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A REFINED BLOCH GROUP AND THE THIRD HOMOLOGY OF SL_2 OF A FIELD

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ABSTRACT. We use the properties of the refined Bloch group to study the structure of $H_3(SL_2(F), \mathbb{Z})$ for a field F . We compute this group up to 2-torsion when F is a local field with finite residue field of odd order, and we show that for any global field F it is not finitely-generated.

1. INTRODUCTION

In [19], Chih-Han Sah quotes S. Lichtenbaum who mentions our lack of knowledge of the precise structure of $H_3(SL_2(\mathbb{Q}), \mathbb{Z})$ as an example of the unsatisfactory state of our understanding of the homology of linear groups. This was nearly twenty five years ago, and to the author's knowledge the precise structure of this group is still unknown. (Observe, however, that $H_3(SL_n(\mathbb{Q}), \mathbb{Z}) \cong K_3^{\text{ind}}(\mathbb{Q}) \cong \mathbb{Z}/24$ for all $n \geq 3$ - by the results of [8], for example.)

In this article, we study the structure of the third homology of SL_2 of fields by using the properties of the *refined Bloch group* of the field, which was introduced in [7]. We are particularly interested in understanding $H_3(SL_2(F), \mathbb{Z})$ as a *functor* of F , and its possible relation to other functors in K -theory and algebraic geometry.

What are now referred to as *Bloch groups* of fields first appeared in the work of S. Bloch in the late 1970s (see [2] for the lecture notes) as a way of constructing explicit maps - and, in particular, regulators - on K_3 of fields. In the 1980s, they were studied by Dupont and Sah (under the name *scissors congruence group*) because of their connection with 3-dimensional hyperbolic geometry ([4], [19]). This connection is still actively studied today: Bloch groups of number fields are targets for *Bloch invariants* of certain finite-volume oriented hyperbolic 3-manifolds ([15], [16], [5]). These invariants are amenable to explicit calculation and are related in a known way to the Chern-Simons invariant. There are also intriguing connections between Bloch groups, conformal field theories and even modular form theory ([22],[13]).

The precise relationship between the Bloch group and K -theory was established via their mutual connection to the homology of linear groups. These connections were greatly clarified and exploited in the work of Suslin ([21]): For a field F with at least 4 elements, the pre-Bloch group, $\mathcal{P}(F)$, of the field is an abelian group with generators $[x]$, $x \in F^\times \setminus \{1\}$, subject to a family of 5-term relations. The Bloch group, $\mathcal{B}(F)$, is a subgroup of $\mathcal{P}(F)$ which also arises naturally as a quotient of $H_3(GL_2(F), \mathbb{Z})$. Suslin has shown that this extends to a surjection from $H_3(GL_3(F), \mathbb{Z})$ to $\mathcal{B}(F)$ ([21, section 3]). Of course, $K_3(F)$ admits a Hurewicz homomorphism to $H_3(GL(F), \mathbb{Z}) = H_3(GL_3(F), \mathbb{Z})$, and hence, by composition, a map to $\mathcal{B}(F)$. Using calculations of the homology of $GL_n(F)$ ($n = 2, 3$) as well as the homotopy theory of the plus construction, Suslin proved that there is a natural short exact sequence

$$0 \rightarrow \text{Tor}_1^{\mathbb{Z}}(\widetilde{\mu}_F, \mu_F) \rightarrow K_3^{\text{ind}}(F) \rightarrow \mathcal{B}(F) \rightarrow 0$$

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where $\widetilde{\text{Tor}}_1^{\mathbb{Z}}(\mu_F, \mu_F)$ is the unique nontrivial extension of $\text{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)$ by $\mathbb{Z}/2$ and

$$K_3^{\text{ind}}(F) = \text{Coker}(K_3^M(F) \rightarrow K_3(F)).$$

These results of Suslin were extended to finite fields (with at least 4 elements) in [7, section 7].

In a letter to Sah (see [19, section 4]), Suslin asked the question whether the composite $H_3(\text{SL}_2(F), \mathbb{Z}) \rightarrow H_3(\text{SL}(F), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(F)$ induces an isomorphism

$$H_3(\text{SL}_2(F), \mathbb{Z})_{F^\times} \cong K_3^{\text{ind}}(F).$$

The current state of knowledge on this question is that the map is surjective ([8]) and, if we let $A[\frac{1}{2}]$ denote $A \otimes \mathbb{Z}[\frac{1}{2}]$ for any abelian group A , that the induced map

$$H_3(\text{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_{F^\times} \rightarrow K_3^{\text{ind}}(F)[\frac{1}{2}]$$

is an isomorphism ([12]).

The homology groups of the special linear groups $\text{SL}_n(F)$ are naturally modules over the group ring $\mathbb{Z}[F^\times]$ via the short exact sequences

$$1 \longrightarrow \text{SL}_n(F) \longrightarrow \text{GL}_n(F) \xrightarrow{\det} F^\times \longrightarrow 1.$$

Since the scalar matrices $a \cdot I_n$ are central and have determinant a^n , it follows that $(F^\times)^n$ acts trivially on $H_k(\text{SL}_n(F), \mathbb{Z})$. In particular, the groups $H_k(\text{SL}_2(F), \mathbb{Z})$ are modules for the group ring $R_F := \mathbb{Z}[F^\times / (F^\times)^2]$.

When $n > k$ (or $n \geq k$ when k is odd) in the groups $H_k(\text{SL}_n(F), \mathbb{Z})$, we are in the range of stability (see [19], [8] and [9]) and this module structure is necessarily trivial. But below the range of stability, the module structure appears to be nontrivial and interesting. For example, the unstable groups $H_{2n}(\text{SL}_{2n}(F), \mathbb{Z})$ are modules over the Grothendieck-Witt ring of the field (which is a quotient of R_F) and surject onto the even Milnor-Witt K -theory groups of the field F ([9]).

In [7] the author defined a *refined pre-Bloch group*, $\mathcal{RP}(F)$, of a field F and a subgroup, the *refined Bloch group*, $\mathcal{RB}(F)$, which was shown to have the following properties:

- (1) [7, Theorem 4.3] The group $\mathcal{RB}(F)$ is an R_F -module and there is a natural surjective homomorphism of R_F -modules

$$H_3(\text{SL}_2(F), \mathbb{Z}) \twoheadrightarrow \mathcal{RB}(F)$$

- (2) This induces a commutative diagram (of $R_F[\frac{1}{2}]$ -modules) with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)[\frac{1}{2}] & \longrightarrow & H_3(\text{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathcal{RB}(F)[\frac{1}{2}] \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)[\frac{1}{2}] & \longrightarrow & K_3^{\text{ind}}(F)[\frac{1}{2}] & \longrightarrow & \mathcal{B}(F)[\frac{1}{2}] \longrightarrow 0 \end{array}$$

Here R_F acts trivially on the bottom row and furthermore

- (3) [7, Corollary 5.1] On taking F^\times -coinvariants, the top row becomes isomorphic to the bottom row. In particular, $\mathcal{RB}(F)[\frac{1}{2}]_{F^\times} \cong \mathcal{B}(F)[\frac{1}{2}]$ and

$$I_F \mathcal{RB}(F)[\frac{1}{2}] \cong I_F H_3(\text{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) = \text{Ker}(H_3(\text{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow K_3^{\text{ind}}(F)[\frac{1}{2}])$$

where I_F denotes the augmentation ideal of the group ring R_F .

- (4) [7, Lemma 5.2] The group $\mathcal{I}_F \mathcal{RB}(F)[\frac{1}{2}]$ is also the kernel of the stabilization homomorphism

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}[\frac{1}{2}]).$$

The cokernel of this map is the third Milnor K -group $K_3^{\mathrm{M}}(F)[\frac{1}{2}]$ and the image is isomorphic to $K_3^{\mathrm{ind}}(F)[\frac{1}{2}]$ ([8, Theorem 4.7, Section 5]).

The main result of the current article (Theorem 4.9 and its corollaries) tells us that given a valuation v on the field F with residue field k , there are surjective reduction or specialization homomorphisms

$$\mathcal{RP}(F) \twoheadrightarrow \widehat{\mathcal{RP}}(k)$$

where $\widehat{\mathcal{RP}}(k)$ is a certain quotient of $\mathcal{RP}(k)$.

In particular, if $a \in F^\times$ and $v(a)$ is not a multiple of 2, there is a specialization homomorphism which induces a surjection

$$e_a^- \mathcal{RB}(F)[\frac{1}{2}] \twoheadrightarrow \widehat{\mathcal{P}}(k)[\frac{1}{2}]$$

where

$$e_a^- = \frac{1 - \langle a \rangle}{2} \in \mathcal{I}_F[\frac{1}{2}]$$

and $\langle a \rangle$ denotes the square class of a in \mathbf{R}_F and $\widehat{\mathcal{P}}(k)$ is a certain quotient of the classical pre-Bloch group, $\mathcal{P}(k)$, of the residue field.

Using these results, we can prove that $\mathcal{I}_F \mathcal{RB}(F)[\frac{1}{2}]$ is large if F is a field with many valuations. In particular, it follows that $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$ is not finitely generated for any global field F . By contrast, if F is a global field then $\mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}) = K_3^{\mathrm{ind}}(F)$ is well-known to be finitely generated. More precisely, if F is a global field whose class group has odd order then there is a natural surjection

$$\mathcal{I}_F \mathcal{RB}(F)[\frac{1}{2}] \twoheadrightarrow \bigoplus_v \mathcal{P}(k_v)[\frac{1}{2}]$$

where v runs through all the finite places of F and k_v is the residue field at v (see Corollary 5.2). Let n' denote the odd part of the nonzero integer n . By the results of [7], if \mathbb{F}_q is the finite field with q elements, then $\mathcal{P}(\mathbb{F}_q)[\frac{1}{2}]$ is finite cyclic of order $(q + 1)'$.

As another application, we also use the techniques developed to construct explicitly non-trivial \mathbb{F}_3 -vector spaces of known dimension inside groups of the form $\mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ where \mathcal{O}_S is a ring of S -integers corresponding to a set S of primes of the field F , and thus to give lower bounds on the 3-ranks of such groups. Since there is still very little in the literature at present by way of explicit calculations of the homology or cohomology of S -arithmetic groups of this type, we hope that the techniques developed here will be useful in proving more general results of this type in the future. In particular, our results point to the importance of the \mathcal{O}_S^\times -module structure in calculations of this type.

Finally, we use these specialization maps, together with the basic algebraic properties of the refined Bloch groups, developed in section 3 below, to give a calculation of $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])$ for local fields F with finite residue field k of odd order (Theorem 6.19). In particular, for such fields (with some minor restrictions if $\mathbb{Q}_3 \subset F$) we have

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\mathrm{ind}}(F)[\frac{1}{2}] \oplus \mathcal{P}(k)[\frac{1}{2}].$$

Here the right-hand side is an \mathbf{R}_F module with F^\times acting trivially on the first factor while any uniformizer acts as -1 on the second factor.

To return to our opening remarks, it follows from the results here that there is a natural surjection

$$H_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}] = H_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}]) \longrightarrow K_3^{\mathrm{ind}}(\mathbb{Q})[\frac{1}{2}] \oplus \left(\bigoplus_p \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}] \right).$$

It is natural to ask whether this is an isomorphism and, furthermore, what adjustments, if any, need to be made to obtain a corresponding statement with integral coefficients.

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2. REVIEW OF BLOCH GROUPS

2.1. Preliminaries and Notation. For a field F , we let G_F denote the multiplicative group, $F^\times/(F^\times)^2$, of nonzero square classes of the field. For $x \in F^\times$, we will let $\langle x \rangle \in G_F$ denote the corresponding square class. Let R_F denote the integral group ring $\mathbb{Z}[G_F]$ of the group G_F . We will use the notation $\langle\langle x \rangle\rangle$ for the basis elements, $\langle x \rangle - 1$, of the augmentation ideal \bar{I}_F of R_F .

For any $a \in F^\times$, we will let p_a^+ and p_a^- denote respectively the elements $1 + \langle a \rangle$ and $1 - \langle a \rangle$ in R_F .

We let e_a^+ and e_a^- denote respectively the mutually orthogonal idempotents

$$e_a^+ := \frac{p_a^+}{2} = \frac{1 + \langle a \rangle}{2}, \quad e_a^- := \frac{p_a^-}{2} = \frac{1 - \langle a \rangle}{2} \in R_F[\frac{1}{2}].$$

(Of course, these operators depend only on the class of a in G_F .)

For any abelian group A we will let $A[\frac{1}{2}]$ denote $A \otimes \mathbb{Z}[\frac{1}{2}]$. For an integer n , we will let n' denote the odd part of n . Thus if A is a finite abelian group of order n , then $A[\frac{1}{2}]$ is a finite abelian group of order n' . For an additive abelian group A , $A/2$ denotes $A \otimes \mathbb{Z}/2$. However, for multiplicative groups M we will use the notation M/M^2 . If n is a positive integer and A an abelian group, $A_{(n)}$ will denote the set $\{a \in A \mid na = 0\}$.

2.2. The classical Bloch group. For a field F , with at least 4 elements, the *pre-Bloch group*, $\mathcal{P}(F)$, is the group generated by the elements $[x]$, $x \in F^\times \setminus \{1\}$, subject to the relations

$$R_{x,y} : [x] - [y] + [y/x] - \left[(1 - x^{-1})/(1 - y^{-1}) \right] + [(1 - x)/(1 - y)] \quad x \neq y.$$

Let $S_{\mathbb{Z}}^2(F^\times)$ denote the group

$$\frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle x \otimes y + y \otimes x \mid x, y \in F^\times \rangle}$$

and denote by $x \circ y$ the image of $x \otimes y$ in $S_{\mathbb{Z}}^2(F^\times)$.

The map

$$\lambda : \mathcal{P}(F) \rightarrow S_{\mathbb{Z}}^2(F^\times), \quad [x] \mapsto (1 - x) \circ x$$

is well-defined, and the *Bloch group of F* , $\mathcal{B}(F) \subset \mathcal{P}(F)$, is defined to be the kernel of λ .

2.3. The refined Bloch group. The *refined pre-Bloch group*, $\mathcal{RP}(F)$, of a field F which has at least 4 elements, is the R_F -module with generators $[x]$, $x \in F^\times$ subject to the relations $[1] = 0$ and

$$S_{x,y} : 0 = [x] - [y] + \langle x \rangle [y/x] - \langle x^{-1} - 1 \rangle \left[(1 - x^{-1})/(1 - y^{-1}) \right] + \langle 1 - x \rangle [(1 - x)/(1 - y)], \quad x, y \neq 1$$

Of course, from the definition it follows immediately that $\mathcal{P}(F) = (\mathcal{RP}(F))_{F^\times} = H_0(F^\times, \mathcal{RP}(F))$.

For any field F we define the R_F -module

$$RS_{\mathbb{Z}}^2(F^\times) := \mathcal{I}_F^2 \times_{\text{Sym}_{\mathbb{F}_2}^2(G_F)} S_{\mathbb{Z}}^2(F^\times) \subset \mathcal{I}_F^2 \oplus S_{\mathbb{Z}}^2(F^\times)$$

where $S_{\mathbb{Z}}^2(F^\times)$ has the trivial R_F -module structure.

As an R_F -module, $RS_{\mathbb{Z}}^2(F^\times)$ is generated by the elements

$$[a, b] := (\langle\langle a \rangle\rangle \langle\langle b \rangle\rangle, a \circ b) \in RS_{\mathbb{Z}}^2(F^\times).$$

The *refined Bloch-Wigner homomorphism* Λ to be the R_F -module homomorphism

$$\Lambda : \mathcal{RP}(F) \rightarrow RS_{\mathbb{Z}}^2(F), \quad [x] \mapsto [1 - x, x]$$

which is well-defined by [7, proof of Theorem 4.3].

In view of the definition of $RS_{\mathbb{Z}}^2(F^\times)$, we can express $\Lambda = (\lambda_1, \lambda_2)$ where $\lambda_1 : \mathcal{RP}(F) \rightarrow \mathcal{I}_F^2$ is the map $[x] \mapsto \langle\langle 1 - x \rangle\rangle \langle\langle x \rangle\rangle$, and λ_2 is the composite

$$\mathcal{RP}(F) \twoheadrightarrow \mathcal{P}(F) \xrightarrow{\lambda} S_{\mathbb{Z}}^2(F^\times).$$

Finally, we can define the *refined Bloch group* of the field F (with at least 4 elements) to be the R_F -module

$$\mathcal{RB}(F) := \text{Ker}(\Lambda : \mathcal{RP}(F) \rightarrow RS_{\mathbb{Z}}^2(F^\times)).$$

2.4. The fields \mathbb{F}_2 and \mathbb{F}_3 . Throughout this paper it will be convenient for us to have (refined and classical) pre-Bloch and Bloch groups for the fields with 2 and 3 elements. For this reason, we introduce the following *ad hoc* definitions.

$\mathcal{P}(\mathbb{F}_2) = \mathcal{RP}(\mathbb{F}_2) = \mathcal{RB}(\mathbb{F}_2) = \mathcal{B}(\mathbb{F}_2)$ is simply an additive group of order 3 with distinguished generator, denoted $C_{\mathbb{F}_2}$.

$\mathcal{RP}(\mathbb{F}_3)$ is the cyclic $R_{\mathbb{F}_3}$ -module generated by the symbol $[-1]$ and subject to the one relation

$$0 = 2 \cdot ([-1] + \langle -1 \rangle [-1]).$$

The homomorphism

$$\Lambda : \mathcal{RP}(\mathbb{F}_3) \rightarrow RS_{\mathbb{Z}}^2(\mathbb{F}_3^\times) = \mathcal{I}_{\mathbb{F}_3}^2 = 2 \cdot \mathbb{Z} \langle\langle -1 \rangle\rangle$$

is the $R_{\mathbb{F}_3}$ -homomorphism sending $[-1]$ to $\langle\langle -1 \rangle\rangle^2 = -2 \langle\langle -1 \rangle\rangle$.

Then $\mathcal{RB}(\mathbb{F}_3) = \text{Ker}(\Lambda)$ is the submodule of order 2 generated by $[-1] + \langle -1 \rangle [-1]$.

Furthermore, we let $\mathcal{P}(\mathbb{F}_3) = \mathcal{RP}(\mathbb{F}_3)_{\mathbb{F}_3^\times}$. This is a cyclic \mathbb{Z} -module of order 4 with generator $[-1]$. Let $\lambda : \mathcal{P}(\mathbb{F}_3) \rightarrow S_{\mathbb{Z}}^2(\mathbb{F}_3^\times) = \text{Sym}_{\mathbb{F}_2}^2(\mathbb{F}_3^\times)$ be the map $[-1] \mapsto -1 \circ -1$. Then $\mathcal{B}(\mathbb{F}_3) := \text{Ker}(\lambda)$ is cycle of order 2 with generator $2[-1]$ and the natural map $\mathcal{RB}(\mathbb{F}_3) \rightarrow \mathcal{B}(\mathbb{F}_3)$ is an isomorphism.

2.5. The refined Bloch Group and $H_3(SL_2(F), \mathbb{Z})$. We recall some results from [7]: The main result there is

Theorem 2.1. *Let F be an infinite field.*

There is a natural complex

$$0 \rightarrow \text{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \rightarrow H_3(SL_2(F), \mathbb{Z}) \rightarrow \mathcal{RB}(F) \rightarrow 0.$$

which is exact everywhere except possibly at the middle term. The middle homology is annihilated by 4.

In particular, for any infinite field there is a natural short exact sequence

$$0 \rightarrow \text{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)[\frac{1}{2}] \rightarrow H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{RB}(F)[\frac{1}{2}] \rightarrow 0.$$

The following result is Corollary 5.1 in [7]:

Lemma 2.2. *Let F be an infinite field. Then the natural map $\mathcal{RB}(F) \rightarrow \mathcal{B}(F)$ is surjective and the induced map $\mathcal{RB}(F)_{F^\times} \rightarrow \mathcal{B}(F)$ has a 2-primary torsion kernel.*

Now for any field F , let

$$H_3(\mathrm{SL}_2(F), \mathbb{Z})_0 := \mathrm{Ker}(H_3(\mathrm{SL}_2(F), \mathbb{Z}) \rightarrow K_3^{\mathrm{ind}}(F))$$

and

$$\mathcal{RB}_0(F) := \mathrm{Ker}(\mathcal{RB}(F) \rightarrow \mathcal{B}(F))$$

Lemma 2.3. [7, Lemma 5.2] *Let F be an infinite field. Then*

- (1) $H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 = \mathcal{RB}(F)[\frac{1}{2}]_0$
- (2) $H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 = \mathcal{I}_F H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])$ and $\mathcal{RB}_0(F)[\frac{1}{2}] = \mathcal{I}_F \mathcal{RB}(F)[\frac{1}{2}]$.
- (3) $H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 = \mathrm{Ker}(H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(\mathrm{SL}_3(F), \mathbb{Z}[\frac{1}{2}]))$
 $= \mathrm{Ker}(H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(\mathrm{GL}_2(F), \mathbb{Z}[\frac{1}{2}]))$.

On the other hand, the corresponding results for finite fields are as follows (the results in [7] apply to fields with at least 4 elements, but it is straightforward to verify that they extend to the fields \mathbb{F}_2 and \mathbb{F}_3 with the definitions supplied above):

Lemma 2.4. [7, Lemma 7.1]. *For a finite field k the natural map $\mathcal{RP}(k) \rightarrow \mathcal{P}(k)$ induces an isomorphism $\mathcal{RB}(k) \cong \mathcal{B}(k)$.*

For a field F , we let $\mathrm{Tor}_1^{\mathbb{Z}}(\widetilde{\mu_F}, \mu_F)$ denote the unique nontrivial extension of $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)$ by $\mathbb{Z}/2$ if the characteristic of F is not 2, and $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)$ in characteristic 2.

Theorem 2.5. [7, Corollary 7.5] *There is a natural short exact sequence*

$$0 \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\widetilde{\mu_{\mathbb{F}_q}}, \mu_{\mathbb{F}_q}) \rightarrow H_3(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[1/p]) \rightarrow \mathcal{B}(\mathbb{F}_q) \rightarrow 0$$

for any finite field \mathbb{F}_q of order $q = p^f$.

Furthermore, there is a natural isomorphism

$$H_3(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[1/p]) \cong K_3^{\mathrm{ind}}(\mathbb{F}_q).$$

Lemma 2.6. [7, Section 5,7]

$$\mathcal{B}(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/(q+1)/2, & q \text{ odd} \\ \mathbb{Z}/(q+1), & q \text{ even} \end{cases}$$

and if $K \subset \mathrm{SL}_2(\mathbb{F}_q)$ is a cyclic subgroup of order $(q+1)'$ then the composite map

$$\mathbb{Z}/(q+1)' \cong H_3(K, \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{B}(\mathbb{F}_q)[\frac{1}{2}]$$

is an isomorphism.

Corollary 2.7. *For any prime power q , $\mathcal{P}(\mathbb{F}_q)[\frac{1}{2}]$ is cyclic of order $(q+1)'$.*

Proof. $S_{\mathbb{Z}}^2(\mathbb{F}_q^\times)$ has order dividing 2, so the inclusion $\mathcal{B}(\mathbb{F}_q) \rightarrow \mathcal{P}(\mathbb{F}_q)$ induces an isomorphism $\mathcal{B}(\mathbb{F}_q)[\frac{1}{2}] \cong \mathcal{P}(\mathbb{F}_q)[\frac{1}{2}]$. \square

2.6. **The map $\mathrm{H}_3(G, \mathbb{Z}) \rightarrow \mathcal{RB}(F)$.** If G is any subgroup of $\mathrm{SL}_2(F)$, then by composing the map $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \rightarrow \mathcal{RB}(F)$ of Theorem 2.1 with the map $\mathrm{H}_3(G, \mathbb{Z}) \rightarrow \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$ induced by the inclusion of G into $\mathrm{SL}_2(F)$ we obtain a map $\mathrm{H}_3(G, \mathbb{Z}) \rightarrow \mathcal{RB}(F)$. In [7, Section 6] a recipe is given for calculating this map for subgroups G of $\mathrm{SL}_2(F)$ which don't act transitively on $\mathbb{P}^1(F)$. We recall this calculation in the case that G is a finite cyclic subgroup.

First, given 4 distinct points $x_0, x_1, x_2, x_3 \in \mathbb{P}^1(F)$ we define the *refined cross ratio* $\mathrm{cr}(x_0, x_1, x_2, x_3) \in \mathcal{RP}(F)$ by

$$\mathrm{cr}(x_0, x_1, x_2, x_3) = \begin{cases} \left\langle \frac{(x_2-x_0)(x_0-x_1)}{x_2-x_1} \right\rangle \left[\frac{(x_2-x_1)(x_3-x_0)}{(x_2-x_0)(x_3-x_1)} \right], & \text{if } x_i \neq \infty \\ \langle x_1 - x_2 \rangle \left[\frac{x_1-x_2}{x_1-x_3} \right], & \text{if } x_0 = \infty \\ \langle x_2 - x_0 \rangle \left[\frac{x_3-x_0}{x_2-x_0} \right], & \text{if } x_1 = \infty \\ \langle x_0 - x_1 \rangle \left[\frac{x_3-x_0}{x_3-x_1} \right], & \text{if } x_2 = \infty \\ \left\langle \frac{(x_2-x_0)(x_0-x_1)}{x_2-x_1} \right\rangle \left[\frac{x_2-x_1}{x_2-x_0} \right], & \text{if } x_3 = \infty \end{cases}$$

Lemma 2.8. [7, Section 6] *Suppose that G is a finite cyclic subgroup of $\mathrm{SL}_2(F)$ of order r with generator t . Let $x \in \mathbb{P}^1(F)$ with trivial stabilizer $G_x = 1$, and let $y \in \mathbb{P}^1(F) \setminus G \cdot x$.*

The composite

$$\mathbb{Z}/r \cong \mathrm{H}_3(G, \mathbb{Z}) \rightarrow \mathcal{RB}(F)$$

is given by the formula

$$1 \mapsto \sum_{i=0}^{r-1} \mathrm{cr}(\beta_3^{x,y}(1, t, t^{i+1}, t^{i+2})).$$

where

$$\begin{aligned} \beta_3^{x,y}(1, t, t, t^2) &= 0 \\ \beta_3^{x,y}(1, t, t^{i+1}, t^{i+2}) &= (x, t(x), t^{i+1}(x), t^{i+2}(x)) \text{ for } 1 \leq i \leq r-3 \\ \beta_3^{x,y}(1, t, t^{r-1}, 1) &= (y, t(x), t^{-1}(x), x) - (y, x, t(x), t^{-1}(x)) \\ \beta_3^{x,y}(1, t, t^r, t^{r+1}) &= \beta_3^{x,y}(1, t, 1, t) \\ &= \begin{cases} 0, & y = t(y) \\ (y, t(y), x, t(x)) + (y, t(y), t(x), x), & y \neq t(y) \end{cases} \end{aligned}$$

Furthermore, the resulting map is independent of the particular choice of x and y .

3. SOME ALGEBRA IN $\mathcal{RP}(F)$

In this section we study certain key elements and submodules of the refined pre-Bloch group of a field F .

3.1. **The elements $\psi_i(x)$ and the modules $\mathcal{K}_F^{(i)}$.** We recall the elements

$$\{x\} := [x] + [x^{-1}] \in \mathcal{P}(F)$$

(for $x \in F^\times$). A straightforward calculation - see Suslin [21] - shows that these symbols allow us to define a group homomorphism

$$F^\times \rightarrow \mathcal{P}(F), \quad x \mapsto \{x\}$$

whose kernel contains $(F^\times)^2$; i.e. we have

$$\{x^2\} = 0 \text{ and } \{xy\} = \{x\} + \{y\} \text{ for all } x, y.$$

In particular, these elements satisfy $2\{x\} = 0$ for all x .

We now consider two liftings of these elements in $\mathcal{RP}(F)$: For $x \in F^\times$ we let

$$\psi_1(x) := [x] + \langle -1 \rangle [x^{-1}]$$

and

$$\psi_2(x) := \begin{cases} \langle 1-x \rangle (\langle x \rangle [x] + [x^{-1}]), & x \neq 1 \\ 0, & x = 1 \end{cases}$$

(If $F = \mathbb{F}_2$, we interpret this as $\psi_i(1) = 0$ for $i = 1, 2$. For $F = \mathbb{F}_3$, we have $\psi_1(-1) = \psi_2(-1) = [-1] + \langle -1 \rangle [-1]$.)

The maps $F^\times \rightarrow \mathcal{RP}(F)$, $x \mapsto \psi_i(x)$ are no longer homomorphisms, but are 1-cocycles for the action of F^\times :

Lemma 3.1. *Let F be a field. For $i \in \{1, 2\}$, the map*

$$F^\times \rightarrow \mathcal{RP}(F), x \mapsto \psi_i(x)$$

is a 1-cocycle; i.e. we have

$$\psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x) \text{ for all } x, y \in F^\times.$$

Proof. The statement is trivial for $F = \mathbb{F}_2$ or \mathbb{F}_3 . We can thus assume F has at least 4 elements.

If $x = 1$ or $y = 1$, the required identities are clear. If $x \neq 1$ and $y \neq x^{-1}$ the relation $0 = S_{x,xy} + \langle -1 \rangle S_{x^{-1},x^{-1}y^{-1}}$ in $\mathcal{RP}(F)$ yields the identity

$$\psi_1(x) - \psi_1(xy) + \langle x \rangle \psi_1(y) = 0.$$

Thus we must also prove that $\langle x \rangle \psi_1(x^{-1}) + \psi_1(x) = 0$ for all $x \neq 1$. Fix $x \neq 1$ and choose $y \notin \{1, x^{-1}\}$ (here we use that F has at least 4 elements). Then

$$\langle y \rangle \psi_1(x) = \psi_1(xy) - \psi_1(y) = -\langle xy \rangle \psi_1(x^{-1})$$

and multiplying by $\langle y \rangle$ gives the required identity.

Now, for $x, y \in F^\times$, let

$$Q(x, y) := \langle x \rangle \begin{bmatrix} x \\ y \end{bmatrix} + \langle y \rangle \begin{bmatrix} y \\ x \end{bmatrix} \in \mathcal{RP}(F).$$

Then

$$Q(x, y) = \langle y \rangle \left(\left\langle \frac{x}{y} \right\rangle \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} y \\ x \end{bmatrix} \right) = \langle y \rangle \left\langle 1 - \frac{x}{y} \right\rangle \psi_2 \left(\frac{x}{y} \right) = \langle y - x \rangle \psi_2 \left(\frac{x}{y} \right).$$

For $a, b \neq 1$, the relation $0 = S_{a,b} + S_{b,a}$ in $\mathcal{RP}(F)$ gives the identity

$$Q(a^{-1} - 1, b^{-1} - 1) = Q(a^{-1}, b^{-1}) + Q(1 - a, 1 - b).$$

Thus

$$\langle b^{-1} - a^{-1} \rangle \psi_2 \left(\frac{a^{-1} - 1}{b^{-1} - 1} \right) = \langle b^{-1} - a^{-1} \rangle \psi_2 \left(\frac{b}{a} \right) + \langle a - b \rangle \psi_2 \left(\frac{1 - a}{1 - b} \right)$$

and hence

$$\psi_2 \left(\frac{b^{-1} - 1}{a^{-1} - 1} \right) = \psi_2 \left(\frac{b}{a} \right) + \left\langle \frac{b}{a} \right\rangle \psi_2 \left(\frac{1 - a}{1 - b} \right).$$

Now if we fix $x, y \neq 1$ with $xy \neq 1$, we can solve the equations

$$x = \frac{b}{a}, \quad y = \frac{1-a}{1-b}$$

for a and b and prove the required identity for $\psi_2(\cdot)$. \square

Observe from the definitions that, for $i = 1, 2$, $\psi_i(x^{-1}) = \langle -1 \rangle \psi_i(x)$ for all $x \in F^\times$. In particular, $\langle -1 \rangle \psi_i(-1) = \psi_i(-1)$ for $i = 1, 2$.

The properties enumerated in following proposition will be used often in the remainder of this section.

Proposition 3.2. *For $i \in \{1, 2\}$ we have:*

- (1) $\langle\langle x \rangle\rangle \psi_i(y) = \langle\langle y \rangle\rangle \psi_i(x)$ for all x, y
- (2) $\psi_i(xy^2) = \psi_i(x) + \psi_i(y^2)$ for all x, y
- (3) $\langle\langle x \rangle\rangle \psi_i(y^2) = 0$ for all x, y
- (4) $2 \cdot \psi_i(-1) = 0$ for all i
- (5) $\psi_i(x^2) = -\langle\langle x \rangle\rangle \psi_i(-1)$ for all x
- (6) $2 \cdot \psi_i(x^2) = 0$ for all x and if -1 is a square in F then $\psi_i(x^2) = 0$ for all x .
- (7) $\langle\langle x \rangle\rangle \langle\langle y \rangle\rangle \psi_i(-1) = 0$ for all x, y
- (8) $\langle -1 \rangle \langle\langle x \rangle\rangle \psi_i(y) = \langle\langle x \rangle\rangle \psi_i(y)$ for all x, y
- (9) Let

$$\epsilon(F) := \begin{cases} 1, & -1 \in (F^\times)^2 \\ 2, & -1 \notin (F^\times)^2 \end{cases}$$

The map $G_F \rightarrow \mathcal{RP}(F), \langle x \rangle \mapsto \epsilon(F) \psi_i(x)$ is a well-defined 1-cocycle.

Proof. The identities $\psi_i(1) = 0$ and $\psi_i(x^{-1}) = \langle -1 \rangle \psi_i(x)$ follow from the definition of $\psi_i(\cdot)$. More generally, let M be an R_F -module and let $\psi : F^\times \rightarrow M$ be a 1-cocycle satisfying

$$\psi(1) = 0 \text{ and } \psi(x^{-1}) = \langle -1 \rangle \psi(x) \text{ for all } x \in F^\times.$$

- (1) By the cocycle condition, for all $x, y \in F^\times$ we have

$$\psi(xy) = \langle x \rangle \psi(y) + \psi(x) = \langle y \rangle \psi(x) + \psi(y).$$

Rearranging this latter equality, we deduce: $\langle\langle x \rangle\rangle \psi(y) = \langle\langle y \rangle\rangle \psi(x)$ for all x, y .

- (2) For all x, y we have

$$\psi(xy^2) = \langle y^2 \rangle \psi(x) + \psi(y^2) = \psi(x) + \psi(y^2)$$

since $\langle y^2 \rangle = 1$ in R_F .

- (3) From (1) together with the fact that $\langle\langle y^2 \rangle\rangle = 0$ for all y , we deduce:

$$\langle\langle x \rangle\rangle \psi(y^2) = \langle\langle y^2 \rangle\rangle \psi(x) = 0.$$

- (4) We have $\langle -1 \rangle \psi(-1) = \psi(-1)$ and thus

$$0 = \psi(1) = \psi(-1 \cdot -1) = \psi(-1) + \langle -1 \rangle \psi(-1) = 2\psi(-1).$$

- (5) For all x we have

$$\psi(x) = \psi\left(\frac{1}{x} \cdot x^2\right) = \psi\left(\frac{1}{x}\right) + \psi(x^2) = \langle -1 \rangle \psi(x) + \psi(x^2).$$

Thus

$$\psi(x^2) = -\langle\langle -1 \rangle\rangle \psi(x) = -\langle\langle x \rangle\rangle \psi(-1).$$

(6) The first statement follows from (4) and (5). For the second, observe that for any x we have

$$\psi(x^2) = -\langle\langle -1 \rangle\rangle \psi(x)$$

and $\langle\langle -1 \rangle\rangle = 0$ if -1 is a square.

(7) This statement follows from (3) and (5).

(8) This is a restatement of (7); namely

$$\langle\langle -1 \rangle\rangle \langle\langle x \rangle\rangle \psi(y) = \langle\langle x \rangle\rangle \langle\langle y \rangle\rangle \psi(-1) = 0.$$

(9) By (6), $\epsilon(F)\psi(x^2) = 0$ in M for all x and thus $\epsilon(F)\psi(xy^2) = \epsilon(F)\psi(x)$ for all x, y . Thus the proposed map is well-defined (and is thus clearly a 1-cocycle). \square

Now, for $i \in \{1, 2\}$, let $\mathcal{K}_F^{(i)}$ denote the \mathbf{R}_F -submodule of $\mathcal{RP}(F)$ generated by the set $\{\psi_i(x) \mid x \in F^\times\}$.

Lemma 3.3. *Let F be a field. Then for $i \in \{1, 2\}$*

$$\lambda_1(\mathcal{K}_F^{(i)}) = \mathfrak{p}_{-1}^+(\mathcal{I}_F) \subset \mathcal{I}_F^2$$

and $\text{Ker}(\lambda_1|_{\mathcal{K}_F^{(i)}})$ is annihilated by 4.

Proof. We use the identities

$$\langle\langle a \rangle\rangle \langle\langle b \rangle\rangle = \langle\langle ab \rangle\rangle - \langle\langle a \rangle\rangle - \langle\langle b \rangle\rangle, \quad \langle\langle -1 \rangle\rangle \langle\langle a \rangle\rangle = \langle\langle -a \rangle\rangle - \langle\langle -1 \rangle\rangle, \quad \langle\langle ab^2 \rangle\rangle = \langle\langle a \rangle\rangle$$

in \mathcal{I}_F .

Thus

$$\begin{aligned} \lambda_1(\psi_1(x)) &= \lambda_1([x]) + \langle\langle -1 \rangle\rangle \lambda_1([x^{-1}]) \\ &= \langle\langle x \rangle\rangle \langle\langle 1-x \rangle\rangle + \langle\langle -1 \rangle\rangle \langle\langle x \rangle\rangle \langle\langle x(x-1) \rangle\rangle \\ &= \langle\langle x(1-x) \rangle\rangle - \langle\langle x \rangle\rangle - \langle\langle 1-x \rangle\rangle + \langle\langle -1 \rangle\rangle (\langle\langle x-1 \rangle\rangle - \langle\langle x \rangle\rangle - \langle\langle x(x-1) \rangle\rangle) \\ &= \langle\langle x(1-x) \rangle\rangle - \langle\langle x \rangle\rangle - \langle\langle 1-x \rangle\rangle + \langle\langle 1-x \rangle\rangle - \langle\langle -x \rangle\rangle - \langle\langle x(1-x) \rangle\rangle + \langle\langle -1 \rangle\rangle \\ &= \langle\langle -1 \rangle\rangle - \langle\langle x \rangle\rangle - \langle\langle -x \rangle\rangle = \langle\langle -x \rangle\rangle \cdot \langle\langle x \rangle\rangle \\ &= -\mathfrak{p}_{-1}^+ \cdot \langle\langle x \rangle\rangle \end{aligned}$$

Thus $\lambda_1(\mathcal{K}_F^{(1)}) = \mathfrak{p}_{-1}^+(\mathcal{I}_F)$.

For $x \neq 1$ we have $\psi_2(x) = \langle\langle x(1-x) \rangle\rangle [x] + \langle\langle 1-x \rangle\rangle [x^{-1}]$ and thus

$$\begin{aligned} \lambda_1(\psi_2(x)) &= \langle\langle x(1-x) \rangle\rangle \langle\langle x \rangle\rangle \langle\langle 1-x \rangle\rangle + \langle\langle 1-x \rangle\rangle \langle\langle x \rangle\rangle \langle\langle x(x-1) \rangle\rangle \\ &= \langle\langle x(1-x) \rangle\rangle (\langle\langle x(1-x) \rangle\rangle - \langle\langle x \rangle\rangle - \langle\langle 1-x \rangle\rangle) + \langle\langle 1-x \rangle\rangle (\langle\langle x-1 \rangle\rangle - \langle\langle x(x-1) \rangle\rangle - \langle\langle x \rangle\rangle) \\ &= -\langle\langle 1-x \rangle\rangle - \langle\langle x \rangle\rangle + \langle\langle x(1-x) \rangle\rangle + \langle\langle -1 \rangle\rangle - \langle\langle -x \rangle\rangle - \langle\langle x(1-x) \rangle\rangle + \langle\langle 1-x \rangle\rangle \\ &= \langle\langle -1 \rangle\rangle - \langle\langle x \rangle\rangle - \langle\langle -x \rangle\rangle = \langle\langle x \rangle\rangle \cdot \langle\langle -x \rangle\rangle = -\mathfrak{p}_{-1}^+ \cdot \langle\langle x \rangle\rangle. \end{aligned}$$

Thus $\lambda_1(\mathcal{K}_F^{(2)}) = \mathfrak{p}_{-1}^+(\mathcal{I}_F)$ also.

For the second statement, recall that for any group G and any $\mathbb{Z}[G]$ -module M a 1-cocycle $\rho : G \rightarrow M$ gives rise to $\mathbb{Z}[G]$ -homomorphism $\mathcal{I}_G \rightarrow M$, defined by $g-1 \mapsto \rho(g)$. Thus, for $i \in \{1, 2\}$, we have a well-defined \mathbf{R}_F -homomorphism

$$\mathcal{I}_F \rightarrow \mathcal{RP}(F), \quad \langle\langle x \rangle\rangle \mapsto 2\psi_i(x).$$

Combining this with the inclusion $\mathfrak{p}_{-1}^+(\mathcal{I}_F) \rightarrow \mathcal{I}_F$ we obtain an \mathbf{R}_F -module homomorphism $\mu : \mathfrak{p}_{-1}^+(\mathcal{I}_F) \rightarrow \mathcal{K}_F^{(i)}$ sending $\mathfrak{p}_{-1}^+ \langle\langle x \rangle\rangle = \langle\langle x \rangle\rangle + \langle\langle -1 \rangle\rangle \langle\langle x \rangle\rangle$ to $2\psi_i(x) + \langle\langle -1 \rangle\rangle 2\psi_i(x)$. However,

for any $x \in F^\times$ we have $2\langle\langle -1 \rangle\rangle\psi_i(x) = 2\langle\langle x \rangle\rangle\psi_i(-1) = 0$ by Proposition 3.2 (4), so that $2\langle -1 \rangle\psi_i(x) = 2\psi_i(x)$ and thus $2\psi_i(x) + \langle -1 \rangle 2\psi_i(x) = 4\psi_i(x)$.

It follows that $\mu \circ (\lambda_1|_{\mathcal{K}_F^{(i)}})$ is just multiplication by 4, and the result is proved. \square

Remark 3.4. Since $p_{-1}^+ \mathcal{I}_F$ is a free abelian group, it follows that, as an abelian group, $\mathcal{K}_F^{(i)}$ decomposes as a direct sum $A \oplus (\mathcal{K}_F^{(i)})_{\text{tors}}$ where A is a free abelian group and 4 annihilates $(\mathcal{K}_F^{(i)})_{\text{tors}} = \text{Ker}(\lambda_1|_{\mathcal{K}_F^{(i)}})$.

Furthermore, if $\langle\langle -1 \rangle\rangle\psi_i(x) = 0$ for all x (for example, if $-1 \in (F^\times)^2$) then $\psi_i(x^2) = 0$ for all x and the map $G_F \rightarrow \mathcal{RP}(F)$, $\langle x \rangle \mapsto \psi_i(x)$ is already a well-defined 1-cocycle. The above arguments then show that $(\mathcal{K}_F^{(i)})_{\text{tors}} = \text{Ker}(\lambda_1|_{\mathcal{K}_F^{(i)}})$ is annihilated by 2.

3.2. The constants C_F and D_F . In the classical pre-Bloch group $\mathcal{P}(F)$ the expression $[x] + [1-x]$ is known to be independent of $x \in F \setminus \{0, 1\}$ (see, for example, [21, Section 1]). Furthermore, this constant has order dividing 6. We consider now an analogous constant in $\mathcal{RP}(F)$.

Let F be a field with at least 4 elements. For $x \in F^\times$, $x \neq 1$ we let

$$\tilde{C}(x) := [x] + \langle -1 \rangle [1-x] \text{ and } C(x) = \tilde{C}(x) + \langle\langle 1-x \rangle\rangle \psi_1(x).$$

Lemma 3.5. *Let F be a field with at least 4 elements. Then $C(x)$ is constant; i.e. for all $x, y \in F \setminus \{0, 1\}$ we have $C(x) = C(y)$ in $\mathcal{RP}(F)$.*

Proof. In $\mathcal{RP}(F)$ we have

$$\begin{aligned} 0 &= S_{x,y} + \langle -1 \rangle S_{1-x,1-y} \\ &= [x] - [y] + \langle x \rangle \left[\frac{y}{x} \right] - \langle x^{-1} - 1 \rangle \left[\frac{x^{-1} - 1}{y^{-1} - 1} \right] + \langle 1-x \rangle \left[\frac{1-x}{1-y} \right] \\ &\quad + \langle -1 \rangle [1-x] - \langle -1 \rangle [1-y] + \langle x-1 \rangle \left[\frac{1-y}{1-x} \right] - \langle 1-x^{-1} \rangle \left[\frac{y^{-1} - 1}{x^{-1} - 1} \right] + \langle -x \rangle \left[\frac{x}{y} \right] \\ &= \tilde{C}(x) - \tilde{C}(y) + \langle x \rangle \psi_1 \left(\frac{y}{x} \right) - \langle 1-x^{-1} \rangle \psi_1 \left(\frac{1-y^{-1}}{1-x^{-1}} \right) + \langle x-1 \rangle \psi_1 \left(\frac{y-1}{x-1} \right) \\ &= (\tilde{C}(x) - \psi_1(x) + \psi_1(1-x^{-1}) - \psi_1(x-1)) - (\tilde{C}(y) - \psi_1(y) + \psi_1(1-y^{-1}) - \psi_1(y-1)) \end{aligned}$$

(using the cocycle property of $\psi_1(\cdot)$ to obtain the last line). Furthermore

$$\begin{aligned} \psi_1(1-x^{-1}) - \psi_1(x-1) - \psi_1(x) &= \psi_1((x-1)x^{-1}) - \psi_1(x-1) - \psi_1(x) \\ &= \langle x-1 \rangle \psi_1(x^{-1}) - \psi_1(x) = \langle 1-x \rangle \psi_1(x) - \psi_1(x) = \langle\langle 1-x \rangle\rangle \psi_1(x). \end{aligned}$$

\square

Definition 3.6. Thus, for a given field F with at least 4 elements, we will denote by C_F the common value of the expressions $C(x)$ for $x \in F \setminus \{0, 1\}$.

Furthermore, we let $C_{\mathbb{F}_2}$ denote the distinguished generator of $\mathcal{RP}(\mathbb{F}_2) = \mathcal{RB}(\mathbb{F}_2)$ and we let $C_{\mathbb{F}_3} := \psi_1(-1) = (1 + \langle -1 \rangle)[-1] \in \mathcal{RB}(\mathbb{F}_3)$.

For any field F , we let $\Phi(T)$ denote the polynomial $T^2 - T + 1 \in F[T]$. Observe that if F has characteristic 3 then -1 is a root of $\Phi(T)$. In any other characteristic, a root of $\Phi(T)$ will be of the form $-\zeta$, where ζ is a primitive cube root of unity.

Corollary 3.7. *Let F be a field.*

- (1) $\langle -1 \rangle C_F = C_F$
- (2) If $\Phi(T)$ has a root in F , then $C_F = \psi_1(-1)$.
- (3) For any field F , $C_F \in \mathcal{RB}(F)$; i.e $\Lambda(C_F) = 0$.

Proof. (1) For $x \notin \{0, 1\}$ we have $C_F = C(1-x) = \langle -1 \rangle C(x) = \langle -1 \rangle C_F$ (since $\langle -1 \rangle \langle\langle 1-x \rangle\rangle \psi_1(x) = \langle\langle 1-x \rangle\rangle \psi_1(x)$ by Proposition 3.2 (8)).

- (2) \mathbb{F}_2 contains no root of $\Phi(T)$. If $F = \mathbb{F}_3$, then -1 is a root of $\Phi(T)$, but the result is true by definition. So we can assume that F has at least 4 elements.

Let x be a root of $\Phi(T)$. Then $1-x = x^{-1}$ and thus

$$\begin{aligned} C_F = C(x) &= [x] + \langle -1 \rangle [1-x] + \langle\langle 1-x \rangle\rangle \psi_1(x) \\ &= \psi_1(x) + \langle\langle x^{-1} \rangle\rangle \psi_1(x) \\ &= \psi_1(x) + \langle\langle x \rangle\rangle \psi_1(x) = \langle x \rangle \psi_1(x). \end{aligned}$$

Now $x^3 = -1$, so that $x = -1 \cdot x^4$ and hence $\langle x \rangle = \langle -1 \rangle$.

Furthermore, using Proposition 3.2 (2) and (6), it follows that

$$= \psi_1(x) = \psi_1(-1 \cdot x^4) = \psi_1(-1) + \psi_1(x^4) = \psi_1(-1) + 2\psi_1(x^2) = \psi_1(-1)$$

Thus $C_F = \langle -1 \rangle \psi_1(-1) = \psi_1(-1)$ as claimed.

- (3) Fix $x \in F^\times \setminus \{1\}$. Recall that $\lambda_1(\psi_1(x)) = -p_{-1}^+ \langle\langle x \rangle\rangle$ (see the proof of Lemma 3.3). Thus

$$\begin{aligned} \lambda_1(C_F) &= \lambda_1([x]) + \langle -1 \rangle \lambda_1([1-x]) - \langle\langle 1-x \rangle\rangle \lambda_1(\psi_1(x)) \\ &= \langle\langle 1-x \rangle\rangle \langle\langle x \rangle\rangle + \langle -1 \rangle \langle\langle x \rangle\rangle \langle\langle 1-x \rangle\rangle - \langle\langle 1-x \rangle\rangle p_{-1}^+ \langle\langle x \rangle\rangle \\ &= p_{-1}^+ \langle\langle 1-x \rangle\rangle \langle\langle x \rangle\rangle - p_{-1}^+ \langle\langle 1-x \rangle\rangle \langle\langle x \rangle\rangle = 0. \end{aligned}$$

On the other hand, recalling that F^\times acts trivially on $\mathbb{S}_{\mathbb{Z}}^2(F^\times)$, we have

$$\begin{aligned} \lambda_2(C_F) &= \lambda_2([x]) + \langle -1 \rangle \lambda_2([1-x]) - \langle\langle 1-x \rangle\rangle \lambda_2(\psi_1(x)) \\ &= (1-x) \circ x + x \circ (1-x) = 0 \text{ in } \mathbb{S}_{\mathbb{Z}}^2(F^\times) \end{aligned}$$

proving the result. □

Definition 3.8. For any field F , we define the element $D_F \in \mathcal{RB}(F)$ by $D_F := 2C_F$.

Observe that $D_F = (1 + \langle -1 \rangle)C_F$ by Corollary 3.7 (1).

Lemma 3.9. For any $x \in F \setminus \{0, 1\}$ we have

$$D_F = [x] + \langle -1 \rangle \left[\frac{1}{1-x^{-1}} \right] - \psi_1 \left(\frac{1}{1-x} \right)$$

Proof.

$$\begin{aligned} D_F = 2C_F &= C(x) + C \left(\frac{1}{1-x} \right) \\ &= [x] + \langle -1 \rangle [1-x] + \langle\langle 1-x \rangle\rangle \psi_1(x) + \left[\frac{1}{1-x} \right] + \langle -1 \rangle \left[\frac{1}{1-x^{-1}} \right] + \langle\langle 1-x^{-1} \rangle\rangle \psi_1 \left(\frac{1}{1-x} \right). \end{aligned}$$

However,

$$\langle -1 \rangle [1-x] + \left[\frac{1}{1-x} \right] = \psi_1 \left(\frac{1}{1-x} \right)$$

and

$$\langle\langle 1-x \rangle\rangle \psi_1(x) = \left\langle\left\langle \frac{1}{1-x} \right\rangle\right\rangle \psi_1(x) = \langle\langle x \rangle\rangle \psi_1\left(\frac{1}{1-x}\right).$$

Thus

$$\begin{aligned} D_F &= [x] + \langle -1 \rangle \left[\frac{1}{1-x^{-1}} \right] + \left(1 + \langle\langle x \rangle\rangle + \left\langle\left\langle 1-x^{-1} \right\rangle\right\rangle \right) \psi_1\left(\frac{1}{1-x}\right) \\ &= [x] + \langle -1 \rangle \left[\frac{1}{1-x^{-1}} \right] + \left(\langle\langle x \rangle\rangle + \langle 1-x^{-1} \rangle - 1 \right) \psi_1\left(\frac{1}{1-x}\right). \end{aligned}$$

So it remains to prove that

$$\langle x \rangle \psi_1\left(\frac{1}{1-x}\right) = -\langle 1-x^{-1} \rangle \psi_1\left(\frac{1}{1-x}\right)$$

for any $x \neq 0, 1$.

Now, by the cocycle property,

$$\langle x \rangle \psi_1\left(\frac{1}{1-x}\right) = \psi_1\left(\frac{x}{1-x}\right) - \psi_1(x).$$

However, for any $a \in F^\times$, we have $\psi_1(a) = \langle -1 \rangle \psi_1(-a) - \psi_1(-1)$, and thus for any $a, b \in F^\times$ we have

$$\psi_1(a) - \psi_1(b) = \langle -1 \rangle \psi_1(-a) - \langle -1 \rangle \psi_1(-b) = \psi_1(-a^{-1}) - \psi_1(-b^{-1}).$$

It follows that

$$\langle x \rangle \psi_1\left(\frac{1}{1-x}\right) = \psi_1\left(-\frac{1-x}{x}\right) - \psi_1(-x^{-1}) = \psi_1(1-x^{-1}) - \psi_1(-x^{-1}).$$

On the other hand, using the cocycle property again,

$$\begin{aligned} \langle 1-x^{-1} \rangle \psi_1\left(\frac{1}{1-x}\right) &= \psi_1\left(\frac{1-x^{-1}}{1-x}\right) - \psi_1(1-x^{-1}) \\ &= \psi_1(-x^{-1}) - \psi_1(1-x^{-1}) \end{aligned}$$

completing the proof. □

Lemma 3.10. *For any field F we have $3D_F = 6C_F = 0$.*

Proof. Fix $x \neq 0, 1$. Then

$$\begin{aligned} 3D_F &= D\left(\frac{1}{x}\right) + D\left(\frac{1}{1-x^{-1}}\right) + D(1-x) \\ &= \left[\frac{1}{x}\right] + \langle -1 \rangle \left[\frac{1}{1-x}\right] - \left[\frac{1}{1-x^{-1}}\right] - \langle -1 \rangle [1-x^{-1}] \\ &\quad + \left[\frac{1}{1-x^{-1}}\right] + \langle -1 \rangle [x] - [1-x] - \langle -1 \rangle \left[\frac{1}{1-x}\right] \\ &\quad + [1-x] + \langle -1 \rangle [1-x^{-1}] - \left[\frac{1}{x}\right] - \langle -1 \rangle [x] \\ &= 0. \end{aligned}$$

□

Remark 3.11. Examples show that this is best possible. Under the natural map $\mathcal{RP}(F) \rightarrow \mathcal{P}(F)$ the image of C_F is the constant $C := [x] + [1 - x] \in \mathcal{B}(F)$. It can be shown that this element has order 6 for example when $F = \mathbb{R}$ ([21, Section 1]) or when F is a finite field with q elements and $q \equiv -1 \pmod{12}$ ([7, Lemma 7.11]).

Theorem 3.12. *Let F be a field. Then*

(1) *For all $x \in F^\times$,*

$$\langle\langle x \rangle\rangle D_F = \psi_1(x) - \psi_2(x).$$

(2) *Let E be the field extension obtained from F by adjoining a root of $\Phi(T)$. Then*

$$\langle\langle x \rangle\rangle D_F = 0 \text{ if } \pm x \in N_{E/F}(E^\times) \subset F^\times.$$

Proof. We consider first the case of a finite field F . The results in the final section of [7] show that the natural map $\mathcal{RP}(F) \rightarrow \mathcal{P}(F)$ induces an isomorphism $\mathcal{RB}(F) \cong \mathcal{B}(F)$. Now $D_F \in \mathcal{RB}(F)$ and thus $\langle\langle x \rangle\rangle D_F = 0$ for all x . Similarly for all x , $\psi_1(x) - \psi_2(x) \in \mathcal{RB}(F)$ and this maps to $\{x\} - \{x\} = 0$ in $\mathcal{B}(F)$.

Thus, we can assume without loss that F is an infinite field.

Let

$$t := \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SL}_2(F).$$

So

$$t^2 = t^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

and for $x \in \mathbb{P}^1(F)$ we have $t(x) = 1 - x^{-1}$ and $t^{-1}(x) = (1 - x)^{-1}$.

Thus $t(x) = x$ if and only if $\Phi(x) = 0$.

We now choose $x \in F^\times \setminus \{1\}$ with $t(x) \neq x$ (i.e. x not a root of $\Phi(T)$) and $y \in F^\times \setminus \{1\}$ satisfying $t(y) \neq y$ and $y \notin \{x, t(x), t^{-1}(x)\}$.

By Lemma 2.8, the natural composite map

$$\mathbb{Z}/3 = \mathrm{H}_3(\langle t \rangle, \mathbb{Z}) \rightarrow \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \rightarrow \mathcal{RB}(F)$$

is given by the formula

$$1 \mapsto \sum_{i=0}^2 \mathrm{cr}(\beta_3^{x,y}(1, t, t^{i+1}, t^{i+2})) := C(x, y).$$

Thus, using Lemma 2.8 again, we have

$$\begin{aligned} C(x, y) &= \mathrm{cr}(\beta_3^{x,y}(1, t, t^{-1}, 1)) + \mathrm{cr}(\beta_3^{x,y}(1, t, 1, t)) \\ &= \mathrm{cr}(y, t(x), t^{-1}(x), x) - \mathrm{cr}(y, x, t(x), t^{-1}(x)) + \mathrm{cr}(y, t(y), x, t(x)) + \mathrm{cr}(y, t(y), t(x), x) \\ &= \left\langle \frac{(t^{-1}(x) - y)(y - t(x))}{t^{-1}(x) - t(x)} \right\rangle \left[\frac{(t^{-1}(x) - t(x))(x - y)}{(t^{-1}(x) - y)(x - t(x))} \right] - \left\langle \frac{(t(x) - y)(y - x)}{t(x) - x} \right\rangle \left[\frac{(t(x) - x)(t^{-1}(x) - y)}{(t(x) - y)(t^{-1}(x) - x)} \right] \\ &\quad + \left\langle \frac{(x - y)(y - t(y))}{x - t(y)} \right\rangle \left[\frac{(x - t(y))(t(x) - y)}{(x - y)(t(x) - t(y))} \right] + \left\langle \frac{(t(x) - y)(y - t(y))}{t(x) - t(y)} \right\rangle \left[\frac{(t(x) - t(y))(x - y)}{(t(x) - y)(x - t(y))} \right] \end{aligned}$$

Now let

$$a = a(x, y) = \frac{(x - y)(t(x) - y)}{t(x) - x}.$$

Multiplying $C(x, y)$ by the square class $\langle a \rangle$ gives

$$\begin{aligned} \langle a \rangle C(x, y) &= \left\langle \frac{(t^{-1}(x) - t(x))(x - y)}{(t^{-1}(x) - y)(x - t(x))} \right\rangle \left[\frac{(t^{-1}(x) - t(x))(x - y)}{(t^{-1}(x) - y)(x - t(x))} \right] - \langle -1 \rangle \left[\frac{(t(x) - x)(t^{-1}(x) - y)}{(t(x) - y)(t^{-1}(x) - x)} \right] \\ &+ \left\langle \frac{(y - t(y))(x - t(x))}{(x - t(y))(y - t(x))} \right\rangle \left[\frac{(x - t(y))(t(x) - y)}{(x - y)(t(x) - t(y))} \right] + \left\langle \frac{(y - t(y))(t(x) - x)}{(x - y)(t(x) - t(y))} \right\rangle \left[\frac{(t(x) - t(y))(x - y)}{(t(x) - y)(x - t(y))} \right] \end{aligned}$$

Now let

$$s = s(x, y) := \frac{(t^{-1}(x) - t(x))(x - y)}{(t^{-1}(x) - y)(x - t(x))}$$

and

$$u = u(x, y) := \frac{(x - y)(t(x) - t(y))}{(x - t(y))(t(x) - y)}.$$

Thus

$$\begin{aligned} \langle a \rangle C(x, y) &= \langle s \rangle [s] - \langle -1 \rangle \left[\frac{1}{1 - s} \right] + \langle 1 - u \rangle [u^{-1}] + \langle u^{-1} - 1 \rangle [u] \\ &= \langle s \rangle [s] - \langle -1 \rangle \left[\frac{1}{1 - s} \right] + \psi_2(u). \end{aligned}$$

Now

$$t^{-1}(x) - t(x) = \frac{x^2 - x + 1}{x(x - 1)}, \quad t^{-1}(x) - y = \frac{1 - y + xy}{1 - x}, \quad x - t(x) = -\frac{x^2 - x + 1}{x}$$

so that

$$s = \frac{x - y}{1 - y + xy}.$$

On the other hand,

$$u = \frac{(x - y)^2}{(xy - y + 1)(xy - x + 1)} = s \cdot \frac{s}{s - 1} = \frac{s}{1 - s^{-1}}.$$

Hence $C(x, y) = \langle a \rangle E$ where

$$E = E(x, y) = \langle s \rangle [s] - \langle -1 \rangle \left[\frac{1}{1 - s} \right] + \psi_2\left(\frac{s}{1 - s^{-1}}\right).$$

Now, from the definition of $\psi_2(s)$ and the identity $\langle -1 \rangle \psi_2(a) = \psi_2(a^{-1})$, we have

$$\langle s - 1 \rangle \psi_2(s^{-1}) = \langle 1 - s \rangle \psi_2(s) = \langle s \rangle [s] + [s^{-1}].$$

Furthermore

$$D_F = D(s^{-1}) = [s^{-1}] + \langle -1 \rangle \left[\frac{1}{1 - s} \right] - \psi_1\left(\frac{1}{1 - s^{-1}}\right).$$

It follows that

$$E = \langle s - 1 \rangle \psi_2(s^{-1}) - D_F - \psi_1\left(\frac{1}{1 - s^{-1}}\right) + \psi_2\left(\frac{s}{1 - s^{-1}}\right).$$

But

$$\langle s - 1 \rangle = \left\langle \frac{s}{1 - s^{-1}} \right\rangle.$$

Thus, using the cocycle property of $\psi_2(\cdot)$ we get

$$\begin{aligned} E &= \left\langle \frac{s}{1-s^{-1}} \right\rangle \psi_2(s^{-1}) + \psi_2\left(\frac{s}{1-s^{-1}}\right) - D_F - \psi_1\left(\frac{1}{1-s^{-1}}\right) \\ &= \psi_2\left(\frac{1}{1-s^{-1}}\right) - \psi_1\left(\frac{1}{1-s^{-1}}\right) - D_F. \end{aligned}$$

Now let

$$r := \frac{1}{1-s^{-1}} = \frac{x-y}{x-1-xy}$$

and observe that

$$a = \frac{(x-y)(t(x)-y)}{t(x)-x} = \frac{(x-y)(x-1-xy)}{x-1-x^2} = \frac{(x-y)(x-1-xy)}{-\Phi(x)}$$

and hence $\langle a \rangle = \langle -\Phi(x)r \rangle$.

Thus

$$(1) \quad C = C(x, y) = \langle -\Phi(x)r \rangle (\psi_2(r) - \psi_1(r) - D_F) \in \mathcal{RB}(F)$$

has order dividing 3 and is independent of the choice of x and y .

For the remainder of the proof, we will suppose that x satisfies $(x+1)\Phi(x) \neq 0$. Since for any $r \in F^\times \setminus \{x^{-1}, x/(x-1)\}$ we can solve for y

$$y = \frac{rx - r - x}{rx - 1} \in F^\times$$

then clearly r can assume any nonzero value other than x^{-1} and $x/(x-1)$ by appropriate choice of y .

In particular, taking $r = -1$ and multiplying both sides of the equation by 4 we see that

$$C = -\langle \Phi(x) \rangle D_F$$

since $4\psi_i(-1) = 0$ while $4C = C$ and $4D_F = D_F$.

It follows that $\langle \Phi(x) \rangle D_F$ is independent of x . Using the identity

$$\Phi(x_1)\Phi(x_2) = (x_1 + x_2 - 1)^2 \Phi\left(\frac{x_1x_2 - 1}{x_1 + x_2 - 1}\right)$$

it follows that

$$\langle \Phi(x_1) \rangle \langle \Phi(x_2) \rangle = \left\langle \Phi\left(\frac{x_1x_2 - 1}{x_1 + x_2 - 1}\right) \right\rangle (= \langle \Phi(z) \rangle, \text{ say}).$$

Thus, from

$$\langle \Phi(x_2) \rangle D_F = \langle \Phi(z) \rangle D_F$$

we deduce, on multiplying by $\langle \Phi(x_1) \rangle$, that

$$\langle \Phi(z) \rangle D_F = \langle \Phi(x_1) \rangle \langle \Phi(z) \rangle D_F$$

and thus that $\langle \Phi(x_1) \rangle D_F = D_F$. Since we also have $\langle -1 \rangle D_F = 0$ (by Corollary 3.7 (1)), it follows more generally that

$$\langle \langle \pm \Phi(x)z^2 \rangle \rangle D_F = 0$$

for any z . Note that if $E \neq F$ then a norm from E will be an element of the form $\Phi(a)b^2$ for some $a, b \in F^\times$. Thus, statement (2) of the theorem follows immediately.

Now choose $r = b^2$ in formula (1). This gives

$$C = \langle \Phi(x) \rangle (\psi_2(b^2) - \psi_1(b^2)) - D_F.$$

Multiplying both sides by 4 again shows that $C = -D_F$.

Using this, formula (1) says that

$$-D_F = \langle -\Phi(x)r \rangle (\psi_2(r) - \psi_1(r) - D_F)$$

or, equivalently,

$$\langle -\Phi(x)r \rangle D_F = D_F + \psi_1(r) - \psi_2(r)$$

and hence

$$\langle\langle -\Phi(x)r \rangle\rangle D_F = \psi_1(r) - \psi_2(r)$$

for all r . Since $\langle\langle -\Phi(x) \rangle\rangle D_F = 0$ for all x and since

$$\langle\langle -\Phi(x)r \rangle\rangle = \langle\langle -\Phi(x) \rangle\rangle \langle\langle r \rangle\rangle + \langle\langle -\Phi(x) \rangle\rangle + \langle\langle r \rangle\rangle$$

it follows that

$$\langle\langle r \rangle\rangle D_F = \psi_1(r) - \psi_2(r)$$

for all $r \in F^\times$, proving statement (1) of the theorem. \square

Remark 3.13. We will see below that statement (2) is in general best possible. For example if F is a local or global field not containing a primitive cube root of unity, ζ_3 , then often we have $\langle\langle x \rangle\rangle D_F \neq 0$ (and hence $\psi_1(x) \neq \psi_2(x)$) when x is not a norm from $E = F(\zeta_3)$.

Remark 3.14. Observe that the element t chosen in the last proof lies in the image of $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(F)$.

Now it is well known that $\mathrm{SL}_2(\mathbb{Z})$ can be expressed as an amalgamated product $C_4 *_C C_6$. Here C_4 is cyclic of order 4 with generator a and C_6 is cyclic of order 6 with generator b with

$$a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Thus $b^2 = t$ is our matrix of order 3. A straightforward direct calculation, using this decomposition (see, for example, [10, Theorem 4.1.1]), shows that $H_3(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ is cyclic of order 12 and that the inclusion $G := \langle t \rangle \rightarrow \mathrm{SL}_2(\mathbb{Z})$ induces an isomorphism $H_3(G, \mathbb{Z}) \cong H_3(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})_{(3)}$.

It follows that for any field F , the image of the natural map $H_3(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})_{(3)} \rightarrow \mathcal{RB}(F)_{(3)}$ is the cyclic subgroup generated by $-D_F$.

4. VALUATIONS AND SPECIALIZATION

In this section, we prove the existence of specialization or reduction maps from the refined pre-Bloch group of a field to the refined pre-Bloch group of the residue field of a valuation. We use these specialization maps to obtain lower bounds for the groups $\mathcal{I}_F \mathcal{RB}(F)$ for fields with valuations.

4.1. Some preliminary definitions. In this subsection we define certain quotients of the classical and refined Bloch and pre-Bloch groups which we will use in the remainder of the paper.

For any field F we will let

$$\widetilde{\mathcal{RP}}(F) := \mathcal{RP}(F) / \mathcal{K}_F^{(1)}.$$

Observe that for any x we have $\lambda_1(\psi_1(x)) = \langle\langle -x \rangle\rangle \cdot \langle\langle x \rangle\rangle \in \mathcal{I}_F^2$, by the proof of Lemma 3.3, and

$$\begin{aligned} \lambda_2(\psi_1(x)) &= (1-x) \circ x + (1-x^{-1}) \circ x^{-1} \\ &= (1-x) \circ x - \left(\frac{1-x}{-x} \right) \circ x = (1-x) \circ x - (1-x) \circ x + (-x) \circ x \\ &= (-x) \circ x \in \mathcal{S}_{\mathbb{Z}}^2(F^\times). \end{aligned}$$

It follows that $\Lambda(\psi_1(x)) = [-x, x] \in \text{RS}_{\mathbb{Z}}^2(F^\times)$ so that $\Lambda(\mathcal{K}_F^{(1)})$ is the \mathbb{R}_F -submodule of $\text{RS}_{\mathbb{Z}}^2(F^\times)$ generated by the symbols $[-x, x]$. Thus we set

$$\widetilde{\text{RS}}_{\mathbb{Z}}^2(F^\times) := \text{RS}_{\mathbb{Z}}^2(F^\times) / \Lambda(\mathcal{K}_F^{(1)})$$

and let

$$\widetilde{\Lambda} : \widetilde{\mathcal{RP}}(F) \rightarrow \widetilde{\text{RS}}_{\mathbb{Z}}^2(F^\times)$$

be the resulting \mathbb{R}_F -homomorphism induced by Λ . Finally we let $\widetilde{\mathcal{RB}}(F) := \text{Ker}(\widetilde{\Lambda})$.

Lemma 4.1. *For any field F the natural map $\mathcal{RP}(F) \rightarrow \widetilde{\mathcal{RP}}(F)$ induces a surjection $\mathcal{RB}(F) \rightarrow \widetilde{\mathcal{RB}}(F)$ whose kernel is annihilated by 4. In particular,*

$$\mathcal{RB}(F)[\frac{1}{2}] \cong \widetilde{\mathcal{RB}}(F)[\frac{1}{2}].$$

Proof. The surjectivity of the map is clear from the definitions. On the other hand, the kernel of the map $\mathcal{RB}(F) \rightarrow \widetilde{\mathcal{RB}}(F)$ is $\mathcal{RB}(F) \cap \mathcal{K}_F^{(1)}$ which is contained in $\text{Ker}(\lambda_1|_{\mathcal{K}_F^{(1)}})$ and which in turn is annihilated by 4 by Lemma 3.3. \square

For finite fields, the results of [7] allow us to be more precise:

Lemma 4.2. *Let \mathbb{F}_q be a finite field with q elements.*

- (1) *If q is even or if $q \equiv 1 \pmod{4}$ then $\mathcal{B}(\mathbb{F}_q) = \mathcal{RB}(\mathbb{F}_q) = \widetilde{\mathcal{RB}}(\mathbb{F}_q)$. This group is cyclic of order $(q+1)$ when q is even and $(q+1)/2$ when $q \equiv 1 \pmod{4}$.*
- (2) *If $q \equiv 3 \pmod{4}$ then $\widetilde{\mathcal{RB}}(\mathbb{F}_q) \cong \mathcal{B}(\mathbb{F}_q) / \langle \{-1\} \rangle$ is cyclic of order $(q+1)/4$.*

Proof. The cases $q = 2$ or $q = 3$ are immediate.

When $q \geq 4$, by [7, Lemma 7.1], the natural map $\mathcal{RP}(\mathbb{F}_q) \rightarrow \mathcal{P}(\mathbb{F}_q)$ induces an isomorphism $\mathcal{RB}(\mathbb{F}_q) \cong \mathcal{B}(\mathbb{F}_q)$ and for all x the image of $\psi_1(x) \in \mathcal{RB}(\mathbb{F}_q)$ is $\{x\} \in \mathcal{B}(\mathbb{F}_q)$, which has order divisible by 2.

If q is even or if $q \equiv 1 \pmod{4}$ then $\mathcal{B}(\mathbb{F}_q)$ is cyclic of order $q+1$ or $(q+1)/2$ respectively, and hence is of odd order.

On the other hand, if $q \equiv 3 \pmod{4}$ then [7, Lemma 7.8] shows that $\mathcal{K}_{\mathbb{F}_q}^{(1)} = \langle \psi_1(-1) \rangle$ is cyclic of order 2 and is contained in $\mathcal{RB}(\mathbb{F}_q) \cong \mathcal{B}(\mathbb{F}_q)$. \square

We let \mathcal{K}_F denote the \mathbb{R}_F -submodule $\mathcal{K}_F^{(1)} + \mathcal{K}_F^{(2)}$ of $\mathcal{RP}(F)$.

Lemma 4.3. *For any field F we have*

$$\mathcal{K}_F = \mathcal{K}_F^{(1)} + \mathcal{I}_F C_F$$

and

$$\mathcal{K}_F \cong \mathcal{K}_F^{(1)} \oplus \mathcal{I}_F D_F.$$

Proof. By Theorem 3.12, for any field F we have an identity of \mathbb{R}_F -submodules of $\mathcal{RP}(F)$

$$\mathcal{K}_F = \mathcal{K}_F^{(1)} + \mathcal{K}_F^{(2)} = \mathcal{K}_F^{(1)} + \mathcal{I}_F D_F.$$

On the other hand since

$$\begin{aligned} D_F &\equiv \left[\frac{1}{x} \right] + \langle -1 \rangle \left[\frac{1}{1-x} \right] \pmod{\mathcal{K}_F^{(1)}} \\ C_F &\equiv [x] + \langle -1 \rangle [1-x] \pmod{\mathcal{K}_F^{(1)}} \\ \text{and } [x^{-1}] &\equiv -\langle -1 \rangle [x] \pmod{\mathcal{K}_F^{(1)}}, \end{aligned}$$

it follows that $C_F \equiv -\langle -1 \rangle D_F \equiv -D_F \pmod{\mathcal{K}_F^{(1)}}$ (using Corollary 3.7 (1)) and thus

$$\mathcal{K}_F^{(1)} + I_F C_F = \mathcal{K}_F^{(1)} + I_F D_F.$$

Finally, suppose that $a \in \mathcal{K}_F^{(1)} \cap I_F D_F$. Then $3a = 0$ since $3D_F = 0$ and $4a = 0$ since $a \in \mathcal{K}_F^{(1)} \cap \mathcal{RB}(F)$, so that $a = 0$. Thus $\mathcal{K}_F^{(1)} \cap I_F D_F = 0$. \square

For any field F we set

$$\widehat{\mathcal{RP}}(F) := \mathcal{RP}(F)/\mathcal{K}_F = \widetilde{\mathcal{RP}}(F)/I_F D_F \text{ and } \widehat{\mathcal{RB}}(F) = \widetilde{\mathcal{RB}}(F)/I_F D_F.$$

Corollary 4.4. *For any field F there is natural short exact sequence*

$$0 \rightarrow I_F D_F \rightarrow \widetilde{\mathcal{RB}}(F) \rightarrow \widehat{\mathcal{RB}}(F) \rightarrow 0.$$

If F is a finite field, by Lemma 2.4, $\mathcal{RB}(F) \cong \mathcal{B}(F)$ and hence the action of G_F on $\mathcal{RB}(F)$ is trivial and thus we have

Corollary 4.5. *For any finite field F , we have $\widehat{\mathcal{RB}}(F) = \widetilde{\mathcal{RB}}(F)$.*

We will use the following identities repeatedly below:

Lemma 4.6. *For any field F and any $x \in F^\times$, we have*

$$\langle x \rangle [x] = \langle -1 \rangle [x] = -[x^{-1}]$$

in $\widehat{\mathcal{RP}}(F)$.

Proof. Since $\psi_1(x) = 0$ in $\widehat{\mathcal{RP}}(F)$, it follows that $\langle -1 \rangle [x] = -[x^{-1}]$. Since $\psi_2(x) = 0$, we have $\langle x \rangle [x] = -[x^{-1}]$ also. \square

We will also need to consider the corresponding quotient of the classical pre-Bloch group:

Let \mathcal{S}_F denote the (2-torsion) subgroup of $\mathcal{P}(F)$ generated by the elements $\{x\}$, and let

$$\widehat{\mathcal{P}}(F) := \frac{\mathcal{P}(F)}{\mathcal{S}_F}$$

Thus, the natural map $\mathcal{RP}(F) \rightarrow \mathcal{P}(F)$ induces an isomorphism $\widehat{\mathcal{RP}}(F)_{F^\times} \cong \widehat{\mathcal{P}}(F)$.

Recall that \mathcal{D}_F denotes the R_F -module $R_F D_F$. Finally, we let

$$\overline{\mathcal{RP}}(F) := \frac{\mathcal{RP}(F)}{\mathcal{K}_F^{(1)} + \mathcal{D}_F} = \frac{\widetilde{\mathcal{RP}}(F)}{\mathcal{D}_F} = \frac{\widehat{\mathcal{RP}}(F)}{\mathbb{Z} \cdot D_F}$$

and

$$\overline{\mathcal{RB}}(F) := \frac{\widetilde{\mathcal{RB}}(F)}{\mathcal{D}_F}.$$

In section 6 below, we will also need the corresponding classical version:

$$\overline{\mathcal{P}}(k) := \frac{\mathcal{P}(k)}{\mathbb{Z}D_k + \mathcal{S}_k} = \frac{\widehat{\mathcal{P}}(k)}{\mathbb{Z}D_k}.$$

Note that $\mathcal{D}_F = I_F D_F + \mathbb{Z}D_F$ and that the sum $\mathcal{K}_F^{(1)} + \mathcal{D}_F$ is direct (by the argument of Lemma 4.3). Thus we have

Lemma 4.7. *For any field F there are short exact sequences*

$$0 \rightarrow \mathcal{D}_F \rightarrow \widehat{\mathcal{RB}}(F) \rightarrow \overline{\mathcal{RB}}(F) \rightarrow 0$$

and

$$0 \rightarrow \mathbb{Z}D_F \rightarrow \widehat{\mathcal{RB}}(F) \rightarrow \overline{\mathcal{RB}}(F) \rightarrow 0.$$

4.2. Valuations. Now let Γ be an ordered (additive) abelian group and let $v : F^\times \rightarrow \Gamma$ be a valuation, which we will always assume to be surjective. As usual, let

$$\mathcal{O} = \mathcal{O}_v = \{0\} \cup \{x \in F^\times \mid v(x) \geq 0\}$$

be the valuation ring, let

$$\mathcal{M} = \mathcal{M}_v = \{0\} \cup \{x \in F^\times \mid v(x) > 0\}$$

be the maximal ideal, let $k = k_v = \mathcal{O}/\mathcal{M}$ be the residue field and

$$U = U_v = \{x \in F^\times \mid v(x) = 0\} = \mathcal{O} \setminus \mathcal{M}.$$

Also let f_v be the group homomorphism $U \rightarrow k^\times$, $u \mapsto \bar{u} := u + \mathcal{M}$, and let $U_1 = U_{1,v} = \text{Ker}(f_v) = 1 + \mathcal{M}$.

Since Γ is a torsion-free group we have a short exact sequence of \mathbb{F}_2 -vector spaces

$$(2) \quad 1 \rightarrow U/U^2 \rightarrow G_F \rightarrow \Gamma/2 \rightarrow 0.$$

We have homomorphisms of commutative rings

$$\begin{array}{ccc} \mathbb{Z}[U/U^2] & \hookrightarrow & \mathbf{R}_F \\ \downarrow & & \\ \mathbf{R}_k & & \end{array}$$

Thus, if M is any \mathbf{R}_k module

$$M_F = M_{F,v} := \mathbf{R}_F \otimes_{\mathbb{Z}[U/U^2]} M$$

is an \mathbf{R}_F -module.

We will require the following result several times in the next section:

Lemma 4.8. *Let F be a field with valuation $v : F^\times \rightarrow \Gamma$ and corresponding residue field k . Let $a, b \in F$ and suppose that $v(a) \equiv v(b) \pmod{2\Gamma}$. Then $\langle a \rangle \otimes C_k = \langle b \rangle \otimes C_k$ in $\widehat{\mathcal{RP}}(k)_F$.*

Proof. We have $v(a) = v(b) + 2\gamma$ for some $\gamma \in \Gamma$. Choose $c \in F^\times$ with $v(c) = \gamma$. Then $u = a/(bc^2) \in U$. So $\langle a \rangle - \langle b \rangle = \langle b \rangle \langle\langle u \rangle\rangle$ in \mathbf{R}_F . Thus

$$\langle a \rangle \otimes C_k - \langle b \rangle \otimes C_k = \langle a \rangle \otimes \langle\langle \bar{u} \rangle\rangle C_k = 0$$

since $\langle\langle \bar{u} \rangle\rangle C_k \in \mathcal{I}_k C_k$ and hence represents 0 in $\widehat{\mathcal{RP}}(k)$. □

4.3. The specialization homomorphisms.

Theorem 4.9. *Let F be a field with valuation v and corresponding residue field k .*

Then there is a surjective \mathbf{R}_F -module homomorphism

$$S_v : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{RP}}(k)_F$$

$$[a] \mapsto \begin{cases} 1 \otimes [\bar{a}], & v(a) = 0 \\ 1 \otimes C_k, & v(a) > 0 \\ -(1 \otimes C_k), & v(a) < 0 \end{cases}$$

Proof. Let Z_1 denote the set of symbols of the form $[x]$, $x \neq 1$ and let $T : \mathbf{R}_F[Z_1] \rightarrow \widehat{\mathcal{RP}}(k)_F$ be the unique \mathbf{R}_F -homomorphism given by

$$[a] \mapsto \begin{cases} 1 \otimes [\bar{a}], & v(a) = 0 \\ 1 \otimes C_k, & v(a) > 0 \\ -(1 \otimes C_k), & v(a) < 0 \end{cases}$$

We must prove that $T(S_{x,y}) = 0$ for all $x, y \in F^\times \setminus \{1\}$.

Through the remainder of this proof we will adopt the following notation: Given $x, y \in F^\times \setminus \{1\}$, we let

$$u = \frac{y}{x} \text{ and } w = \frac{1-x}{1-y}.$$

Note that

$$\frac{1-x^{-1}}{1-y^{-1}} = \frac{y}{x} \cdot \frac{x-1}{y-1} = uw.$$

Thus, with this notation, $S_{x,y}$ becomes $[x] - [y] + \langle x \rangle [u] - \langle x^{-1} - 1 \rangle [uw] + \langle 1 - x \rangle [w]$.

We divide the proof into several cases:

Case (i): $v(x), v(y) \neq 0$

Subcase (a): $v(x) = v(y) > 0$.

Then $1-x, 1-y \in U_1$ and hence $w \in U_1$, so that $\bar{w} = 1$ and $\overline{uw} = \bar{u}$. Thus

$$T(S_{x,y}) = 1 \otimes C_k - 1 \otimes C_k + \langle x \rangle \otimes [\bar{u}] - \langle x^{-1} - 1 \rangle \otimes [\bar{u}].$$

However, $x^{-1} - 1 = x^{-1}(1-x)$, so that $\langle x^{-1} - 1 \rangle \otimes \bar{u} = \langle x \rangle \otimes \langle 1 - \bar{x} \rangle [\bar{u}] = \langle x \rangle \otimes [\bar{u}]$, and thus $T(S_{x,y}) = 0$ as required.

Subcase (b): $v(x) = v(y) < 0$.

Then $u \in U$ and $uw \in U_1$ so that $\bar{w} = \bar{u}^{-1}$. Thus

$$T(S_{x,y}) = -1 \otimes C_k + 1 \otimes C_k + \langle x \rangle \otimes [\bar{u}] + \langle 1 - x \rangle \otimes [\bar{u}^{-1}].$$

But $1-x = -x(1-x^{-1})$ and $1-x^{-1} \in U_1$, so that the last term is $\langle -x \rangle \otimes [\bar{u}^{-1}]$ and hence $T(S_{x,y}) = \langle x \rangle \otimes \psi_1(\bar{u}) = 0$ in $\widehat{\mathcal{RP}}(k)_F$.

Subcase (c): $v(x) > v(y) > 0$.

Then $w \in U_1$ and $v(u), v(uw) < 0$. So

$$T(S_{x,y}) = 1 \otimes C_k - 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k.$$

But since $x^{-1} - 1 = x^{-1}(1-x)$ and $1-x \in U_1$ it follows that $\langle x^{-1} - 1 \rangle \otimes C_k = \langle x \rangle \otimes C_k$ and hence $T(S_{x,y}) = 0$.

Subcase (d): $v(x) > 0 > v(y)$.

Then $v(u) = v(y) - v(x) < 0$, $v(w) = -v(1-y) = -v(y(y^{-1}-1)) = -v(y) > 0$ and $v(uw) = v(u) + v(w) = -v(x) < 0$. So

$$T(S_{x,y}) = 1 \otimes C_k + 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1 - x \rangle \otimes C_k.$$

But $1-x \in U_1$ and $\langle x^{-1} - 1 \rangle = \langle x \rangle \langle 1 - x \rangle$. So this gives $T(S_{x,y}) = 1 \otimes 3C_k$. However, $3C_k = -3D_k = 0$ in $\widehat{\mathcal{RP}}(k)$.

Subcase (e): $0 > v(x) > v(y)$.

Then $v(u) = v(y) - v(x) < 0$ and $uw \in U_1$, so that $v(w) = -v(u) > 0$ and $\bar{w} = \bar{u}^{-1}$. Thus

$$T(S_{x,y}) = -(1 \otimes C_k) + 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle 1 - x \rangle \otimes C_k.$$

Now $1-x = -x(1-x^{-1})$ and $1-x^{-1} \in U_1$, so the last term is $\langle x \rangle \otimes \langle -1 \rangle C_k = \langle x \rangle \otimes C_k$ by Corollary 3.7 (1). This gives $T(S_{x,y}) = 0$ as required.

Subcases (f),(g),(h): The corresponding calculations when $v(y) > v(x)$ are almost identical.

Case (ii): $x, y \in U_1$.

Subcase (a): $v(1-x) \neq v(1-y)$.

Then $u \in U_1$ and $v(w) = v(uw) \neq 0$. So

$$T(S_{x,y}) = \pm \left(\langle x^{-1} - 1 \rangle \otimes C_k - \langle 1-x \rangle \otimes C_k \right) = 0$$

since $\langle x^{-1} - 1 \rangle = \langle x^{-1} \rangle \langle 1-x \rangle$ and $x^{-1} \in U_1$.

Subcase (b): $v(1-x) = v(1-y)$.

Then $u \in U_1$ and $w, uw \in U$ with $u\bar{w} = \bar{w}$. So

$$T(S_{x,y}) = - \langle x^{-1} - 1 \rangle \otimes [\bar{w}] + \langle 1-x \rangle \otimes [\bar{w}]$$

which is 0 by the same argument as the previous (sub)case.

Case (iii): $x \in U_1, v(y) \neq 0$.

Then $v(u) = v(y)$. Observe that $v(1-y) = \min(v(1), v(y)) = \min(0, v(y)) \leq 0$. Of course, $v(1-x) > 0$.

Thus $v(w) = v(1-x) - v(1-y) > 0$ and $v(uw) = v(u) + v(w) = v(1-x) + v(y) - v(1-y) > 0$ since $v(y) - v(1-y) \geq 0$. So

$$T(S_{x,y}) = \pm(1 \otimes C_k - 1 \otimes C_k) - \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1-x \rangle \otimes C_k = 0$$

since $x \in U_1$ and thus $\langle x^{-1} - 1 \rangle = \langle 1-x \rangle$.

Case (iv): $v(x) \neq 0, y \in U_1$

Arguing as in the last case, $v(w), v(uw) < 0$ in this case.

Subcase (a): $v(x) > 0$

Then $v(u) = -v(x) < 0$. So

$$T(S_{x,y}) = 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k - \langle 1-x \rangle \otimes C_k = 0$$

using Lemma 4.8 together with the identities $v(1-x) = 0 = v(1)$ and $v(x^{-1} - 1) = -v(x)$.

Subcase (b): $v(x) < 0$.

Then $v(u) > 0$ and

$$T(S_{x,y}) = -1 \otimes C_k + \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k - \langle 1-x \rangle \otimes C_k.$$

This vanishes by Lemma 4.8 together with the identities $v(1-x) = v(x)$ and $v(x^{-1} - 1) = 0 = v(1)$.

Case (v): $x \in U \setminus U_1$ and $v(y) \neq 0$.

Subcase (a): $v(y) > 0$.

Then $v(u) = v(y) > 0$. Since $1-x \in U$ and $1-y \in U_1$, it follows that $v(w) = 0$, $\bar{w} = 1 - \bar{x}$ in k and $v(uw) > 0$. Thus

$$\begin{aligned} T(S_{x,y}) &= 1 \otimes [\bar{x}] - 1 \otimes C_k + \langle x \rangle \otimes C_k - \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1-x \rangle \otimes [1 - \bar{x}] \\ &= 1 \otimes ([\bar{x}] + \langle 1 - \bar{x} \rangle [1 - \bar{x}] - C_k) \end{aligned}$$

using Lemma 4.8 to eliminate terms. Now $\langle 1 - \bar{x} \rangle [1 - \bar{x}] = \langle -1 \rangle [1 - \bar{x}]$ in $\widehat{\mathcal{RP}}(k)$ by Lemma 4.6, and hence $[\bar{x}] + \langle 1 - \bar{x} \rangle [1 - \bar{x}] = C_k$ in $\widehat{\mathcal{RP}}(k)$. Thus we conclude that $T(S_{x,y}) = 0$ as required.

Subcase (b): $v(y) < 0$

We have $v(u) = v(y) < 0$ and $v(w) = -v(1-y) = -v(y) > 0$. Thus $v(uw) = v(u) + v(w) = v(y) - v(y) = 0$. Furthermore, since $1 - y^{-1} \in U_1$, $u\bar{w} = 1 - \bar{x}^{-1}$. Thus (using Lemma 4.8 again)

$$\begin{aligned} S(T_{x,y}) &= 1 \otimes [\bar{x}] + 1 \otimes [k] - \langle x \rangle \otimes C_k - \langle x^{-1} - 1 \rangle \otimes [1 - \bar{x}^{-1}] + \langle 1 - x \rangle \otimes C_k \\ &= 1 \otimes (C_k + [\bar{x}] - \langle \bar{x}^{-1} - 1 \rangle [1 - \bar{x}^{-1}]) \\ &= 1 \otimes (C_k + [\bar{x}] - \langle \bar{x}^{-1} - 1 \rangle [1 - \bar{x}^{-1}]). \end{aligned}$$

Now $\langle \bar{x}^{-1} - 1 \rangle [1 - \bar{x}^{-1}] = [1 - \bar{x}^{-1}]$ in $\widehat{\mathcal{RP}}(k)$ by Lemma 4.6, and $[\bar{x}] = -\langle -1 \rangle [\bar{x}^{-1}]$ since $\psi_1(\bar{x}) = 0$. Thus

$$[\bar{x}] - \langle \bar{x}^{-1} - 1 \rangle [1 - \bar{x}^{-1}] = -\langle -1 \rangle ([\bar{x}^{-1}] + \langle -1 \rangle [1 - \bar{x}^{-1}]) = -\langle -1 \rangle C_k = -C_k \text{ in } \widehat{\mathcal{RP}}(k)$$

and again $T(S_{x,y}) = 0$ as required.

Case (vi): $y \in U \setminus U_1$ and $v(x) \neq 0$

Subcase (a): $v(x) > 0$

we have $v(u) = -v(x) < 0$, $v(w) = 0$ and $1 - x \in U_1$ so that $\bar{w} = (1 - \bar{y})^{-1}$. Finally $v(uw) = v(u) < 0$. Thus

$$\begin{aligned} T(S_{x,y}) &= 1 \otimes C_k - 1 \otimes [\bar{y}] - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1 - x \rangle \otimes \left[\frac{1}{1 - \bar{y}} \right] \\ &= 1 \otimes \left(C_k - [\bar{y}] + \left[\frac{1}{1 - \bar{y}} \right] \right) \end{aligned}$$

using Lemma 4.8 again (after observing that $v(x^{-1} - 1) = -v(x)$ when $v(x) > 0$). However

$$\left[\frac{1}{1 - \bar{y}} \right] = -\langle -1 \rangle [1 - \bar{y}]$$

in $\widehat{\mathcal{RP}}(k)$, and hence $T(S_{x,y}) = 1 \otimes (C_k - C_k) = 0$.

Subcase (b): $v(x) < 0$

We have $v(u) = -v(x) > 0$ and $v(w) = v(1-x) = v(x) < 0$ and $v(uw) = v(u) + v(w) = 0$. Furthermore, since $1 - x^{-1} \in U_1$, $u\bar{w} = 1/(1 - \bar{y}^{-1})$. Thus

$$\begin{aligned} T(S_{x,y}) &= -(1 \otimes C_k) - 1 \otimes [\bar{y}] + \langle x \rangle \otimes C_k - \langle x^{-1} - 1 \rangle \otimes \left[\frac{1}{1 - \bar{y}^{-1}} \right] - \langle 1 - x \rangle \otimes C_k \\ &= 1 \otimes \left(-C_k - [\bar{y}] - \langle -1 \rangle \left[\frac{1}{1 - \bar{y}^{-1}} \right] \right) \end{aligned}$$

using Lemma 4.8 and the fact that $\langle x^{-1} - 1 \rangle = \langle -1 \rangle \langle 1 - x^{-1} \rangle$ and $1 - x^{-1} \in U_1$.

But in $\widehat{\mathcal{RP}}(k)$ we have, by Lemma 4.6,

$$-[\bar{y}] - \langle -1 \rangle \left[\frac{1}{1 - \bar{y}^{-1}} \right] = \langle -1 \rangle [\bar{y}^{-1}] + [1 - \bar{y}^{-1}] = C_k.$$

Case (vii): $x \in U \setminus U_1$ and $y \in U_1$.

We have $v(u) = 0$ and $\bar{u} = \bar{x}^{-1}$. Furthermore, $v(w) = -v(1 - y) < 0$ and thus $v(uw) = v(w) < 0$ also. Thus

$$\begin{aligned} T(S_{x,y}) &= 1 \otimes [\bar{x}] + \langle x \rangle \otimes [\bar{x}^{-1}] + \langle x^{-1} - 1 \rangle \otimes C_k - \langle 1 - x \rangle \otimes C_k \\ &= 1 \otimes \left([\bar{x}] + \langle \bar{x} \rangle [\bar{x}^{-1}] \right) \text{ (by Lemma 4.8)} \\ &= 1 \otimes \langle \bar{x} \rangle \langle 1 - \bar{x} \rangle \psi_2(\bar{x}) = 0. \end{aligned}$$

Case (viii): $x \in U_1$ and $y \in U \setminus U_1$

We have $u \in U$ and $\bar{u} = \bar{y}$, $v(w) = v(1 - x) > 0$ and $v(uw) = v(w) > 0$. So

$$T(S_{x,y}) = -(1 \otimes [\bar{y}]) + \langle x \rangle \otimes [\bar{y}] - \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1 - x \rangle \otimes C_k = 0$$

by Lemma 4.8 and the fact that $x \in U_1$.

Case (ix): $x, y \in U \setminus U_1$

In this case $\bar{x}, \bar{y} \in k^\times \setminus \{1\}$ and

$$T(S_{x,y}) = 1 \otimes S_{\bar{x}, \bar{y}} = 1 \otimes 0.$$

□

Given a valuation v on a field F and an element $a \in F^\times$ with $v(a) \neq 0$, we will let $\epsilon_v(a)$ denote $\text{sign}(v(a)) \in \{\pm 1\}$.

Suppose that $\rho : \mathbf{R}_F \rightarrow \mathbf{R}_k$ is a ring homomorphism satisfying $\rho(\langle u \rangle) = \langle \bar{u} \rangle$ for any $u \in U$. Then by composing S_v with the surjective \mathbf{R}_F -module homomorphism

$$\widehat{\mathcal{RP}}(k)_F \rightarrow \widehat{\mathcal{RP}}(k), \quad x \otimes y \mapsto \rho(x)y.$$

we obtain:

Corollary 4.10. *Let $\rho : \mathbf{R}_F \rightarrow \mathbf{R}_k$ be any ring homomorphism satisfying $\rho(\langle u \rangle) = \langle \bar{u} \rangle$ if $u \in U$. Then ρ induces a \mathbf{R}_F -module structure on $\widehat{\mathcal{RP}}(k)$ and there is a surjective homomorphism of \mathbf{R}_F -modules $S = S_\rho : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{RP}}(k)$ determined by*

$$[a] \mapsto \begin{cases} [\bar{a}], & a \in U_v \\ \epsilon_v(a)C_k, & a \notin U_v \end{cases}$$

Now, if we choose a splitting, $j : \Gamma/2 \rightarrow G_F$ of the sequence (2), we obtain a ring isomorphism

$$\mathbf{R}_F \cong \mathbb{Z}[U/U^2 \times j(\Gamma/2)] \cong \mathbb{Z}[U/U^2] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma/2].$$

On the other hand, any character $\chi : \Gamma/2 \rightarrow \mu_2 = \{1, -1\}$ induces a ring homomorphism $\mathbb{Z}[\Gamma/2] \rightarrow \mathbb{Z}$. Thus, given a splitting j and a character χ , we obtain a ring homomorphism

$$\rho = \rho_{j,\chi} : \mathbf{R}_F \rightarrow \mathbf{R}_k, \langle u \cdot j(\gamma) \rangle \mapsto \chi(\gamma) \langle \bar{u} \rangle$$

and a corresponding surjective specialization map

$$S = S_{j,\chi} : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{RP}}(k)$$

which is also an \mathbf{R}_F -homomorphism.

Example 4.11. For example, if v is a discrete valuation, then $\Gamma = \mathbb{Z}$ and $\Gamma/2 = \mathbb{Z}/2$. Any choice, π , of uniformizing parameter, determines a splitting $\mathbb{Z}/2 \rightarrow G_F$, $1 \mapsto \langle \pi \rangle$. There are two characters, ϵ , on $\mathbb{Z}/2$, namely 1 and -1 . Thus we get ring homomorphisms

$$\rho_{\pi,\epsilon} : \mathbf{R}_F \rightarrow \mathbf{R}_k, \langle u\pi^r \rangle \mapsto \epsilon^r \langle \bar{u} \rangle.$$

and corresponding specialization homomorphisms

$$S_{\pi,\epsilon} : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{RP}}(k).$$

In general, the specialization homomorphisms $S_{j,\chi}$ do not restrict to homomorphisms $\mathcal{RB}(F) \rightarrow \widehat{\mathcal{RB}}(k)$ of Bloch groups.

To see this, we begin with the following observation:

Lemma 4.12. *For a field F and $x \in F^\times$ we have*

$$\mathfrak{p}_{-1}^+ \langle\langle x \rangle\rangle \widehat{\mathcal{RP}}(F) \subset \widehat{\mathcal{RB}}(F).$$

Proof. Since G_F acts trivially on $S_{\mathbb{Z}}^2(F^\times)$ we have $\langle\langle x \rangle\rangle S_{\mathbb{Z}}^2(F^\times) = 0$ and thus

$$\begin{aligned} \widetilde{\Lambda}(\langle\langle x \rangle\rangle \mathfrak{p}_{-1}^+[a]) &= (\langle\langle x \rangle\rangle \langle\langle 1-a \rangle\rangle \mathfrak{p}_{-1}^+ \langle\langle a \rangle\rangle, 0) \\ &= (\langle\langle x \rangle\rangle \langle\langle 1-a \rangle\rangle \langle\langle -a \rangle\rangle \langle\langle a \rangle\rangle, 0) \\ &= \langle\langle x \rangle\rangle \langle\langle 1-a \rangle\rangle [-a, a] = 0 \in \mathbf{RS}_{\mathbb{Z}}^2(F^\times). \end{aligned}$$

□

Corollary 4.13. *Let F be a field with valuation $v : F^\times \rightarrow \Gamma$, with corresponding residue field k . Let $j : \Gamma/2 \rightarrow G_F$ be a splitting, and let χ be a nontrivial character on $\Gamma/2$.*

Then the image of

$$S_{j,\chi}|_{\mathcal{RB}(F)} : \mathcal{RB}(F) \rightarrow \widehat{\mathcal{RP}}(k)$$

contains $2\mathfrak{p}_{-1}^+ \widehat{\mathcal{RP}}(k)$.

Proof. First observe that $S_{j,\chi}$ induces a homomorphism

$$\widehat{\mathcal{RP}}(F) \rightarrow \widehat{\mathcal{RP}}(k)$$

since

$$S_{j,\chi}(\psi_1(x)) = \begin{cases} \psi_1(\bar{x}), & v(x) = 0 \\ 0, & v(x) \neq 0 \end{cases}$$

Suppose that $\gamma \in \Gamma$ with $\chi(\gamma) = -1$ and let $\pi := j(\gamma)$. Since $\widehat{\mathcal{RP}}(k)$ is an \mathbf{R}_F -module via the homomorphism $\rho_{j,\chi}$, $\langle\langle \pi \rangle\rangle$ acts on $\widehat{\mathcal{RP}}(k)$ as multiplication by -1 . Thus $\langle\langle \pi \rangle\rangle \widehat{\mathcal{RP}}(k) = 2\widehat{\mathcal{RP}}(k)$ and hence $S_{j,\chi}$ induces a surjective homomorphism

$$\langle\langle \pi \rangle\rangle \mathfrak{p}_{-1}^+ \widehat{\mathcal{RP}}(F) \rightarrow 2\mathfrak{p}_{-1}^+ \widehat{\mathcal{RP}}(k).$$

□

Remark 4.14. As just noted, the maps $S_{j,\chi} : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{RP}}(k)$ descend to maps $\widehat{\mathcal{RP}}(F) \rightarrow \widehat{\mathcal{RP}}(k)$. However, they do not usually induce maps $\widehat{\mathcal{RP}}(F) \rightarrow \widehat{\mathcal{RP}}(k)$. In fact, below we will use the homomorphisms $S_{j,\chi}$ to detect the non-triviality of $I_F C_F$ for global and (some) local fields.

Corollary 4.15. *Let F be a field with valuation $v : F^\times \rightarrow \Gamma$ and quadratically closed residue field k . Let $j : \Gamma/2 \rightarrow G_F$ be a splitting, and let $\chi : \Gamma/2 \rightarrow \mu_2$ be a nontrivial character. Then $S_{j,\chi}$ induces a surjective map*

$$\mathcal{RB}(F) \twoheadrightarrow \mathcal{RP}(k).$$

Proof. Since k is quadratically closed, $\mathcal{RP}(k) = \mathcal{P}(k)$ and $\psi_i(x) = \{x\} = 0$ for all x for $i = 1, 2$. Thus $\widehat{\mathcal{RP}}(k) = \mathcal{P}(k)$ and $\mathfrak{p}_{-1}^+ \widehat{\mathcal{RP}}(k) = \mathfrak{p}_{-1}^+ \mathcal{P}(k) = 2\mathcal{P}(k)$. However, since every element of k^\times is a square, $\mathcal{P}(k)$ is 2-divisible ([4, Lemma 5.8]) and thus $2\mathfrak{p}_{-1}^+ \mathcal{RP}(k) = 4\mathcal{P}(k) = \mathcal{P}(k) = \mathcal{RP}(k)$. □

Corollary 4.16. *Let F be a field with valuation $v : F^\times \rightarrow \Gamma$ and corresponding residue field k . Let $j : \Gamma/2 \rightarrow G_F$ be a splitting, and let $\chi : \Gamma/2 \rightarrow \mu_2$ be a nontrivial character. Choose $\gamma \in \Gamma$ with $\chi(\gamma) = -1$ and let $\pi = j(\gamma)$.*

Then the map $S_{j,\chi}$ induces a surjective homomorphism of $\mathbf{R}_F[\frac{1}{2}]$ -modules

$$e_\pi^- e_{-1}^+ \left(\mathcal{RB}(F)[\frac{1}{2}] \right) \rightarrow e_{-1}^+ \left(\widehat{\mathcal{RP}}(k)[\frac{1}{2}] \right)$$

In particular, if $-1 \in (F^\times)^2$, then there is a surjective homomorphism

$$e_\pi^- \left(\mathcal{RB}(F)[\frac{1}{2}] \right) \rightarrow \widehat{\mathcal{RP}}(k)[\frac{1}{2}]$$

Proof. Since

$$e_\pi^- e_{-1}^+ \left(\widehat{\mathcal{RP}}(F)[\frac{1}{2}] \right) \subset \widehat{\mathcal{RB}}(F)[\frac{1}{2}]$$

by Lemma 4.12, it follows that

$$e_\pi^- e_{-1}^+ \left(\widehat{\mathcal{RP}}(F)[\frac{1}{2}] \right) = e_\pi^- e_{-1}^+ \left(\widehat{\mathcal{RB}}(F)[\frac{1}{2}] \right) \cong e_\pi^- e_{-1}^+ \left(\mathcal{RB}(F)[\frac{1}{2}] \right)$$

(using Lemma 4.1) □

Remark 4.17. In general, the specialization maps $S_{j,\chi}$ depend on the choice of splitting j and character χ .

To spell this out: If $\gamma \in \Gamma$ then the \mathbf{R}_F -module structure on $\widehat{\mathcal{RP}}(k)$ associated to $S_{j,\chi}$ requires that $\langle j(\gamma) \rangle$ acts as multiplication by $\chi(\gamma)$. If j' is another splitting, then we will have $j(\gamma) = j'(\gamma) \cdot u$ for some $u \in U$, and the action of $\langle j(\gamma) \rangle$ with respect to the module-structure associated to the pair (j', γ) will be multiplication by $\langle \bar{u} \rangle \chi(\gamma)$. Note that $\langle \bar{u} \rangle \in \mathbf{R}_k$.

We can free ourselves of the dependence on the choice of splitting j (or the choice of uniformizer π , in the case of a discrete valuation) by composing $S_{j,\chi}$ with the natural surjective map

$$\widehat{\mathcal{RP}}(k) \rightarrow \widehat{\mathcal{P}}(k).$$

Since \mathbf{R}_k acts trivially on $\mathcal{P}(k)$, we get well-defined surjective specialization maps

$$S_\chi : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{P}}(k)$$

which depend only on the choice of character $\chi : \Gamma/2 \rightarrow \mu_2$.

It is important to note that although $\widehat{\mathcal{P}}(k)$ is a trivial \mathbf{R}_k -module, the \mathbf{R}_F -module structure induced from a character χ is not usually trivial.

Thus, if χ is not the trivial character and if $\pi \in G_F$ satisfies $\chi(v(\pi)) = -1$, then π acts as multiplication by -1 on $\widehat{\mathcal{P}}(k)$ and S_χ induces a surjective homomorphism

$$e_\pi^- \left(\mathcal{RB}(F)[\frac{1}{2}] \right) \rightarrow \widehat{\mathcal{P}}(k)[\frac{1}{2}]$$

5. APPLICATIONS: GLOBAL FIELDS

From the existence of these specialization maps it follows that if F is field with many valuations, then $H_3(\mathrm{SL}_2(F), \mathbb{Z})_0$ must be large. For example:

Theorem 5.1. *Let F be a field and let \mathcal{V} be a family of discrete values on F satisfying*

- (1) *For any $x \in F^\times$, $v(x) = 0$ for all but finitely many $v \in \mathcal{V}$*
- (2) *The map*

$$F^\times \rightarrow \bigoplus_{v \in \mathcal{V}} \mathbb{Z}/2, \quad a \mapsto \{v(a)\}_v$$

is surjective.

Then there is a natural surjective homomorphism

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow \bigoplus_{v \in \mathcal{V}} \widehat{\mathcal{P}}(k_v)[\frac{1}{2}].$$

Proof. We have

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 = \mathcal{RB}(F)_0[\frac{1}{2}] = \mathcal{I}_F \mathcal{RB}(F)[\frac{1}{2}] = \sum_{a \in G_F} e_a^- \mathcal{RB}(F)[\frac{1}{2}]$$

by Lemma 2.3.

We denote by $S_{v,-1}$ the specialization map $\mathcal{RP}(F) \rightarrow \widehat{\mathcal{P}}(k_v)$ corresponding to the character $\epsilon = -1$ (see Remark 4.17).

Note that if $a \in F^\times$ with $v(a) \equiv 0 \pmod{2}$, then for any $y \in \mathcal{RP}(F)$, $S_v(e_a^- y) = e_a^- S_v(y) = 0$ since $\langle a \rangle$ acts trivially on $\mathcal{P}(k_v)$. Thus, for any $a \in F^\times$, $y \in \mathcal{RP}(F)$, we have $S_{v,-1}(e_a^- y) = 0$ for almost all v . It follows that for any $x \in \mathcal{I}_F \mathcal{RB}(F)[\frac{1}{2}]$, $S_{v,-1}(x) = 0$ for almost all $v \in \mathcal{V}$.

Thus the maps $S_{v,-1}$ induce a well-defined $\mathbf{R}_F[\frac{1}{2}]$ -homomorphism

$$S : \mathcal{RB}(F)[\frac{1}{2}]_0 \rightarrow \bigoplus_v \widehat{\mathcal{P}}(k_v)[\frac{1}{2}], \quad x \mapsto \{S_{v,-1}(x)\}_v.$$

Finally, let $y = \{y_v\}_v \in \bigoplus_v \widehat{\mathcal{P}}(k_v)[\frac{1}{2}]$. Let $T = \{v \in \mathcal{V} \mid y_v \neq 0\}$. For each $v \in T$, choose $x_v \in \mathcal{RB}(F)[\frac{1}{2}]$ with $S_{v,-1}(x_v) = y_v$.

For each $v \in T$, choose $\pi_v \in F^\times$ satisfying $w(\pi_v) \equiv \delta_{w,v} \pmod{2}$ for all $w \in \mathcal{V}$. Then

$$y = S \left(\sum_{v \in T} e_{\pi_v}^- \cdot x_v \right).$$

□

Since \mathcal{S}_k is a 2-torsion group, we have $\widehat{\mathcal{P}}(k)[\frac{1}{2}] = \mathcal{P}(k)[\frac{1}{2}]$ for a finite field k , and we deduce:

Corollary 5.2. *Let F be a global field and let S be a finite set of places of F . Let*

$$\mathcal{O}_S = \{a \in F^\times \mid v(a) \geq 0 \text{ for all } v \notin S\} \cup \{0\}.$$

Suppose that $\mathrm{Cl}(\mathcal{O}_S)/2 = 0$. Let \mathcal{V} be the set of all finite places of F not in S . Then there is a surjective homomorphism of \mathbf{R}_F -modules

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow \bigoplus_{v \in \mathcal{V}} \mathcal{P}(k_v)[\frac{1}{2}].$$

Here, if π_v is a uniformizer for v , then the square class of π_v acts as -1 on the factor $\mathcal{P}(k_v)[\frac{1}{2}]$ on the right.

Remark 5.3. Note that it follows that the groups $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$ are not finitely generated, for any global field F . By contrast, when F is a global field, the groups $\mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}) \cong K_3^{\mathrm{ind}}(F)$ are well-known to be finitely generated. Furthermore the stabilization map $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \rightarrow \mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z})$ is known to be surjective in this case, by the results of [8].

Remark 5.4. Observe that in the case $F = \mathbb{Q}$ the short exact sequence

$$0 \rightarrow \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}]) \rightarrow K_3^{\mathrm{ind}}(\mathbb{Q})[\frac{1}{2}] \rightarrow 0$$

is split, since the composite

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}]) \rightarrow K_3^{\mathrm{ind}}(\mathbb{Q})[\frac{1}{2}] = \mathcal{B}(\mathbb{Q})[\frac{1}{2}] = \mathbb{Z} \cdot D_{\mathbb{Q}}$$

is an isomorphism (of groups of order 3). Thus there is a natural surjective homomorphism

$$H_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}]) \twoheadrightarrow K_3^{\mathrm{ind}}(\mathbb{Q})[\frac{1}{2}] \oplus \left(\bigoplus_p \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}] \right)$$

When k is algebraically closed, $\widehat{\mathcal{P}}(k) = \mathcal{P}(k) = \mathcal{RP}(k)$ and $\mathcal{P}(k)$ is uniquely divisible – i.e. a \mathbb{Q} -vector space – by the results of Suslin ([21]).

Corollary 5.5. *Let k be an algebraically closed field and let X be an algebraic variety over k , with rational function field $k(X)$. Suppose that X is regular in codimension 1 and that $\mathrm{Cl}(X)/2 = 0$. Let V be the set of codimension 1 subvarieties.*

The theorem says that there is a surjective homomorphism of $\mathbf{R}_{k(X)}$ -modules

$$H_3(\mathrm{SL}_2(k(X)), \mathbb{Q})_0 \rightarrow \bigoplus_{v \in V} \mathcal{P}(k)$$

Remark 5.6. Note that this shows that $H_3(\mathrm{SL}_2(F), \mathbb{Z})_0$ may have a large torsion-free part.

For any global field F we have

$$H_3(\mathrm{SL}_2(F), \mathbb{Z}) = \varinjlim_S H_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$$

(where S varies over all finite sets of places). It is natural to inquire to what extent the results described above for fields may extend to rings of S -integers \mathcal{O}_S (at least when S is large). For example, it follows from the equation just quoted that the square classes of \mathcal{O}_S^\times must act nontrivially on the group $H_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ for large S .

As noted in the introduction, there are very few explicit calculations of the homology or cohomology of S -arithmetic groups such as $\mathrm{SL}_2(\mathcal{O}_S)$, in spite of their great interest in geometry and number theory. (Over number fields, the homology groups are known to be finitely-generated ([3]), and calculation of homology and cohomology are essentially equivalent by the universal coefficient theorem.) For example, the cohomology of the groups $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}[1/p])$ (p a prime) are known (see [1]) and there has been extensive work on the calculation of the integral cohomology of the groups $\mathrm{SL}_2(\mathcal{O})$ where \mathcal{O} is the ring of integers in an imaginary quadratic field (see [18], [17],[20]). Little is known, on the other hand, about the integral (co)homology of $\mathrm{SL}_2(\mathbb{Z}[1/N])$ when N has at least 2 prime factors (see [6] and [11] for some partial results).

Here we give an example of how the module structure and specialization maps developed above can allow us to construct nontrivial homology classes in the groups $H_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$:

Let

$$t := \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and let G be the cyclic group of order 3 generated by t . For any ring A , we will let $\mathbb{D} := \mathbb{D}_A$ denote the image of the generator of $H_3(G, \mathbb{Z})$ in $H_3(\mathrm{SL}_2(A), \mathbb{Z})$ (see Remark 3.14 above).

Thus if A is a subring of the field F , \mathbb{D}_A maps to $-D_F$ under the map $H_3(\mathrm{SL}_2(A), \mathbb{Z}) \rightarrow \mathcal{RB}(F)$.

Theorem 5.7. *Let F be a number field. Let S be a nonempty set of finite primes of F such that $\mathrm{Cl}^S(F)/2 = 0$, where $\mathrm{Cl}^S(F)$ denotes the subgroup of $\mathrm{Cl}(\mathcal{O}_F)$ generated by the primes of S .*

Then there is a natural surjective map of $\mathbb{F}_3[\mathcal{O}_S^\times/2]$ -modules

$$H_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})_{(3)} \rightarrow \bigoplus_{v \in S} \mathcal{P}(k_v)_{(3)}.$$

Proof. For $v \in S$, let R_v denote the composite

$$H_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \longrightarrow H_3(\mathrm{SL}_2(F), \mathbb{Z}) \longrightarrow \mathcal{RB}(F) \xrightarrow{S_{v,-1}} \widehat{\mathcal{P}}(k_v).$$

Observe that the associated action of \mathcal{O}_S^\times on $\widehat{\mathcal{P}}(k_\nu)$ is described as follows: For $a \in \mathcal{O}_S^\times$, $\langle a \rangle$ acts as multiplication by $(-1)^{v(a)}$.

By [7], section 7, $\widehat{\mathcal{P}}(k)_{(3)} = \mathcal{P}(k)_{(3)}$ is generated by $D_k = 2C_k$ for any finite field k .

Our hypothesis on $\mathrm{Cl}^S(F)$ guarantees that the map

$$\mathcal{O}_S^\times \rightarrow \bigoplus_{\nu \in S} \mathbb{Z}/2$$

is surjective, and thus for each $w \in S$, there exists $u_w \in \mathcal{O}_S^\times$ satisfying $v(u_w) \equiv \delta_{\nu,w} \pmod{2}$ for all $\nu \in S$.

Finally, for each $w \in S$, let

$$\mathbb{D}_{w,S} := e_{u_w}^- \cdot \mathbb{D}_{\mathcal{O}_S} \in \mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})_{(3)}$$

(see Remark 3.14) and hence

$$R_\nu(\mathbb{D}_{w,S}) = -e_{u_w}^- \cdot D_{k_\nu} = -\delta_{\nu,w} D_{k_\nu}.$$

□

Corollary 5.8. *Let F be a number field not containing ζ_3 . Let S be a finite set of primes of F for which $\mathrm{Cl}^S(F)/2 = 0$. Let $r(S) = |\{\nu \in S \mid |k_\nu| \equiv -1 \pmod{3}\}|$. Then*

$$3\text{-rank}(\mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})) \geq 1 + r(S).$$

Proof. The 3-rank of $\bigoplus_{\nu \in S} \mathcal{P}(k_\nu)_{(3)}$ is $r(S)$ since $\mathcal{P}(k_\nu)_{(3)} = \mathcal{B}(k_\nu)_{(3)}$, and $\mathcal{B}(k_\nu)$ is cyclic of order $(|k_\nu| + 1)/2$ or $|k_\nu| + 1$ by the results of [7].

On the other hand, letting u_w be as in the proof of Theorem 5.7, the element

$$\mathbb{D}_+ := \left(\prod_{w \in S} e_{u_w}^+ \right) \mathbb{D}_{\mathcal{O}_S} \in \mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})_{(3)}$$

lies in the kernel of the map

$$\bigoplus_{\nu} R_\nu : \mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})_{(3)} \rightarrow \bigoplus_{\nu \in S} \mathcal{B}(k_\nu)_{(3)}$$

However, for any place ν of the field F , the image of \mathbb{D}_+ in $\mathcal{RB}(F)$ maps to $-D_{k_\nu}$ under the specialization map $S_{\nu,1} : \mathcal{RB}(F) \rightarrow \widehat{\mathcal{P}}(k_\nu)$ corresponding to the trivial character $\chi = 1$. If we choose ν for which 3 divides $|k_\nu| + 1$ then D_{k_ν} has order 3. It follows that \mathbb{D}_+ has order 3.

□

6. THE THIRD HOMOLOGY OF SL_2 OF LOCAL FIELDS

In this section, we use the properties of the refined Bloch group to calculate $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$ up to 2-torsion when F is a local field – i.e. is complete with respect to discrete value – with finite residue field of odd characteristic (Theorem 6.19).

6.1. Preliminary results. We will begin by recalling some of the relevant facts about structure of the multiplicative group of a local field.

Let F be a field with discrete value v and residue field k . Let π be a uniformizer. Then the homomorphism $v : F^\times \rightarrow \mathbb{Z}$ splits and we have

$$F^\times = U \cdot \pi^{\mathbb{Z}} \cong U \times \mathbb{Z}.$$

Thus

$$G_F \cong U/U^2 \times \mathbb{Z}/2.$$

Now suppose that F is complete with respect to v and that $2 \in U$ (i.e. $2 \neq 0$ in k). Then Hensel's Lemma ([14, Chapter II, 4.6]) implies that, for any $u \in U$, if $\bar{u} \in (k^\times)^2$ then $u \in U^2$. In particular, in this case $U_1 = U_1^2$ and $U/U^2 \cong k^\times/(k^\times)^2 = G_k$.

It follows that if F is complete with respect to v and if k is finite of odd order, then

$$G_F \cong G_k \times \mathbb{Z}/2$$

has order 4 with generators $\langle \pi \rangle$ and $\langle u \rangle$ where u is any nonsquare unit.

Suppose that L/F is an extension of local fields. Let π be a uniformizer of F . Then there is an induced extension of residue fields k_L/k_F and $[L : F] = v_L(\pi)[k_L : k_F]$ for any uniformizer π of F ([14, Chapter II, 6.8]).

An extension L/F of local fields is said to be *unramified* if a uniformizer π of F is also a uniformizer in L . For example, if ζ_n is a primitive n -th root of unity in some separable closure of F and if $(n, \text{char}(k)) = 1$, then $F(\zeta_n)/F$ is an unramified extension ([14, Chapter II, Proposition 7.12]). Furthermore, Hensel's Lemma implies that, when $(n, \text{char}(k)) = 1$, $\zeta_n \in F$ if and only if k contains a primitive n th root of unity.

Local class field theory gives us valuable information about norm groups $N_{L/F}(L^\times)$ of finite algebraic extensions L/F of local fields:

- (1) If L/F is a finite abelian extension of local fields of degree d , then $F^\times/N_{L/F}(L^\times)$ is a group of order d ([14, Chapter 5, Theorem 1.3]).
- (2) If L/F is a finite unramified extension of local fields, then $N_{L/F}(U_L) = U_F$ ([14, Chapter V, Corollary 1.2]).
- (3) The theory of the Hilbert symbol tells us that if $(n, \text{char}(k)) = 1$ and if $\zeta_n \in F$, then $u \in U_F$ is a norm from $F(\sqrt[n]{b})$ if and only if

$$\bar{u}^{v(b) \frac{k^\times - 1}{n}} = 1 \text{ in } k^\times.$$

(See [14, Chapter V, 3.2 (iii) and 3.4].)

As above, E denotes the field obtained from F by adjoining a root of $\Phi(T) = T^2 - T + 1$.

Lemma 6.1. *Let F be a local field with finite residue field k of order $q = p^f$. Suppose that $\mathbb{Q}_3 \not\subset F$. Then*

- (1) $\langle\langle a \rangle\rangle D_F = 0$ if and only if $a \in \langle -1 \rangle \cdot N_{E/F}(E^\times)$.
- (2) For any uniformizer, π , of F the specialization map $S_{\pi, -1} : \mathcal{RP}(F) \rightarrow \widehat{\mathcal{P}}(k)$ induces an isomorphism of \mathbb{R}_F -modules

$$\mathcal{I}_F \cdot D_F \cong \mathbb{Z} \cdot D_k = \mathcal{P}(k)_{(3)}.$$

Proof. We have already seen that if $a \in \langle -1 \rangle \cdot N_{E/F}(E^\times)$ then $\langle\langle a \rangle\rangle D_F = 0$ (Theorem 3.12). In other words, the action of \mathbb{R}_F on D_F factors through the group ring $\mathbb{Z}[F^\times / \langle -1 \rangle \cdot N_{E/F}(E^\times)]$.

If $\text{char}(F) = 3$, then $E = F$ and hence $D_F = 0$ by Corollary 3.7 (2). Similarly, $D_k = 0$. So the result holds trivially.

So we may assume that $\text{char}(F), \text{char}(k) \neq 3$. In this case $E = F(\sqrt{-3}) = F(\zeta_3)$. By the facts just quoted, E/F is unramified and every unit is a norm. In particular, $-1 \in N_{E/F}(E^\times)$.

If $\zeta_3 \in F$, then $\langle\langle a \rangle\rangle D_F = 0$ for all a , and also $D_k = 0$. The result holds again for trivial reasons.

Otherwise, $\zeta_3 \notin F$ and $F^\times/N_{E/F}(E^\times)$ is cyclic of order 2 generated by the class of a uniformizer π . It follows that

$$\mathcal{I}_F D_F = \mathbb{Z} \langle\langle \pi \rangle\rangle D_F$$

has order 1 or 3. However, since $\zeta_3 \notin k$, $q \not\equiv 1 \pmod{3}$ so $3|q+1$ and D_k has order 3. $S_{\pi,-1}(D_F) = D_k \neq 0$ and since π acts as -1 on the right, $S_{\pi,-1}(\langle\langle\pi\rangle\rangle D_F) = -2D_k = D_k \neq 0$, so that the statements of the lemma follow immediately. \square

Lemma 6.2. *Suppose that $\mathbb{Q}_3 \subset F$. Then $-1 \in N_{E/F}(E^\times)$ if and only if $[F : \mathbb{Q}_3]$ is even.*

Proof. By (3) above (with $n = 2$ and $u = -3$),

$$\begin{aligned} -1 \in N_{E/F}(E^\times) &\iff (-1)^{v(3)\frac{3^f-1}{2}} = 1 \\ &\iff v(3)\frac{3^f-1}{2} \text{ is even} \\ &\iff v(3)f \text{ is even.} \end{aligned}$$

However, by the result on the degree of local field extensions quoted before Lemma 6.1 above, $[F : \mathbb{Q}_3] = v(3)f$. \square

Corollary 6.3. *If $[F : \mathbb{Q}_3]$ is odd then*

$$\mathcal{I}_F D_F = 0 = \mathbb{Z} \cdot D_k.$$

Proof. Under these hypotheses, $-1 \notin N_{E/F}(E^\times)$ and thus $F^\times = \langle -1 \rangle \cdot N_{E/F}(E^\times)$ (since $[F^\times : N_{E/F}(E^\times)] = 2$). Thus $\mathcal{I}_F D_F = 0$ by Theorem 3.12. But $\text{char}(k) = 3$ and hence $D_k = 0$ also. \square

Remark 6.4. Of course, if $\mathbb{Q}_3 \subset F$ and $-1 \in N_{E/F}(E^\times)$ then our results do not rule out the possibility that $\mathcal{I}_F D_F$ has order 3 (rather than 1), but this module, if nontrivial, cannot be detected by $S_{\pi,-1}$ (since $D_k = 0$ in this case).

The following lemma will be central to our computations below:

Lemma 6.5. *In any field F we have*

$$e_b^- [a] = 0 \text{ in } \widehat{\mathcal{RP}}(F)[\tfrac{1}{2}]$$

whenever $a, b \in F^\times$ with $a \equiv -b \pmod{(F^\times)^2}$.

Proof. Let $a \in F^\times$, $a \neq 1$.

Then $\langle a \rangle [a] = \langle -1 \rangle [a]$ in $\widehat{\mathcal{RP}}(F)$ by Lemma 4.6. Thus $\langle -a \rangle [a] = [a]$ and hence $\langle\langle -a \rangle\rangle [a] = 0$. It follows that $e_{-a}^- [a] = 0$ in $\widehat{\mathcal{RP}}(F)[\frac{1}{2}]$.

Of course, e_x^- only depends on the square class of x , so the result follows. \square

For the remainder of this section we let F be a local field complete with respect to the discrete valuation v , with residue field k of order $q = p^f$, with p odd. Furthermore, for simplicity, we will suppose that if $\mathbb{Q}_3 \subset F$ then $[F : \mathbb{Q}_3]$ is odd.

Recall that G_F has order 4: if π is a uniformizing parameter and if u is a nonsquare unit $G_F = \{1, \langle\pi\rangle, \langle u \rangle, \langle u\pi \rangle\}$. (If $q \equiv 3 \pmod{4}$, we can take $u = -1$.)

We let \widehat{G}_F denote the group of characters $\text{Hom}(G_F, \mu_2) = \text{Hom}(G_F, \{\pm 1\})$ of G_F . Given $\chi \in \widehat{G}_F$, we have the associated idempotent

$$e_\chi = \frac{1}{4} \sum_{g \in G_F} \chi(g) \langle g \rangle.$$

Observe that if χ is a *nontrivial* character on G_F , then

$$e_\chi = \prod_{a \in \chi^{-1}(-1)} e_a^- \in \mathcal{R}_F[\tfrac{1}{2}].$$

If M is an $\mathbf{R}_F[\frac{1}{2}]$ -module and if $\chi \in \widehat{G}_F$, then $M_\chi := e_\chi(M)$ is a submodule and $g \cdot m = \chi(g)m$ for all $g \in G_F$. Observe that the functor $M \rightarrow M_\chi$ is an exact functor on the category of $\mathbf{R}_F[\frac{1}{2}]$ -modules.

For the rest of this section we fix the following: Let π be a uniformizing parameter for F and let u be a fixed nonsquare unit, which we take to be -1 in the case $q \equiv 3 \pmod{4}$. Clearly a nontrivial character in \widehat{G}_F is determined by $\chi^{-1}(-1)$. We label the four characters as follows: χ_1 is the trivial character, χ is the character with $\chi^{-1}(-1) = \{\langle \pi \rangle, \langle u\pi \rangle\}$, ψ is the character with $\psi^{-1}(-1) = \{\langle u\pi \rangle, \langle u \rangle\}$ and ψ' is the remaining character.

Note that for any $\mathbf{R}_F[\frac{1}{2}]$ -module M , we have a decomposition

$$M = M_{\chi_1} \oplus M_\chi \oplus M_\psi \oplus M_{\psi'}$$

where

$$M_{\chi_1} = M^{G_F} \cong M_{G_F} = H_0(F^\times, M)$$

and

$$\mathcal{I}_F M = M_\chi \oplus M_\psi \oplus M_{\psi'}.$$

Since $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)$ is a trivial G_F -module, from the exact sequence

$$0 \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)[\frac{1}{2}] \rightarrow H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{RB}(F)[\frac{1}{2}] \rightarrow 0$$

it follows that

$$H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_\rho = \mathcal{RB}(F)[\frac{1}{2}]_\rho$$

for any $\rho \neq \chi_1$.

On the other hand,

$$H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_{\chi_1} = H_0(F^\times, H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])) = K_3^{\mathrm{ind}}(F)[\frac{1}{2}].$$

Thus we have a $\mathbf{R}_F[\frac{1}{2}]$ -module decomposition

$$H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\mathrm{ind}}(F)[\frac{1}{2}] \oplus \mathcal{RB}(F)[\frac{1}{2}]_\chi \oplus \mathcal{RB}(F)[\frac{1}{2}]_\psi \oplus \mathcal{RB}(F)[\frac{1}{2}]_{\psi'}.$$

Our goal in the remainder of this section is to show that the last two factors on the right are zero and that the second factor is isomorphic, via $S_{\pi,-1}$, to $\mathcal{P}(k)[\frac{1}{2}]$.

In order to do this, we make a couple of reductions:

Lemma 6.6. *Let F be as stated. Then*

- (1) $(\mathcal{D}_F)_\psi = 0 = (\mathcal{D}_F)_{\psi'}$.
- (2) $(\mathcal{D}_F)_\chi = \mathcal{I}_F \mathcal{D}_F$.

Proof. Note that

$$\mathcal{I}_F \mathcal{D}_F = (\mathcal{D}_F)_\chi \oplus (\mathcal{D}_F)_\psi \oplus (\mathcal{D}_F)_{\psi'}.$$

So we must prove that the last two factors are 0.

Recall that $e_\psi = e_u^- e_{u\pi}^-$ and $e_{\psi'} = e_u^- e_\pi^-$. Now $e_u^- = -\langle u \rangle / 2$ and our conditions guarantee that $u \in \langle -1 \rangle N_{E/F}(E^\times)$ so that $\langle u \rangle D_F = 0$ by Theorem 3.12. \square

In view of Corollary 4.4 and Lemma 4.7 we have:

Corollary 6.7. $\mathcal{RB}(F)[\frac{1}{2}]_\psi = \overline{\mathcal{RB}}(F)[\frac{1}{2}]_\psi$ and $\mathcal{RB}(F)[\frac{1}{2}]_{\psi'} = \overline{\mathcal{RB}}(F)[\frac{1}{2}]_{\psi'}$.

Observe that $S_{\pi,-1}$ induces a well-defined surjective homomorphism

$$\bar{S}_{\pi,-1} : \overline{\mathcal{RP}}(F) \rightarrow \overline{\mathcal{P}}(k).$$

Lemma 6.8. *The homomorphism*

$$S_{\pi,-1} : \mathcal{RB}(F)[\frac{1}{2}]_{\chi} \rightarrow \widehat{\mathcal{P}}(k)[\frac{1}{2}]$$

is an isomorphism if and only if the homomorphism

$$\bar{S}_{\pi,-1} : \overline{\mathcal{RB}}(F)[\frac{1}{2}]_{\chi} \rightarrow \overline{\mathcal{P}}(k)[\frac{1}{2}]$$

is an isomorphism.

Proof. Using Lemma 6.6 (2), we have a commutative diagram of R_F -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_F D_F & \longrightarrow & \mathcal{RB}(F)[\frac{1}{2}]_{\chi} & \longrightarrow & \overline{\mathcal{RB}}(F)[\frac{1}{2}]_{\chi} \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow S_{\pi,-1} & & \downarrow \bar{S}_{\pi,-1} \\ 0 & \longrightarrow & \mathbb{Z}D_k & \longrightarrow & \widehat{\mathcal{P}}(k)[\frac{1}{2}] & \longrightarrow & \overline{\mathcal{P}}(k)[\frac{1}{2}] \longrightarrow 0 \end{array}$$

in which the left vertical arrow is an isomorphism by Lemma 6.1 (2). \square

6.2. S_3 -dynamics. Again, in this section, F denotes a field complete with respect to a discrete value v with finite residue field k of odd order q .

For any field L , the transformations

$$\sigma, \tau : \mathbb{P}^1(L) \rightarrow \mathbb{P}^1(L), \quad \sigma(x) = x^{-1} \text{ and } \tau(x) = 1 - x$$

determine an action of the symmetric group on 3 letters S_3 . $\{0, 1, \infty\}$ is an orbit for this action, and hence S_3 acts on $\mathbb{P}^1(L) \setminus \{0, 1, \infty\} = F^{\times} \setminus \{1\}$.

Lemma 6.9. *If $\alpha \in R_F$ and if $\alpha[x] = 0$ in $\overline{\mathcal{RP}}(F)$. Then $\alpha[\gamma(x)] = 0$ for all $\gamma \in S_3$.*

Proof. In $\overline{\mathcal{RP}}(F)$ we have $D_F = 0$ and $\psi_1(x) = 0$ for all x , so that

$$[x^{-1}] = [1 - x] = -\langle -1 \rangle [x].$$

It follows that if $\alpha[x] = 0$ for some $\alpha \in R_F$, then $\alpha[\sigma(x)] = \alpha[\tau(x)] = 0$ also. The statement follows since σ and τ generate S_3 . \square

Remark 6.10. We will apply this below in the case where ρ is a character of G_F and $\alpha = e_{\rho} \in R_F[\frac{1}{2}]$

Let $\bar{R} = (k^{\times})^2$ and let $\bar{N} = k^{\times} \setminus (k^{\times})^2$. Let

$$\bar{R}_1 = \{a \in \bar{R} \setminus \{1\} | \tau(a) \in \bar{R}\}, \quad \bar{R}_{-1} = \{a \in \bar{R} | \tau(a) \in \bar{N}\}.$$

Let

$$\bar{N}_1 = \{a \in \bar{N} | \tau(a) \in \bar{R}\}, \quad \bar{N}_{-1} = \{a \in \bar{N} | \tau(a) \in \bar{N}\}.$$

Thus we have a partition

$$k^{\times} = \{1\} \cup \bar{R}_1 \cup \bar{R}_{-1} \cup \bar{N}_1 \cup \bar{N}_{-1}$$

Let U be the group of units of F and let U_1 be the kernel of the reduction map $U \rightarrow k^{\times}$. Then there is a corresponding partition

$$U = R \cup N = U_1 \cup R_1 \cup R_{-1} \cup N_1 \cup N_{-1}$$

where $R := U^2 = \{a \in U | \bar{a} \in \bar{R}\}$, $N = U \setminus R$, $R_1 := \{a \in U | \bar{a} \in \bar{R}_1\} = \{a \in R | \tau(a) \in R\}$, $R_{-1} := \{a \in U | \bar{a} \in \bar{R}_{-1}\}$, $N_1 := \{a \in U | \bar{a} \in \bar{N}_1\}$ and $N_{-1} := \{a \in U | \bar{a} \in \bar{N}_{-1}\}$.

Lemma 6.11. (1) *If $q \equiv 1 \pmod{4}$ then*

$$\begin{aligned}\bar{R}_1 &= \sigma(\bar{R}_1) = \tau(\bar{R}_1) \\ \sigma(\bar{N}_1) &= \bar{N}_{-1} \text{ and } \sigma(\bar{R}_{-1}) = \bar{R}_{-1}\end{aligned}$$

Furthermore

$$|\bar{R}_{-1}| = |\bar{N}_{-1}| = |\bar{N}_1| = \frac{q-1}{4} \text{ and } |\bar{R}_1| = \frac{q-5}{4}.$$

(2) *If $q \equiv 3 \pmod{4}$ then*

$$\begin{aligned}\bar{N}_{-1} &= \sigma(\bar{N}_{-1}) = \tau(\bar{N}_{-1}) \\ \sigma(\bar{R}_1) &= \bar{R}_{-1} \text{ and } \sigma(\bar{N}_1) = \bar{N}_1\end{aligned}$$

Furthermore

$$|\bar{R}_{-1}| = |\bar{N}_1| = |\bar{R}_1| = \frac{q-3}{4} \text{ and } |\bar{N}_{-1}| = \frac{q+1}{4}.$$

Proof. (1) Clearly $\tau(\bar{R}_1) = \bar{R}_1$ by definition. Furthermore, if $a \in \bar{R}_1$ then

$$1 - \frac{1}{a} = (-1) \cdot (1 - a) \cdot \frac{1}{a} \in \bar{R}$$

and thus $a^{-1} = \sigma(a) \in \bar{R}_1$.

It follows that $\sigma(\bar{R}_{-1}) = \bar{R}_{-1}$ since $\sigma(\bar{R}) = \bar{R}$.

Similarly, if $a \in \bar{N}_1$ then $a^{-1} \in \bar{N}$ $1 - a \in \bar{R}$ and thus

$$1 - \frac{1}{a} = (-1) \cdot (1 - a) \cdot \frac{1}{a} \in \bar{N}$$

so that $\sigma(\bar{N}_1) = \bar{N}_{-1}$.

The second statement follows by simple counting since

$$|\bar{N}_1| + |\bar{N}_{-1}| = |\bar{N}| = (q-1)/2.$$

(2) The argument is similar except that this time

$$a - 1 = (-1) \cdot (1 - a) \in \begin{cases} \bar{N}, & 1 - a \in \bar{R} \\ \bar{R}, & 1 - a \in \bar{N} \end{cases}$$

since $-1 \in \bar{N}$.

□

Taking inverse images in $U \subset F$, and noting that $U \setminus U_1$ is closed under the action of S_3 , we obtain :

Corollary 6.12. (1) *If $q \equiv 1 \pmod{4}$ then*

$$\begin{aligned}R_1 &= \sigma(R_1) = \tau(R_1) \\ \sigma(N_1) &= N_{-1} \text{ and } \sigma(R_{-1}) = R_{-1}\end{aligned}$$

(2) *If $q \equiv 3 \pmod{4}$ then*

$$\begin{aligned}N_{-1} &= \sigma(N_{-1}) = \tau(N_{-1}) \\ \sigma(R_1) &= R_{-1} \text{ and } \sigma(N_1) = N_1\end{aligned}$$

We will need the following elementary result below:

Lemma 6.13. *Let k be a finite field of odd order, and let $n \in \bar{N}$. Then there exist $r_1, r_2 \in \bar{R}$ such that $n = r_1 + r_2$.*

Proof. If $q \equiv 1 \pmod{4}$, there exists $r \in \bar{R}$ with $rn \in \bar{N}_1$. Thus $1 - rn = s \in \bar{R}$ and

$$n = \frac{1}{r} + \left(-\frac{s}{r}\right) = r_1 + r_2.$$

If $q \equiv 3 \pmod{4}$ then there exists $r \in \bar{R}$ with $rn \in \bar{N}_{-1}$. Then $1 - nr = m \in \bar{N}$ and thus

$$n = \frac{1}{r} + \left(-\frac{m}{r}\right) = r_1 + r_2.$$

□

6.3. The main result. Let F be a local field with discrete value v and finite residue field k of odd order. Throughout π will denote a choice of uniformizer.

Let $M(F)$ denote the $R_F[\frac{1}{2}]$ -module $e_{-1}^- \mathcal{I}_F[\frac{1}{2}]$.

Lemma 6.14. (1) *If $\rho \neq \chi_1$ then there is a natural short exact sequence of $R_F[\frac{1}{2}]$ -modules*

$$0 \longrightarrow \overline{\mathcal{RB}}(F)[\frac{1}{2}]_\rho \longrightarrow \overline{\mathcal{RP}}(F)[\frac{1}{2}]_\rho \xrightarrow{\bar{\lambda}_1} M(F)_\rho \longrightarrow 0.$$

(2) *If $q \equiv 1 \pmod{4}$ and if $\rho \neq \chi_1$ then*

$$\overline{\mathcal{RB}}(F)[\frac{1}{2}]_\rho = \overline{\mathcal{RP}}(F)[\frac{1}{2}]_\rho.$$

(3) *If $q \equiv 3 \pmod{4}$ then*

$$\overline{\mathcal{RB}}(F)[\frac{1}{2}]_\chi = \overline{\mathcal{RP}}(F)[\frac{1}{2}]_\chi.$$

Proof. There is an exact sequence

$$0 \longrightarrow \overline{\mathcal{RB}}(F) \longrightarrow \overline{\mathcal{RP}}(F) \xrightarrow{\Lambda} \widetilde{RS}_{\mathbb{Z}}^2(F^\times)$$

The kernel of the surjection $\widetilde{RS}_{\mathbb{Z}}^2(F^\times) \rightarrow \mathcal{I}_F^2$ is a trivial R_F -module. Furthermore, since $\mathcal{I}_F/\mathcal{I}_F^2 \cong G_F$ is a 2-torsion group, we have $\mathcal{I}_F^2[\frac{1}{2}] = \mathcal{I}_F[\frac{1}{2}]$. Putting these facts together we obtain an isomorphism of R_F -modules

$$\widetilde{RS}_{\mathbb{Z}}^2(F^\times)[\frac{1}{2}]_\rho \cong \mathcal{I}_F[\frac{1}{2}]_\rho$$

for all $\rho \neq \chi_1$. This in turn induces an isomorphism

$$\widetilde{RS}_{\mathbb{Z}}^2(F^\times)[\frac{1}{2}]_\rho \cong \left(\frac{\mathcal{I}_F[\frac{1}{2}]}{\langle \langle x \rangle \langle -x \rangle \rangle} \right)_\rho = \left(\frac{\mathcal{I}_F[\frac{1}{2}]}{\mathfrak{p}_{-1}^+ \mathcal{I}_F[\frac{1}{2}]} \right)_\rho = \left(\frac{\mathcal{I}_F[\frac{1}{2}]}{\mathfrak{e}_{-1}^+ \mathcal{I}_F[\frac{1}{2}]} \right)_\rho \cong e_{-1}^- \mathcal{I}_F[\frac{1}{2}]_\rho = M(F)_\rho$$

and the homomorphism

$$\overline{\mathcal{RP}}(F)[\frac{1}{2}]_\rho \rightarrow \widetilde{RS}_{\mathbb{Z}}^2(F^\times)[\frac{1}{2}]_\rho \cong M(F)_\rho$$

is induced by the map $\bar{\lambda}_1 : \overline{\mathcal{RP}}(F) \rightarrow M(F)$

$$[x] \mapsto e_{-1}^- \lambda_1([x]) = e_{-1}^- \langle x \rangle \langle 1 - x \rangle.$$

To see that this homomorphism is surjective, observe first that if $q \equiv 1 \pmod{4}$, then $\langle -1 \rangle = 1$ and hence $e_{-1}^- = 0$ in $R_F[\frac{1}{2}]$. Thus $M(F) = 0$ in this case, and there is nothing to prove. This also implies statement (2) of the lemma.

So we can suppose that $q \equiv 3 \pmod{4}$. Then, as a $\mathbb{Z}[\frac{1}{2}]$ -module, $\mathcal{I}_F[\frac{1}{2}]$ is free of rank 3 with basis e_{-1}^- , $e_{-\pi}^-$ and $e_{-\pi^2}^-$. Then $e_{-1}^- (\mathcal{I}_F[\frac{1}{2}])$ is a free $\mathbb{Z}[\frac{1}{2}]$ -module of rank 2 with basis e_{-1}^- and $e_{-\pi}^- e_{-1}^-$ (since, for example, $e_{-1}^- e_{-\pi}^- = -(e_{-1}^- + e_{-\pi}^- e_{-1}^-)$.) As an $R_F[\frac{1}{2}]$ -module, it is thus generated by e_{-1}^- .

Now let $n \in N_{-1}$. Then $1 - n \in N_{-1}$ and

$$\lambda_1 \left(\frac{1}{4} [n] \right) = \frac{1}{4} \langle\langle n \rangle\rangle \langle\langle 1 - n \rangle\rangle = \frac{1}{4} \langle\langle -1 \rangle\rangle \langle\langle -1 \rangle\rangle = (e_{-1}^-)^2 = e_{-1}^-$$

so that the map is surjective.

Finally, observe also that if $q \equiv 3 \pmod{4}$ then $e_\chi e_{-1}^- = 0$ so that $M(F)_\chi = 0$ and statement (3) also follows. \square

We will use the following notation in the remainder of this section: Given a character, ρ of G_F and $a \in F^\times$, we will denote $e_\rho([a]) \in \overline{\mathcal{RP}}(F)[\frac{1}{2}]$ by $[a]_\rho$.

In the each of next few lemmas, the strategy of proof is largely the same. The proof involves showing that $[a]_\rho = 0$ for a belonging to some subset of F^\times . We begin with Lemma 6.5 to show that $[a]_\rho = 0$ for a belonging to one or more square classes, and then use Lemma 6.9 to deduce the vanishing of $[a]_\rho$ for a larger set of a .

In order to follow the arguments, it may be helpful for the reader to keep the following decompositions in mind: Recall that there are four square classes, or cosets modulo $(F^\times)^2$, in the group F^\times . Thus

$$F^\times = (F^\times)^2 \cup u \cdot (F^\times)^2 \cup \pi \cdot (F^\times)^2 \cup u\pi \cdot (F^\times)^2$$

and

$$\begin{aligned} (F^\times)^2 &= \bigcup_{n \in \mathbb{Z}} R\pi^{2n} = R \cup \bigcup_{n \neq 0} R\pi^{2n} = U_1 \cup R_1 \cup R_{-1} \cup \bigcup_{n \neq 0} R\pi^{2n}. \\ u \cdot (F^\times)^2 &= \bigcup_{n \in \mathbb{Z}} N\pi^{2n} = N \cup \bigcup_{n \neq 0} N\pi^{2n} = N_1 \cup N_{-1} \cup \bigcup_{n \neq 0} N\pi^{2n}. \\ \pi \cdot (F^\times)^2 &= \bigcup_{n \in \mathbb{Z}} R\pi^{2n-1}. \\ u\pi \cdot (F^\times)^2 &= \bigcup_{n \in \mathbb{Z}} N\pi^{2n-1}. \\ U_1 &= \bigcup_{m \geq 1} (1 - U\pi^m) = \bigcup_{n \geq 1} (1 - U\pi^{2n}) \cup \bigcup_{n \geq 1} (1 - U\pi^{2n-1}). \end{aligned}$$

We will repeatedly use the *key fact* that $U_1 = U_1^2$ and thus $1 - a$ is a square whenever $v(a) > 0$.

Lemma 6.15. *Let ρ be a character of G_F . Then the following are equivalent:*

- (1) $[a]_\rho = 0$ in $\overline{\mathcal{RP}}(F)$ for all $a \in U_1$
- (2) $[a]_\rho = 0$ in $\overline{\mathcal{RP}}(F)$ for all a satisfying $v(a) \neq 0$.

Proof. Suppose first that $[a]_\rho = 0$ for all $a \in U_1$. Let $b \in F^\times$ with $v(b) > 0$. Then $c := \tau(b) = 1 - b \in U_1$. Thus $[c]_\rho = 0$ by assumption. It follows that $[b]_\rho = [\tau(c)]_\rho = 0$ by Lemma 6.9 with $\alpha = e_\rho$.

On the other hand, if $b \in F^\times$ with $v(b) < 0$, then $v(b^{-1}) > 0$ and thus $[b^{-1}]_\rho = 0$. But then $[b]_\rho = [\sigma(b^{-1})]_\rho = 0$ by Lemma 6.9 again.

Conversely, suppose that $[a]_\rho = 0$ whenever $v(a) \neq 0$ and that $b \in U_1$. Then $v(1 - b) > 0$. Thus $[1 - b]_\rho = 0$ and hence $[b]_\rho = [\tau(1 - b)]_\rho = 0$. \square

Lemma 6.16. *The $R_F[\frac{1}{2}]$ -module homomorphism*

$$S_{\pi, -1} : \mathcal{RP}(F)[\frac{1}{2}]_\chi \rightarrow \widehat{\mathcal{P}}(k)[\frac{1}{2}]$$

is an isomorphism.

Proof. By Lemma 6.8 it is enough to prove that the map

$$\bar{S}_{\pi,-1} : \overline{\mathcal{RP}}(F)[\frac{1}{2}]_{\chi} \rightarrow \overline{\mathcal{P}}(k)[\frac{1}{2}]$$

is an isomorphism.

To prove the result we will show that there is a well-defined $R_F[\frac{1}{2}]$ -module homomorphism

$$T : \overline{\mathcal{P}}(k)[\frac{1}{2}] \rightarrow \overline{\mathcal{RP}}(F)[\frac{1}{2}]_{\chi}, \quad T([\bar{a}]) = [a]_{\chi}$$

(where $a \in U$ maps to $\bar{a} \in k^{\times}$) which is inverse to $\bar{S}_{\pi,-1}$.

To show that T is well-defined we must show that $[au]_{\chi} = [a]_{\chi}$ whenever $a \in U$ and $u \in U_1$.

Since clearly

$$\bar{S}_{\pi,-1} \circ T = \text{Id}_{\widehat{\mathcal{RB}(k)}[\frac{1}{2}]},$$

it is then only necessary to show that T is surjective: we must show that $[a]_{\chi} = 0$ whenever $v(a) \neq 0$ (since if $v(a) = 0$ then $[a]_{\chi} = T([\bar{a}])$ by definition of T , and thus $[a]_{\chi}$ lies in the image of T for all $a \in F^{\times}$).

Note, to begin with, that since $e_{\chi} = e_{\pi}^{-} e_{u\pi}^{-}$, we have

$$[a]_{\chi} = 0 \text{ if } a \equiv -\pi, -u\pi \pmod{(F^{\times})^2}$$

by Lemma 6.5. This is clearly equivalent to $[a]_{\chi} = 0$ if a lies in one of the two square classes $\pi \cdot (F^{\times})^2$ and $u\pi \cdot (F^{\times})^2$. Thus $[a]_{\chi} = 0$ if $a \in U\pi^{2n-1}$ for any $n \in \mathbb{Z}$.

Thus for any $n \geq 1$, $[a]_{\chi} = 0$ if $a \in \tau(U\pi^{2n-1}) = 1 - U\pi^{2n-1} \subset U_1$.

Consider now the case $a = w\pi^{2n}$, $w \in U$ and $n \geq 1$. By considering the relation $(S_{1/\pi, a/\pi})_{\chi}$, we see that

$$0 = \left[\frac{1}{\pi} \right]_{\chi} - \left[\frac{a}{\pi} \right]_{\chi} + \langle \pi \rangle [a]_{\chi} - \langle \pi - 1 \rangle \left[-w\pi^{2n-1} \frac{1 - \pi}{1 - w\pi^{2n-1}} \right]_{\chi} + \langle -\pi \rangle \left[\frac{1}{\pi} \cdot \frac{1 - \pi}{1 - w\pi^{2n-1}} \right]_{\chi} = \langle \pi \rangle [a]_{\chi}$$

since all terms other than the third belong to the square classes $\pi \cdot (F^{\times})^2$ or $u\pi \cdot (F^{\times})^2$.

Thus we deduce that $[a]_{\chi} = 0$ if $a \in U\pi^{2n}$ for any $n \geq 1$. It follows that $[a]_{\chi} = 0$ whenever $v(a) > 0$. By considering $\sigma(a) = 1/a$ it follows that $[a]_{\chi} = 0$ whenever $v(a) \neq 0$. From Lemma 6.15 it follows that $[a]_{\chi} = 0$ whenever $a \in U_1$ also.

Finally, let $a \in U \setminus U_1$ and $w \in U_1$. Then

$$0 = (S_{a,aw})_{\chi} = [a]_{\chi} - [aw]_{\chi} + \langle a \rangle [w]_{\chi} - \langle a^{-1} - 1 \rangle \left[w \cdot \frac{1 - a}{1 - aw} \right]_{\chi} + \langle 1 - a \rangle \left[\frac{1 - a}{1 - aw} \right]_{\chi}$$

which gives $[a]_{\chi} = [aw]_{\chi}$ since the last three terms all lie in U_1 . This proves the Lemma. \square

Lemma 6.17. *Let F be a local field whose residue field k has order q with $q \equiv 1 \pmod{4}$. Then*

$$\overline{\mathcal{RP}}(F)[\frac{1}{2}]_{\psi} = \overline{\mathcal{RP}}(F)[\frac{1}{2}]_{\psi'} = 0.$$

Proof. We treat the case of ψ . Clearly the case of ψ' is identical since it only involves a switch in our choice of uniformizer.

Since $\psi = e_u^{-} e_{\pi}^{-}$ and since $-1 \in (F^{\times})^2$ we have

$$[a]_{\psi} = 0 \text{ if } a \equiv u, \pi \pmod{(F^{\times})^2}$$

by Lemma 6.5.

Thus $[a]_{\psi} = 0$ if $a \in N\pi^{2n} \cup R\pi^{2n-1}$ for any $n \in \mathbb{Z}$.

We must prove that $[a]_{\chi} = 0$ also for $a \in (F^{\times})^2 \cup u\pi \cdot (F^{\times})^2$.

Taking the S_3 -action into account as in the last lemma, it follows that

$$[a]_\psi = 0 \text{ if } a \in 1 - R\pi^{2n-1} \cup 1 - N\pi^{2n} \text{ for any } n \geq 1.$$

Furthermore, if $a \in R_{-1}$ then $\tau(a) \in u \cdot (F^\times)^2$. Thus $[\tau(a)]_\psi = 0$ and hence $[a]_\psi = 0$.

Consider now $a = r\pi^{2n}$ with $r \in R$ and $n \geq 1$. Then we have

$$0 = \left[\frac{\pi}{a} \right]_\psi - [\pi]_\psi + \langle \pi \rangle [a]_\psi - \left[\frac{a - \pi}{1 - \pi} \right]_\psi + \langle \pi \rangle \left[\frac{1}{a} \cdot \frac{a - \pi}{1 - \pi} \right]_\psi$$

which forces $\langle \pi \rangle [a]_\psi = 0 = [a]_\psi$ since the other four terms all belong to the square class $\pi \cdot (F^\times)^2$.

By considering $\sigma(a) = a^{-1}$, we deduce that $[b]_\psi = 0$ for all $b \in R\pi^{2n}$ and all $n \neq 0$.

Next, we consider $a = n\pi^{2m-1}$ with $n \in N$ and $m \geq 1$. By Lemma 6.13, we can write $n = r + s$ with $r, s \in R$.

Let $x = r\pi^{2m-1}$. Observe that

$$\begin{aligned} w &:= \frac{1-x}{1-a} = \frac{1-r\pi^{2m-1}}{1-n\pi^{2m-1}} = (1 - r\pi^{2m-1})(1 + n\pi^{2m-1} + \dots) \\ &= 1 + (n - r)\pi^{2m-1} + \dots = 1 + st\pi^{2m-1} \end{aligned}$$

with $t \in U_1$. Thus $w \in 1 - R\pi^{2n-1}$ since $s \in R$.

Thus we have

$$0 = [x]_\psi - [a]_\psi + \langle \pi \rangle \left[\frac{n}{r} \right]_\psi - \langle \pi \rangle \left[\frac{n}{r} \cdot w \right]_\psi + [w]_\psi = -[a]_\psi$$

since $x \in \pi \cdot (F^\times)^2$, $\frac{n}{r}, \frac{nw}{r} \in N \subset u \cdot (F^\times)^2$ and, as noted above, $w \in 1 - R\pi^{2m-1}$.

By considering $\sigma(a)$ also, we deduce that $[b]_\psi = 0$ for all $b \in N\pi^{2m-1}$ for any $m \in \mathbb{Z}$.

Thus we have shown that $[a]_\psi = 0$ whenever $v(a) \neq 0$ and whenever $a \in R_{-1}$. By Lemma 6.15 it follows that $[a]_\psi = 0$ whenever $a \in U_1$.

It remains to prove that $[a]_\psi = 0$ whenever $a \in R_1$.

Let $a \in R_1$. Observe that $\{as \mid s \in R_{-1}\} \cap R_{-1} \neq \emptyset$ since $|\bar{R}_{-1}| = 1 + |\bar{R}_1|$ by Lemma 6.11 (1). It follows that we can write $a = t/s$ with $s, t \in R_{-1}$.

Observe that $(1-s)/(1-t) \in R_{-1}$ also: Since $1-s, 1-t \in N$ we have $(1-s)/(1-t) \in R$. Since $a = s/t \in R_1$,

$$1 - \frac{s}{t} = \frac{t-s}{t} \in R$$

and thus $s-t, t-s \in R$. Hence

$$1 - \frac{1-s}{1-t} = \frac{s-t}{1-t} \in N$$

so that $(1-s)/(1-t) \in R_{-1}$.

Applying the same argument to s^{-1}, t^{-1} shows that $(1-s^{-1})/(1-t^{-1}) \in R_{-1}$ also. Thus

$$0 = [s]_\psi - [t]_\psi + [a]_\psi - \langle u \rangle \left[\frac{1-s^{-1}}{1-t^{-1}} \right]_\psi + \langle u \rangle \left[\frac{1-s}{1-t} \right]_\psi = [a]_\psi$$

proving the lemma. □

Lemma 6.18. *Let F be a local field with residue field k of order q and $q \equiv 3 \pmod{4}$. Then*

$$\overline{\mathcal{RB}}(F)[\tfrac{1}{2}]_\psi = \overline{\mathcal{RB}}(F)[\tfrac{1}{2}]_{\psi'} = 0.$$

Proof. By Lemma 6.14 it is enough to show that λ_1 induces isomorphisms of $\mathbf{R}_F[\frac{1}{2}]$ -modules

$$\overline{\mathcal{RP}}(F)[\frac{1}{2}]_\psi \cong M(F)_\psi \text{ and } \overline{\mathcal{RP}}(F)[\frac{1}{2}]_{\psi'} \cong M(F)_{\psi'}.$$

As remarked in the proof of Lemma 6.17, it is enough to treat the case of the character ψ .

Observe that $M(F)_\psi$ is a free $\mathbb{Z}[\frac{1}{2}]$ -module of rank 1 with generator $e_\psi = e_{-1}^- e_\pi^-$. Thus, if L is any $\mathbf{R}_F[\frac{1}{2}]$ -module and if $a \in L$, there is a unique $\mathbf{R}_F[\frac{1}{2}]$ -module homomorphism $M(F)_\psi \rightarrow L$ sending e_ψ to $e_\psi(a)$.

Thus we fix $n \in N_{-1}$ and let $\Phi : M(F)_\psi \rightarrow \overline{\mathcal{RP}}(F)[\frac{1}{2}]_\psi$ be the unique $\mathbf{R}_F[\frac{1}{2}]$ -module homomorphism sending e_ψ to $\frac{1}{4}[n]_\psi$.

By our previous remarks, Φ is well-defined and clearly

$$\bar{\lambda}_1 \circ \Phi = \mathrm{Id}_{M(F)_\psi}.$$

So it only remains to show that Φ is surjective. To do this we show that $[a]_\psi = 0$ in $\overline{\mathcal{RP}}(F)[\frac{1}{2}]_\psi$ whenever $a \notin N_{-1}$ and that $[n_1]_\psi = [n_2]_\psi$ for any $n_1, n_2 \in N_{-1}$.

Now

$$[a]_\psi = 0 \text{ if } a \in (F^\times)^2 \cup -\pi \cdot (F^\times)^2.$$

In particular, $[a]_\psi = 0$ if $a \in U_1$, since $U_1 \subset (F^\times)^2$. By Lemma 6.15 it follows that $[a]_\psi = 0$ if $v(a) \neq 0$.

Thus $[a] = 0$ if $a \in \pi \cdot (F^\times)^2$ and $[a] = 0$ if $a \in -1 \cdot (F^\times)^2$ and $v(a) \neq 0$. All of this shows that

$$[a]_\psi = 0 \text{ for any } a \notin N$$

since $N = \{a \in -1 \cdot (F^\times)^2 \mid v(a) = 0\}$.

However, if $a \in N_1$, then $b = \tau(a) \in (F^\times)^2$. Thus $[b]_\psi = 0$ and hence $[a]_\psi = [\tau(b)]_\psi = 0$. So we have shown that $[a]_\psi = 0$ for all $a \notin N_{-1}$.

Finally, let $n_1, n_2 \in N_{-1}$. Then in $\overline{\mathcal{RP}}(F)[\frac{1}{2}]_\psi$ we have

$$0 = [n_1]_\psi - [n_2]_\psi + \langle -1 \rangle \left[\frac{n_2}{n_1} \right]_\psi - \langle -1 \rangle \left[\frac{1 - n_1^{-1}}{1 - n_2^{-1}} \right]_\psi + \left[\frac{1 - n_1}{1 - n_2} \right]_\psi = [n_1]_\psi - [n_2]_\psi$$

since clearly, from the definition of N_{-1} ,

$$\frac{n_2}{n_1}, \quad \frac{1 - n_1}{1 - n_2}, \quad \frac{1 - n_1^{-1}}{1 - n_2^{-1}} \in R.$$

This proves the lemma. □

Putting all of these results together gives the following calculation of the third homology of $\mathrm{SL}_2(F)$:

Theorem 6.19. *Let F be a local field with residue field k of odd order. If $\mathbb{Q}_3 \subset F$, suppose that $[F : \mathbb{Q}_3]$ is odd.*

Then there is an isomorphism of $\mathbf{R}_F[\frac{1}{2}]$ -modules

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\mathrm{ind}}(F)[\frac{1}{2}] \oplus \mathcal{P}(k)[\frac{1}{2}]$$

in which F^\times acts trivially on the first factor, while (the square class of) any uniformizer acts as multiplication by -1 on the second factor.

Proof. As observed in section 6.1 above, there is a $\mathbf{R}_F[\frac{1}{2}]$ -module decomposition

$$\mathbf{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\mathrm{ind}}(F)[\frac{1}{2}] \oplus \mathcal{RB}(F)[\frac{1}{2}]_{\chi} \oplus \mathcal{RB}(F)[\frac{1}{2}]_{\psi} \oplus \mathcal{RB}(F)[\frac{1}{2}]_{\psi'}.$$

Now

$$\mathcal{RB}(F)[\frac{1}{2}]_{\chi} \cong \widehat{\mathcal{P}}(k)[\frac{1}{2}] = \mathcal{P}(k)[\frac{1}{2}]$$

by Lemma 6.8, Lemma 6.14 and Lemma 6.16.

Finally, we have

$$\mathcal{RB}(F)[\frac{1}{2}]_{\psi} = \mathcal{RB}(F)[\frac{1}{2}]_{\psi'} = 0$$

by Corollary 6.7, Lemma 6.17 and Lemma 6.18. □

Remark 6.20. If $\mathbb{Q}_3 \subset F$ and if $[F : \mathbb{Q}_3]$ is even, then the arguments above show that there is a surjective homomorphism of $\mathbf{R}_F[\frac{1}{2}]$ -modules

$$\mathbf{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \twoheadrightarrow K_3^{\mathrm{ind}}(F)[\frac{1}{2}] \oplus \mathcal{P}(k)[\frac{1}{2}]$$

whose kernel has order 1 or 3.

Remark 6.21. It is not difficult to write down an explicit homology class which generates $\mathbf{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 \cong \mathcal{P}(\mathbb{F}_q)[\frac{1}{2}] \cong \mathcal{B}(\mathbb{F}_q)[\frac{1}{2}]$ for a local field F with residue field \mathbb{F}_q of odd order.

Let $r = (q + 1)'$. Let $a \in U \setminus U^2$ and observe that the residue field of $F(\sqrt{a})$ is $\mathbb{F}_q(\sqrt{a}) = \mathbb{F}_{q^2}$. Let z be an element of $\mathbb{F}_q(\sqrt{a})^\times$ of order r . By Hensel's Lemma, there exists $\zeta \in F(\sqrt{a})^\times$ of order r with the property that $\bar{\zeta} = z$; i.e. ζ is a primitive r -th root of unity. Thus, if $\zeta = \alpha + \beta\sqrt{a}$, then $z = \bar{\alpha} + \bar{\beta}\sqrt{a}$.

Now, for any quadratic field extension K/L with $K = L(\sqrt{b})$ for some $b \in L^\times$, there is an injective group homomorphism

$$\kappa : K^\times \rightarrow \mathrm{GL}_2(L), x + y\sqrt{b} \mapsto \begin{bmatrix} x & by \\ y & x \end{bmatrix}$$

with the property that $\det(\kappa(x)) = N_{K/L}(x)$ for all $x \in K^\times$.

Let

$$t = \kappa(\zeta) = \begin{bmatrix} \alpha & a\beta \\ \beta & \alpha \end{bmatrix} \in \mathrm{SL}_2(F).$$

Let Γ_t be a generator of

$$\mathbf{H}_3(\langle t \rangle, \mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}/r.$$

Then the image, γ , of Γ_t under the map

$$\mathbf{H}_3(\langle t \rangle, \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathbf{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])$$

is represented by the cycle

$$\sum_{i=0}^{n-1} 1 \otimes (1, t, t^i, t^{i+1}).$$

Let $\bar{t} \in \mathrm{SL}_2(\mathbb{F}_q)$ be the matrix obtained by reducing the entries of t . Then $\bar{t} = \kappa(z)$ has order r by Lemma 2.6. Thus the class γ maps to a generator of $\mathcal{B}(\mathbb{F}_q)[\frac{1}{2}]$ under the specialization homomorphism $S_{\pi, -1}$ since the diagram

$$\begin{array}{ccccc}
 \mathrm{H}_3(\langle t \rangle, \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathcal{RB}(F)[\frac{1}{2}] \\
 \downarrow \cong & & & & \downarrow S_{\pi-1} \\
 \mathrm{H}_3(\langle \bar{t} \rangle, \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{H}_3(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathcal{B}(\mathbb{F}_q)[\frac{1}{2}]
 \end{array}$$

commutes, because the composite map on the bottom row is an isomorphism (see Lemma 2.6 above).

Let π be a uniformizing parameter and let

$$t_\pi = \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & a\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 1/\pi & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \pi a\beta \\ \beta/\pi & \alpha \end{bmatrix}.$$

Then the homology class

$$e_\pi^-(\gamma) = \frac{1}{2} \sum_{i=0}^{n-1} 1 \otimes \left((1, t, t^i, t^{i+1}) - (1, t_\pi, t_\pi^i, t_\pi^{i+1}) \right)$$

is a generator of $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 = e_\pi^- \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])$ by the arguments of this section.

Remark 6.22. The same techniques as used above apply also to case where the residue field has characteristic 2. However, the number of square classes is of the form 2^k , $k \geq 3$ and the condition $U_1 = U_1^2$ is not satisfied. The ‘ S_3 -dynamics’ – i.e. the manner in which the S_3 orbits intersect with the square classes in F^\times – become more complicated as k grows. The author has confirmed, for example, that

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}_2), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\mathrm{ind}}(\mathbb{Q}_2)[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{F}_2)$$

as expected, but the calculations are long and unedifying.

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