Title	The identity is isolated among composition operators
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Publication date	2004
Publication information	Chu, Cho-Ho, Remo V. Hügli, and Michael Mackey. “The Identity Is Isolated among Composition Operators.” American Mathematical Society, 2004. https://doi.org/10.1090/S0002-9939-04-07474-X.
Publisher	American Mathematical Society
Item record/more information	http://hdl.handle.net/10197/6285
Publisher's statement	First published in Proceedings of the American Mathematical Society 132(11): 3305-3308, 2004, published by the American Mathematical Society
Publisher's version (DOI)	10.1090/S0002-9939-04-07474-X

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 11, Pages 3305–3308
S 0002-9939(04)07474-X
Article electronically published on May 21, 2004

THE IDENTITY IS ISOLATED AMONG COMPOSITION OPERATORS

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(Communicated by Joseph A. Ball)

Abstract. Let $H^\infty(B)$ be the Banach algebra of bounded holomorphic functions on the open unit ball B of a Banach space. We show that the identity operator is an isolated point in the space of composition operators on $H^\infty(B)$. This answers a conjecture of Aron, Galindo and Lindström.

1. Introduction

Let B be the open unit ball of a complex Banach space E, and let $H^\infty(B)$ be the uniform algebra of bounded complex-valued holomorphic functions on B, with the supremum norm $\|f\| = \sup_{x\in B} |f(x)|$. Given any holomorphic self-map ϕ of B, we define the *composition operator* $C_\phi : H^\infty(B) \to H^\infty(B)$ by

$$C_\phi(f) = f \circ \phi \qquad (f \in H^\infty(B)).$$

The collection of such operators with the operator norm topology is denoted by $\mathcal{C}(H^\infty(B))$. This space has been widely studied, and recently, Aron, Galindo and Lindström [1] have determined its path connected components for some special Banach spaces E, thereby extending results of MacCluer, Ohno and Zhao [6] for the case when B is the open unit disc Δ in the complex plane. A main result in [1] is

Theorem 1.1 ([1, Theorem 16])**.** *If $E = C_0(X)$ or E is a Hilbert space, then the composition operators C_ϕ and C_ψ lie in the same path connected component in $\mathcal{C}(H^\infty(B))$ if and only if $\|C_\phi - C_\psi\| < 2$.*

Furthermore, using techniques involving w-strong peak points and determining sets for $H^\infty(B)$ when B belongs to some special Banach spaces, the following result is established.

Theorem 1.2 ([1, Corollary 12])**.** *The identity operator is an isolated point in $\mathcal{C}(H^\infty(B))$ when E is $C_0(X)$ or ℓ_1 or any strictly convex reflexive Banach space.*

Two open questions were raised in [1]. First, does Theorem 1.1 hold when E is a JB^*-triple? A positive answer to this question has been given in [7]. The second conjecture is that Theorem 1.2 holds for every Banach space E. We give a positive

Received by the editors May 7, 2003 and, in revised form, August 12, 2003.
2000 *Mathematics Subject Classification.* Primary 47B38, 46J15, 46G20, 32A10.
Key words and phrases. Composition operator, holomorphic function on Banach space, Carathéodory distance.

answer and a simple proof in this paper. We use only the hyperbolic metric, but do not require w-strong peak points.

2. Proof of the conjecture

The space of all complex-valued homomorphisms on $H^\infty(B)$ forms the maximal ideal space of $H^\infty(B)$ and contains, in particular, the point evaluation functionals $\{\delta_x : x \in B\}$. The pseudo-hyperbolic distance on the maximal ideal space is defined by

$$\beta(m,n) = \sup\{|\hat{f}(n)| : f \in H^\infty(B), \|f\| \le 1, \hat{f}(m) = 0\}$$

where $\hat{f}$ is the Gelfand transform of f. We note from [1, Remark 2] that

$$\|C_\phi - C_\psi\| < 2 \qquad \text{if and only if} \qquad \sup_{x\in B} \beta(\delta_{\phi(x)}, \delta_{\psi(x)}) < 1. \tag{1}$$

The *Carathéodory distance* on B is given by

$$C_B(x,y) = \sup\{\gamma(f(x), f(y)) : f \in H(B,\Delta)\}$$

for x, $y \in B$, where γ is the Poincaré metric on the disc Δ and $H(B,\Delta)$ the space of holomorphic maps from B to Δ. Both C_B and β are contracted by holomorphic functions and preserved by biholomorphic functions.

The metric

$$d_B(x,y) := \sup\{|f(x) - f(y)| : f \in H^\infty(B), \|f\| \le 1\} \qquad (x, y \in B)$$

and its relation to the Carathéodory distance C_B are examined in [7] where it is shown that

$$d_B(x,y) = \frac{2 - 2\sqrt{1 - (\tanh C_B(x,y))^2}}{\tanh C_B(x,y)} \qquad (x, y \in B). \tag{2}$$

We have $d_B(x,y) \ge d_\Delta(h(x), h(y))$ for any $h \in H(B,\Delta)$. Also,

$$d_B(x,y) \le 2\sup\{|f(x)| : f \in H^\infty(B), \|f\| \le 1, f(y) = 0\}. \tag{3}$$

To see this, it suffices to note that for any $f \in H^\infty(B)$ with $\|f\| \le 1$, the function f^y defined by $f^y(x) = \frac{1}{2}(f(x) - f(y))$ is also in the closed unit ball of $H^\infty(B)$.

Let E^* be the dual of a complex Banach space E. We denote the unit spheres of E and E^* by $S(E)$ and $S(E^*)$ respectively. Given $x \in S(E)$, we denote the set of support functionals of x by

$$\operatorname{supp}(x) = \{f \in E^* : \|f\| = f(x) = 1\}.$$

Let $T : E \longrightarrow E$ be a bounded complex linear operator. We recall that the *spatial numerical range* of T is defined by

$$V(T) = \{f(Tx) : x \in S(E), f \in \operatorname{supp}(x)\}$$

(cf. [3]). By [3, Theorem 3.9.4], the numerical radius $v(T)$ of T is given by

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}.$$

We also have, by [3, Theorem 1.4.1],

$$\|T\| \ge v(T) \ge \frac{1}{e}\|T\|. \tag{4}$$

We now prove the conjecture that subsumes Theorem 1.2.

Theorem 2.1. *Let E be a Banach space with open unit ball B. Then the identity operator is an isolated point in the space of composition operators on $H^\infty(B)$.*

Proof. Let $I : H^\infty(B) \longrightarrow H^\infty(B)$ be the identity operator, and suppose that C_ϕ is in the component of I for some holomorphic self-map ϕ of B. We show that ϕ is the identity map on B. We have $\|C_\phi - I\| < 2$ by [5] and $\sup_{x\in B} \beta(\delta_{\phi(x)}, \delta_x) < 1$ by (1). Since

$$\begin{aligned}\beta(\delta_{\phi(x)}, \delta_x) &= \sup\{|\hat{f}(\delta_{\phi(x)})| : f \in H^\infty(B), \|f\| \le 1, \hat{f}(\delta_x) = 0\}\\ &= \sup\{|f(\phi(x))| : f \in H^\infty(B), \|f\| \le 1, f(x) = 0\}\\ &\ge \frac{1}{2} \sup\{|f(\phi(x)) - f(x)| : f \in H^\infty(B), \|f\| \le 1\} \qquad \text{(by (3))}\end{aligned}$$

we see that

$$\sup_{x\in B} d_B(\phi(x), x) < 2$$

and hence (2) gives

$$\sup_{x\in B} C_B(\phi(x), x) < \infty$$

or that

$$\sup_{x\in B, f\in H(B,\Delta)} \gamma(f(x), f(\phi(x))) < \infty.$$

Let $\lambda \in S(E^*)$ be norm-attaining; that is, there exists $x_\lambda \in S(E)$ with $\lambda(x_\lambda) = 1$. Define $\psi : \Delta \to \Delta$ by $\psi(\zeta) = \lambda(\phi(\zeta x_\lambda))$. Then ψ is holomorphic, and we have

$$\begin{aligned}\sup_{\zeta\in\Delta} \gamma(\zeta, \psi(\zeta)) &= \sup_{\zeta\in\Delta} \gamma(\lambda(\zeta x_\lambda), \lambda(\phi(\zeta x_\lambda)))\\ &\le \sup_{x\in B} \gamma(\lambda(x), \lambda(\phi(x)))\\ &\le \sup_{x\in B, f\in H(B,\Delta)} \gamma(f(x), f(\phi(x))) < \infty.\end{aligned}$$

Since $\gamma(\zeta, \psi(\zeta)) = \tanh^{-1} \beta(\zeta, \psi(\zeta))$ on Δ, we have $\sup_\Delta \beta(\zeta, \psi(\zeta)) < 1$ and it follows from the one-dimensional result that $\psi = id_\Delta$. Hence we have

$$\zeta = \lambda(\phi(\zeta x_\lambda)) \qquad (\zeta \in \Delta). \tag{5}$$

In particular, we have $\lambda(\varphi(0)) = 0$. By the Bishop-Phelps theorem [2], the norm-attaining functionals in E^* are norm-dense in E^*. Therefore $\phi(0) = 0$. Let us write (5) in the form

$$id_\Delta = \lambda \circ \phi \circ i_{x_\lambda}$$

where $i_{x_\lambda} : \Delta \to B$ is the map $i_{x_\lambda}(\zeta) = \zeta x_\lambda$. Taking the derivative at $\zeta \in \Delta$ of both sides we obtain

$$1 = \lambda(\phi'(\zeta x_\lambda)(x_\lambda))$$

which gives

$$1 = \lambda(\phi'(0)x_\lambda).$$

The above arguments imply that, for any $x \in S(E)$ and $f \in \operatorname{supp}(x)$, we have $1 = f(\phi'(0)x)$. Let $T = \phi'(0) - I$. We obtain

$$\begin{aligned}V(T) &= \{f(Tx) : x \in S(E), f \in \operatorname{supp}(x)\}\\ &= \{f(\phi'(0)x) - f(x) : x \in S(E), f \in \operatorname{supp}(x)\}\\ &= \{0\}.\end{aligned}$$

It follows from (4) that $\|T\| = 0$. Hence $\phi'(0) = I$. Since we have already established that $\phi(0) = 0$, Cartan's uniqueness theorem asserts that ϕ itself is the identity map on B as required (see [4, Proposition 6.6]). $\square$

Corollary 2.2. *Let E be a Banach space and ψ a biholomorphic self-map of the open unit ball B of E. Then C_ψ is isolated in $\mathcal{C}(H^\infty(B))$.*

Proof. The result is true for $\psi = id$ from above. Now observe that C_ψ is a homeomorphism of $\mathcal{C}(H^\infty(B))$ that takes the identity to the composition operator C_ψ. □

Acknowledgements

The second author wishes to thank the Department of Mathematics of University College Dublin for the hospitality received during his visit there, and the Swiss National Science Foundation for financial support. The third author gratefully acknowledges hospitality received at Queen Mary College, University of London.

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