 1 \bmod 8$, has norm $-1$, see [11].

## 5. Two Artin Symbols

5.1. **First Artin symbol.** Consider $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Let $\epsilon = 1 + \sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^*$. Then $\epsilon$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm $-1$. The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1})$. Set

$$N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1}).$$

Note that $Gal(N_1/\mathbb{Q})$ is the dihedral group of order 8 and $Z(Gal(N_1/\mathbb{Q})) = Gal(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))$ (see [8], Section 3.2).

Only the prime 2 ramifies in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\epsilon})$, and so only the prime 2 ramifies in the compositum $N_1$ over $\mathbb{Q}$. Now as $l \in A$ is unramified in $N_1$ over $\mathbb{Q}$, the Artin symbol $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ is defined for primes $\beta$ of $\mathcal{O}_{N_1}$ containing $l$. Let $\left(\frac{N_1/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ in $Gal(N_1/\mathbb{Q})$. The primes $l \in A$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and ${N_1}^{Z(Gal(N_1/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Thus by Lemma 2.1, we have that $\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{g\}$ for some $g \in Z(Gal(N_1/\mathbb{Q}))$. As $Z(Gal(N_1/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_1/\mathbb{Q}}{l}\right)$. Combining this statement with Addendum (3.7) from [6], we have

**Remark 5.1.**

$$\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{id\} \iff l \text{ splits completely in } N_1$$
$$\iff l \text{ satisfies } \langle 1, 32\rangle.$$

5.2. **Second Artin symbol.** In section 3, we considered

$$N = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p}),$$

the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{2p})$. Similar to the extension $N_1$, we have $Gal(N/\mathbb{Q})$ is the dihedral group of order 8 and $Z(Gal(N/\mathbb{Q})) = Gal(N/\mathbb{Q}(\sqrt{2}, \sqrt{p}))$.

**Proposition 5.2.** *If $l \in A$, then $l$ is unramified in $N$ over $\mathbb{Q}$.*

*Proof.* Since $p \equiv 1 \bmod 8$, the discriminant of $\mathbb{Q}(\sqrt{2p})$ is $8p$. For $l \in A$, we have $\left(\frac{2p}{l}\right) = 1$ and so $l$ is unramified in $\mathbb{Q}(\sqrt{2p})$. We conclude that $l$ is unramified in $N$ over $\mathbb{Q}$. □

As $l \in A$ is unramified in $N$ over $\mathbb{Q}$, the Artin symbol $\left(\frac{N/\mathbb{Q}}{\beta}\right)$ is defined for primes $\beta$ of $\mathcal{O}_N$ containing $l$. Let $\left(\frac{N/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N/\mathbb{Q}}{\beta}\right)$ in $Gal(N/\mathbb{Q})$. The primes $l \in A$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ and $N^{Z(Gal(N/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{p})$. By Lemma 2.1, we have that $\left(\frac{N/\mathbb{Q}}{l}\right) = \{h\}$ for some $h \in Z(Gal(N/\mathbb{Q}))$. As Z(Gal($N/\mathbb{Q}$)) has order 2, there are two possible choices for $\left(\frac{N/\mathbb{Q}}{l}\right)$. Combining this statement and Lemmas 3.2 and 3.3, we have

**Remark 5.3.**

$$\left(\frac{N/\mathbb{Q}}{l}\right) = \{id\} \iff l \text{ splits completely in } N$$
$$\iff l \text{ satisfies } \langle 1, -2p\rangle.$$

$$\left(\frac{N/\mathbb{Q}}{l}\right) \neq \{id\} \iff l \text{ does not split completely in } N$$
$$\iff l \text{ satisfies } \langle p, -2\rangle.$$

## 6. A composite and proof of Theorem 1.2

In this section we consider the composite field $N_1N$. Set

$$\mathfrak{N} = N_1N.$$

Note that $[\mathfrak{N} : \mathbb{Q}] = 32$. As $N_1$ and $N$ are normal extensions of $\mathbb{Q}$, $\mathfrak{N}$ is a normal extension of $\mathbb{Q}$.

For $l \in A$, $l$ is unramified in $\mathfrak{N}$ as it is unramified in $N_1$ and $N$. The Artin symbol $\left(\frac{\mathfrak{N}/\mathbb{Q}}{\beta}\right)$ is now defined for some prime $\beta$ of $\mathcal{O}_{\mathfrak{N}}$ containing $l$. Let $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{\mathfrak{N}/\mathbb{Q}}{\beta}\right)$ in $Gal(\mathfrak{N}/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p}) \subset \mathfrak{N}$, we prove

**Lemma 6.1.** *$Z(Gal(\mathfrak{N}/\mathbb{Q})) = Gal(\mathfrak{N}/M)$ is elementary abelian of order 4.*

*Proof.* For $\sigma \in Gal(\mathfrak{N}/M)$, $\sigma$ can only change the sign of $\sqrt{\epsilon}$ and $\sqrt{\pi}$ as $\epsilon \in M$. Since $\mathfrak{N} = M(\sqrt{\epsilon}, \sqrt{\pi})$, $Gal(\mathfrak{N}/M)$ is elementary abelian of order 4. Now consider the restrictions $r_1 : G_1 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and $r_2 : G_2 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ where $G_1 = Gal(N_1/\mathbb{Q})$ and $G_2 = Gal(N/\mathbb{Q})$. Clearly $r_1\Big|_{Z(G_1)}$ and $r_1\Big|_{Z(G_2)}$ are both trivial. Then by Lemma 2.2, $Z(\mathcal{G})$ is elementary abelian of order 4 where $\mathcal{G} = Gal(\mathfrak{N}/\mathbb{Q})$. Thus $Z(Gal(\mathfrak{N}/\mathbb{Q})) = Gal(\mathfrak{N}/M)$.

□

Now for $l \in A$, $l$ splits completely in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. From Lemma 6.1, $\mathfrak{N}^{Z(Gal(\mathfrak{N}/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. So by Lemma 2.1, we have

$$\left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right) = \{k\} \text{ for some } k \in Gal(\mathfrak{N}/\mathbb{Q}).$$

As $Z(Gal(\mathfrak{N}/\mathbb{Q}))$ has order 4, there are four possible choices for $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right)$. Using Remarks 5.1 and 5.3, we now make the following one to one correspondences.

**Remark 6.2.** (i) $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right) = \{id\} \iff l$ splits completely in $\mathfrak{N}$ $\iff$ $\left\{\begin{array}{l} l \text{ splits completely in} \\ N_1 \text{ and } N \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \end{array}\right\}$.

(ii) $\left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right) \neq \{id\} \iff l$ does not split completely in $\mathfrak{N}$. Now there are three cases:

(1) $\left\{\begin{array}{l} l \text{ splits completely in } N_1, \\ \text{but does not in } N \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \end{array}\right\}$

(2) $\left\{\begin{array}{l} l \text{ splits completely in } N \\ \text{but does not in } N_1 \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \end{array}\right\}$

$$(3)\quad \left\{ \begin{array}{l} l \text{ does split completely} \\ \text{in } N_1 \text{ or } N \end{array} \right\} \iff \left\{ \begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \end{array} \right\}.$$

We can now prove Theorem 1.2

*Proof.* Consider the set $X = \{l \text{ prime} : l \text{ is unramified in } \mathfrak{N} \text{ and} \left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right) = \{k\}\}$ for some $k \in \mathrm{Gal}(\mathfrak{N}/\mathbb{Q})$. By the Čebotarev Density Theorem, the set $X$ has natural density $\frac{1}{32}$ in the set of all primes $l$. Recall

$$A = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = 1\}$$

for some fixed prime $p \equiv 1 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, $A$ has natural density $\frac{1}{8}$ in the set of all primes $l$. Thus $X$ has natural density $\frac{1}{4}$ in $A$. If $p$ satisfies $\langle 1, 32\rangle$, then by Proposition 4.5,

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \end{array} \right\} \text{ or } \left\{ \begin{array}{l} l \text{ does not} \\ \text{satisfy } \langle 1, 32\rangle \end{array} \right\}$$

and

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \iff \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \end{array} \right\}.$$

Using Remark 6.2, we see that for $\mathbb{Q}(\sqrt{pl})$, 4-rank 1 and 4-rank 2 appear with natural density $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ and $\frac{1}{4}$ respectively. A similar argument works if $p$ does not satisfy $\langle 1, 32\rangle$. For $\mathbb{Q}(\sqrt{-pl})$, use Proposition 4.5 and Remark 6.2 to obtain that 4-rank 1 and 2 each appear with natural density $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ in $A$. □

## 7. Proof of Two Corollaries

For squarefree, odd integers d, recall the sets $X = \{d : d = pl\}$ and $Y = \{d : d = -pl\}$ for distinct primes $p$ and $l$. Now consider the sets

$$X_i = \{d : d = pl,\ p \equiv i \bmod 8\},$$
$$Y_i = \{d : -pl,\ p \equiv i \bmod 8\}.$$

Thus $X = X_1 \cup X_3 \cup X_5 \cup X_7$ and $Y = Y_1 \cup Y_3 \cup Y_5 \cup Y_7$. Additionally consider the sets

$$X_{i,j} = \{d : d = pl,\ p \equiv i \bmod 8,\ l \equiv j \bmod 8\},$$
$$Y_{i,j} = \{d : d = -pl,\ p \equiv i \bmod 8,\ l \equiv j \bmod 8\}.$$

Thus, for example, $X_1 = X_{1,1} \cup X_{1,3} \cup X_{1,5} \cup X_{1,7}$ and $Y_7 = Y_{7,1} \cup Y_{7,3} \cup Y_{7,5} \cup Y_{7,7}$.

In Tables 1 and 2 below, for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})})$, we provide cases in which densities of 4-rank values follow from congruence conditions on $p$ and $l$, a condition on the Legendre symbol $\left(\frac{l}{p}\right)$ (if any), and Dirichlet's theorem on primes in arithmetic progressions. In Tables 3 and 4, we provide the same information for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})})$ (compare with [5] or Tables I and II in [9] and [10]).

TABLE 1: $\mathbb{Q}(\sqrt{pl})$

| $p$, $l$ mod 8 | 4-rank | Densities |
|---|---|---|
| 3, 3 | 0 | $\frac{1}{4}$ in $X_3$ |
| 5, 5 | 1 | $\frac{1}{4}$ in $X_5$ |
| 7, 7 | 1 | $\frac{1}{4}$ in $X_7$ |
| 3, 5 | 1 | $\frac{1}{4}$ in $X_3$ and $X_5$ |
| 3, 7 | 1 | $\frac{1}{4}$ in $X_3$ and $X_7$ |
| 5, 7 | 1 | $\frac{1}{4}$ in $X_5$ and $X_7$ |

TABLE 2: $\mathbb{Q}(\sqrt{pl})$

| $p$, $l$ mod 8 | Legendre symbols | 4-rank | Densities |
|---|---|---|---|
| 1, 3 | $\left(\frac{l}{p}\right) = -1$ | 0 | $\frac{1}{8}$ in $X_1$ and $X_3$ |
| | $\left(\frac{l}{p}\right) = 1$ | 1 | $\frac{1}{8}$ in $X_1$ and $X_3$ |
| 1, 5 | $\left(\frac{l}{p}\right) = -1$ | 0 | $\frac{1}{8}$ in $X_1$ and $X_5$ |
| | $\left(\frac{l}{p}\right) = 1$ | 1 | $\frac{1}{8}$ in $X_1$ and $X_5$ |
| 1, 7 | $\left(\frac{l}{p}\right) = -1$ | 1 | $\frac{1}{8}$ in $X_1$ and $X_7$ |
| | $\left(\frac{l}{p}\right) = 1$ | 1 | $\frac{1}{16}$ in $X_1$ and $X_7$ |
| | | 2 | $\frac{1}{16}$ in $X_1$ and $X_7$ |

TABLE 3: $\mathbb{Q}(\sqrt{-pl})$

| $p$,$l$ mod 8 | 4-rank | Densities |
|---|---|---|
| 3, 3 | 1 | $\frac{1}{4}$ in $Y_3$ |
| 5, 5 | 1 | $\frac{1}{4}$ in $Y_5$ |
| 7, 7 | 1 | $\frac{1}{4}$ in $Y_7$ |
| 3, 5 | 0 | $\frac{1}{4}$ in $Y_3$ and $Y_5$ |
| 3, 7 | 0 | $\frac{1}{4}$ in $Y_3$ and $Y_7$ |
| 5, 7 | 0 | $\frac{1}{4}$ in $Y_5$ and $Y_7$ |

TABLE 4: $\mathbb{Q}(\sqrt{-pl})$

| $p$, $l$ mod 8 | Legendre symbols | 4-rank | Densities |
|---|---|---|---|
| 1, 1 | $\left(\frac{l}{p}\right) = -1$ | 1 | $\frac{1}{8}$ in $Y_1$. |
| 1, 3 | $\left(\frac{l}{p}\right) = -1$ | 0 | $\frac{1}{8}$ in $Y_1$ and $Y_3$ |
| | $\left(\frac{l}{p}\right) = 1$ | 1 | $\frac{1}{8}$ in $Y_1$ and $Y_3$ |
| 1, 5 | $\left(\frac{l}{p}\right) = -1$ | 0 | $\frac{1}{8}$ in $Y_1$ and $Y_5$ |
| | $\left(\frac{l}{p}\right) = 1$ | 1 | $\frac{1}{8}$ in $Y_1$ and $Y_5$ |
| 1, 7 | $\left(\frac{l}{p}\right) = -1$ | 0 | $\frac{1}{8}$ in $Y_1$ and $Y_7$ |
| | $\left(\frac{l}{p}\right) = 1$ | 0 | $\frac{1}{16}$ in $Y_1$ and $Y_7$ |
| | | 1 | $\frac{1}{16}$ in $Y_1$ and $Y_7$ |

**Remark 7.1.** By Theorem 1.2, $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$ yields 4-rank 1 and 2 with densities $\frac{3}{32}$ and $\frac{1}{32}$ respectively in $X_1$. By Theorem 1.3, $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = -1$ yields 4-rank 0 and 1 each with density $\frac{1}{16}$ in $X_1$. We can now prove Corollary 1.4.

*Proof.* Regarding the set $X_1$:

- 4-rank 0, 1, and 2 appear with natural densities $\frac{1}{16}$, $\frac{3}{32} + \frac{1}{16} = \frac{5}{32}$, and $\frac{1}{32}$ in $X_{1,1}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,3}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,5}$
- 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $X_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{5}{16}$, $\frac{19}{32}$, and $\frac{3}{32}$ in $X_1$. For the set $X_3$:

- 4-rank 0 and 1 each appear with natural density $\frac{1}{8}$ in $X_{3,1}$
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $X_{3,3}$
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $X_{3,5}$
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $X_{3,7}$.

So 4-rank 0 and 1 appear with natural densities $\frac{3}{8}$ and $\frac{5}{8}$ in $X_3$. Similarly, 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{7}{8}$ in $X_5$ and 4-rank 1 and 2 appear with natural densities $\frac{15}{16}$ and $\frac{1}{16}$ in $X_7$. As each $X_i$ has density $\frac{1}{4}$ in $X$,

- 4-rank 0 appears with natural density $\frac{5}{64} + \frac{3}{32} + \frac{1}{32} = \frac{13}{64}$ in $X$
- 4-rank 1 appears with natural density $\frac{19}{128} + \frac{5}{32} + \frac{7}{32} + \frac{15}{64} = \frac{97}{128}$ in $X$
- 4-rank 2 appears with natural density $\frac{3}{128} + \frac{1}{64} = \frac{5}{128}$ in $X$.

□

**Remark 7.2.** By Theorem 1.2, $p \equiv l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$ yields 4-rank 1 and 2 each with density $\frac{1}{16}$ in $Y_1$. We can now prove Corollary 1.5.

*Proof.* Regarding the set $Y_1$:

- 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $Y_{1,1}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,3}$
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,5}$
- 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$ in $Y_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{3}{8}$, $\frac{9}{16}$, and $\frac{1}{16}$ in $Y_1$. For the set $Y_3$:

- 4-rank 0 and 1 each appear with natural density $\frac{1}{8}$ in $Y_{3,1}$
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $Y_{3,3}$
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $Y_{3,5}$
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $Y_{3,7}$.

So 4-rank 0 and 1 appear with natural densities $\frac{5}{8}$ and $\frac{3}{8}$ in $Y_3$. Similarly, 4-rank 0 and 1 appear with natural densities $\frac{5}{8}$ and $\frac{3}{8}$ in $Y_5$ and 4-rank 0 and 1 appear with natural densities $\frac{11}{16}$ and $\frac{5}{16}$ in $Y_7$. As each $Y_i$ has density $\frac{1}{4}$ in $Y$,

- 4-rank 0 appears with natural density $\frac{3}{32} + \frac{5}{32} + \frac{5}{32} + \frac{11}{64} = \frac{37}{64}$ in $Y$
- 4-rank 1 appears with natural density $\frac{9}{64} + \frac{3}{32} + \frac{3}{32} + \frac{5}{64} = \frac{13}{32}$ in $Y$
- 4-rank 2 appears with natural density $\frac{1}{64}$ in $Y$.

□

## Appendix

The approach of Hurrelbrink and Kolster in [7] led us to write a program in GP/PARI [2] which generates the numerical values in Tables 5-8. The aim is to motivate possible density results for tame kernels of quadratic number fields. In Tables 5 and 6, $p$, $l$, and $r$ are distinct odd primes. In Tables 7 and 8, $d$ is odd and squarefree.

Table 5

| Cardinality | $105 \leq d = plr < 10^6$ | % |
|---|---|---|
| \| 4-rank 0 \| | 8247 | 6.827 |
| \| 4-rank 1 \| | 92544 | 76.605 |
| \| 4-rank 2 \| | 20000 | 16.555 |
| \| 4-rank 3 \| | 16 | .013 |

Table 6

| Cardinality | $-10^6 < d = -plr \leq -105$ | % |
|---|---|---|
| \| 4-rank 0 \| | 67970 | 56.2633 |
| \| 4-rank 1 \| | 50147 | 41.5100 |
| \| 4-rank 2 \| | 2688 | 2.2250 |
| \| 4-rank 3 \| | 2 | .0017 |

TABLE 7

| Cardinality | $3 \leq d < 10^6$ | % |
|---|---|---|
| \| 4-rank 0 \| | 93736 | 23.1284 |
| \| 4-rank 1 \| | 278138 | 68.6278 |
| \| 4-rank 2 \| | 33148 | 8.1789 |
| \| 4-rank 3 \| | 263 | .0649 |

TABLE 8

| Cardinality | $-10^6 < d \leq -3$ | % |
|---|---|---|
| \| 4-rank 0 \| | 251884 | 62.14985 |
| \| 4-rank 1 \| | 148669 | 36.68258 |
| \| 4-rank 2 \| | 4730 | 1.16708 |
| \| 4-rank 3 \| | 2 | .00049 |


## ACKNOWLEDGMENTS

We thank the referee for informing us of the paper by Qin Yue. We also thank Manfred Kolster for his suggestions and comments.


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