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Measuring optical phase digitally in coherent metrology systems.

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ABSTRACT

The accurate measurement of optical phase has many applications in metrology. For biological samples, which appear transparent, the phase data provides information about the refractive index of the sample. In speckle metrology, the phase can be used to estimate stress and strains of a rough surface with high sensitivity. In this theoretical manuscript we compare and contrast the properties of two techniques for estimating the phase distribution of a wavefield under the paraxial approximation: (I) A digital holographic system, and (II) An idealized phase retrieval system. Both systems use a CCD or CMOS array to measure the intensities of the wavefields that are reflected from or transmitted through the sample of interest. This introduces a numerical aspect to the problem. For the two systems above we examine how numerical calculations can limit the performance of these systems leading to a near-infinite number of possible solutions.

Keywords: speckle metrology, phase retrieval, holography, statistical optics

1. INTRODUCTION.

Fourier optics is an important branch of optical theory since it allows the development of relatively simple and intuitive models of optical systems. These models provide significant insight into the characteristics of the underlying optical systems and are also reasonably accurate. Furthermore Fourier optics builds important bridges between disciplines in particular allowing optical systems to be interpreted in terms of signal processing operations that are commonly used in electrical engineering, communication theory and control engineering.[1–6] This type of viewpoint can be extended still further with the development of mixed space/time transforms or mixed space/spatial frequency transforms such as the fractional Fourier transform, the Linear Canonical transform and perhaps even more generally in the form of Wigner distribution functions.[7–13]

In this manuscript we assume that a scalar description of light propagation is valid, and use the Fresnel transform to relate an optical field in one plane to that in another plane, where the optical planes are separated from each other by an axial distance z. The light sources used are assumed to be of a definite mono-chromatic temporal frequency and are both temporally and spatially coherent. Optical elements used to shape the form of the illuminating wavefield are assumed to be 'thin' and operate on an incident optical field, multiplying it by a function describing the optical element over a plane.[2]

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In Fig. 1, we present a schematic of an optical system that is capable of performing a digital holographic measurement. And in Fig. 2 how the setup in Fig. 1 can be modified for phase retrieval. In Fig. 2, we see that the two CCD different arrays are axially displaced from each other. By making a series of intensity measurements in different optical planes, a phase retrieval technique to be used to estimate the phase. These two systems have been compared and contrasted in several publications. It can be shown that the resolution (performance) of digital holographic systems are limited by the extent of the CCD array, and the active area of the sampling pixels. The sampling that occurs at the CCD pixels produces a set of replicas' in the reconstruction domain that can overlap with each other distorting the quality of the reconstruction.[14–16] This implies that under certain conditions it is possible to recover spatial frequencies in the reconstruction domain that are higher than the Nyquist sampling rate of the recording CCD array.[16]

In contrast with PR systems, it seems that the Nyquist sampling rate must be obeyed since the spectral method of propagation is usually used in the PR algorithm.[17] We won't need a reference wave however and the optical setup is much simpler. In many situations these are significant advantages. It is generally said that for PR to work effectively one needs to have a good estimate of the initial phase. When that is the case the algorithms tends to converge to the nearest local minimum which in this case would be one of the global minimums. And hence the algorithm will return the real-physical phase distribution and not a solution that merely satisfies the numerical constraints but has no physical meaning. In this manuscript we will explore this question of multiple phase solutions more closely, by examining a simplified version of the phase retrieval problem.

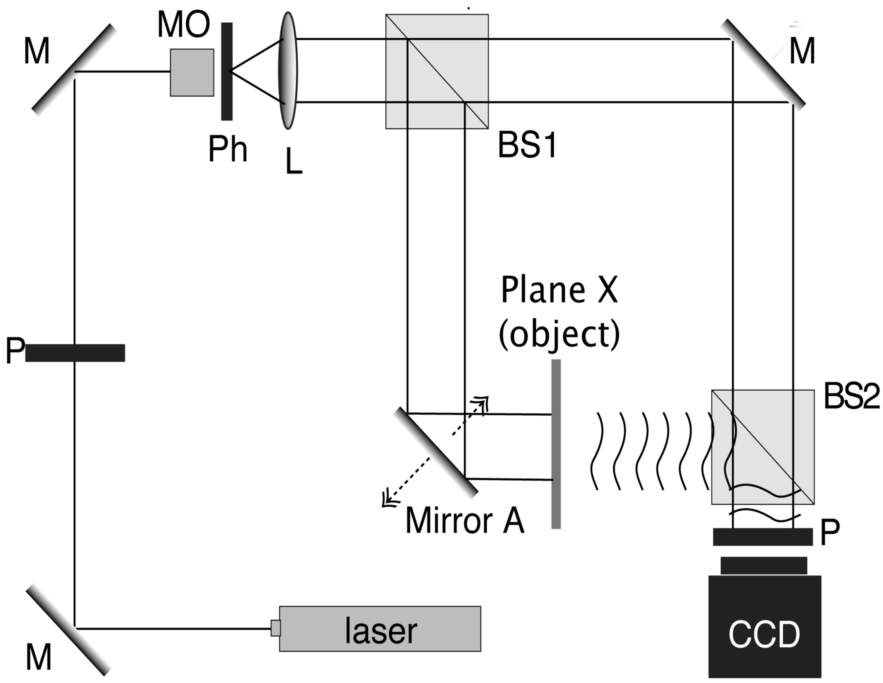

Figure 1. Optical setup for implementing a Fresnel based digital holographic system. MO - microscope objective, BS - beam splitter, M - mirror, P - polarizer, Ph - pinhole.

2. ANALYSIS OF AN IDEALIZED THOUGHT EXPERIMENT

We begin our analysis by examining the idealized optical system depicted in Fig. 2. A set of N distinct point sources, each with a known intensity value and hence known magnitude, a_n, where a_n is the magnitude of point

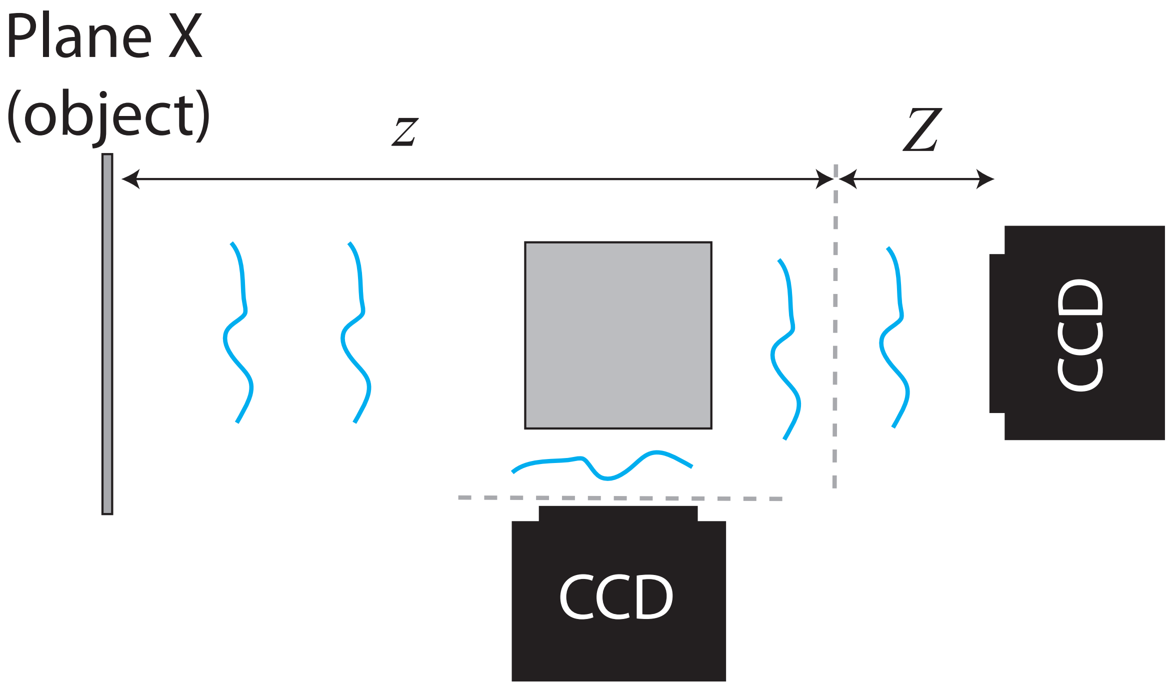

Figure 2. Optical setup for implementing a Fresnel based phase retrieval system, BS is a beam splitter. CCD1 and CCD2 are separated from each other by a distance Z

source n. The phase value, ϕ_n associated with each point source however remains the unknown physical parameter that we wish to measure. So to reiterate the problem, as depicted in Fig. 2 is that we are given a set of N point sources with known amplitudes and unknown phase values. We further assume that these points sources are uniformly spaced a distance Δ from each other. We are also given a second set of intensity measurements, in this case we also record the intensity of the optical field in the Fourier transform plane. As we have noted in the previous section in practical PR measurements two major physical factors play a significant role in limiting the performance of such techniques: the finite extent of the CCD array used to make the intensity measurement and the finite size and number of the CCD pixels which average the light intensity incident upon them.

We have examined such issues in previous chapters for holographic imaging systems, so here we will deliberately take a different approach. We will imagine that we can measure the intensity in the Fourier domain over an infinite extent and with a sampling step size that is arbitrarily high. We will also imagine that only a finite number of perfect point sources in the input plane (with known intensity but unknown phase) contribute to the Fourier plane intensity distribution. Hence in this analysis we exclude any physical imposed limiting factors on the ability of our system to make a measurement. This will allow us to significantly simplify the analysis and to concentrate an another and arguable more important effect: the existence of a very high number of alternative solutions to the PR problem. As we shall see it is possible that many different combinations of phase values can produce identical intensity distributions. Hence this PR problem is said to be ill-posed. We refer back to Fig. 2, where we have indicated that the contributing point sources are Fourier transformed by an imaginary optical Fourier transform system, where $\lambda f = 1$ and hence the Fourier distribution is unscaled. Here this distribution is incident on a material and its intensity distribution is recorded. Later we will perform a unscaled second optical Fourier transform on this intensity distribution and examine its Fourier transform. We note however that although we are performing two successive forward Fourier transforms, we do not arrive back at the input plane distribution since, we perform the second Fourier transform operation on the intensity of the of the Fourier distribution not on its complex amplitude, as in a 4-f imaging system. We thus begin by noting that the Fourier

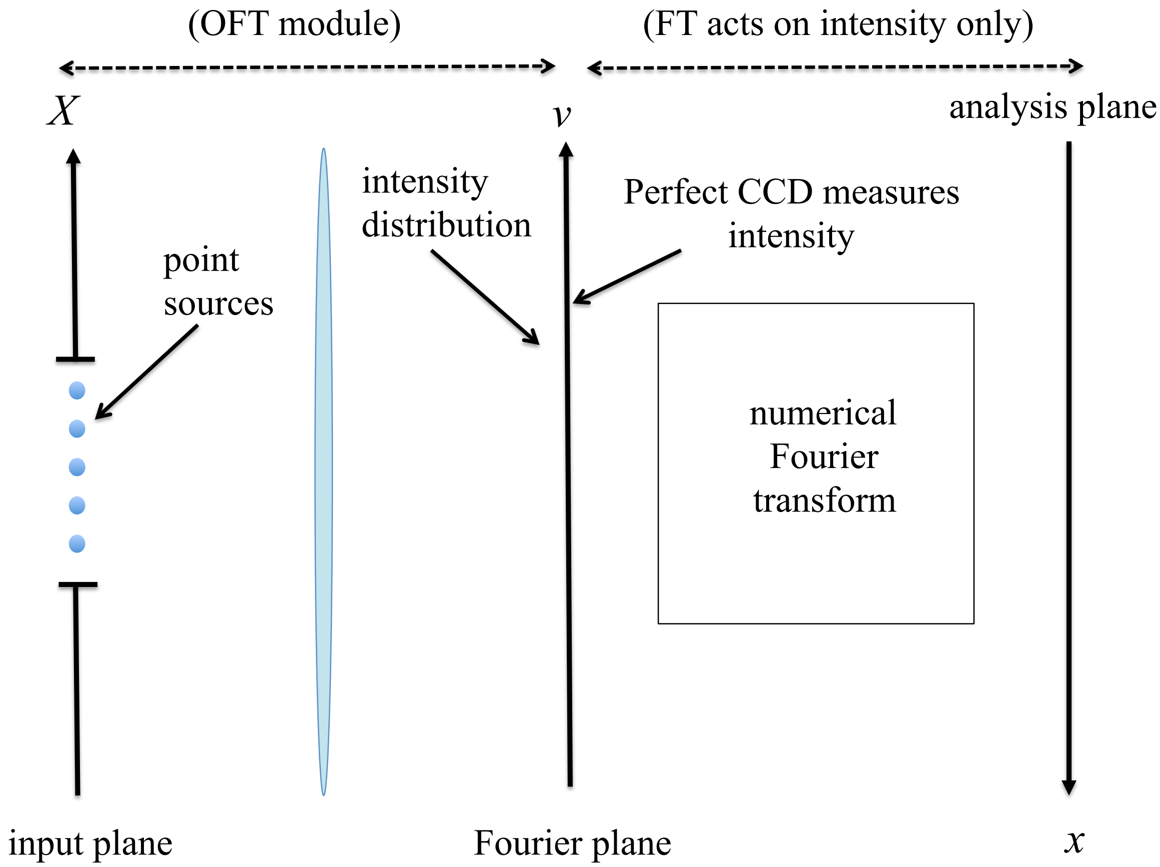

Figure 3. Depiction of PR setup for an idealized ‘Gedanken’ experiment. A set of N point sources with known amplitude and unknown phase are Fourier transformed and the resulting intensity distribution appears in the Fourier plane. This intensity distribution is processed to determine the phase values of the N point sources in its Fourier domain - hence in this ‘Gedanken’ experiment the second OFT (Optical Fourier Transform) module acts on the intensity values only.

transform of a point source is given by

$$FT\left\{\delta\left(x+X_{n}\right)a_{n}\exp\left(j\phi_{n}\right)\right\}(v)=\exp\left(-j2\pi X_{n}v\right)a_{n}\exp\left(j\phi_{n}\right), \tag{1}$$

where X_n is the spatial location of point source 'n' while the parameters, a_n and ϕ_n, refer to its phase and amplitude respectively. Hence for a sum of N distinct point sources we can write

$$FT\left\{\sum_{n=1}^{N}\delta\left(x-X_{n}\right)a_{n}\exp\left(j\phi_{n}\right)\right\}(v)=\sum_{n=1}^{N}\exp\left(-j2\pi X_{n}v\right)a_{n}\exp\left(j\phi_{n}\right). \tag{2}$$

We are however interested in the intensity distribution of the field in the Fourier plane. We remember that the intensity of a complex number is given by that number times its complex conjugate. With this relationship in mind we now write the conjugate expression for Eq. (2)

$$FT\left\{\sum_{n=1}^{N}\delta\left(x-X_{n}\right)a_{n}\exp\left(j\phi_{n}\right)\right\}^{*}(v)=\sum_{n=1}^{N}\exp\left(j2\pi X_{n}v\right)a_{n}\exp\left(-j\phi_{n}\right), \tag{3}$$

where '*' refers to a complex conjugate operation. We can now write an analytical expression for the intensity distribution in the Fourier plane which when simplified can be written in the following form

$$I_{\mathrm{FT}}(v) = \sum_{n=1}^{N}a_{n}^{2}+2\sum_{n=1}^{N-1}\sum_{m=1}^{N-n}a_{m}a_{m+n}\left\{\cos\left[2\pi\left(n\Delta\right)v+\phi_{m}-\phi_{m+n}\right]\right\}. \tag{4}$$

From Eq. (4) we can see that we have a sum of cosines. For each given value of n there is a specific spatial frequency component $f_s = n\Delta$, and there are $N-1$ terms. It is easier to analyze this double summation by initially concentrating on a single spatial frequency component, i.e. for a specific value of n, and we refer to such a component as $I^{n}_{\mathrm{FT}}(v)$. Noting the following trigonometric relationship: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, we can express $I_{\mathrm{FT}}(v)$ as

$$I_{\mathrm{FT}}(v) = \sum_{n=1}^{N}a_{n}^{2}+\sum_{n=1}^{N-1}I^{n}_{\mathrm{FT}}(v), \tag{5}$$

where

$$\begin{aligned}I^{n}_{\mathrm{FT}}(v) &= 2\sum_{m=1}^{N-n}a_{m}a_{m+n}\cos\left[2\pi n\Delta v+\left(\phi_{m}-\phi_{m+n}\right)\right]\\ &= 2\cos\left(2\pi n\Delta v\right)\sum_{m=1}^{N-n}a_{m}a_{m+n}\cos\left(\phi_{m}-\phi_{m+n}\right)\\ &\quad +2\sin\left(2\pi n\Delta v\right)\sum_{m=1}^{N-n}a_{m}a_{m+n}\sin\left(\phi_{m}-\phi_{m+n}\right).\end{aligned} \tag{6}$$

We now consider the inverse Fourier transform of $I^{n}_{\mathrm{FT}}(v)$, which we specifically define

$$\overline{I}_{\mathrm{FT}}(x) = \int_{-\infty}^{\infty}I^{n}_{\mathrm{FT}}(v)\exp\left(+j2\pi xv\right)dv, \tag{7}$$

where are now in a spatial domain once again. We note that in the idealized optical system, see Fig. 3, two forward optical Fourier transforms [defined with a kernel $\exp(-j2\pi vx)$] are performed, whereas in Eq. (7), we have defined an inverse Fourier transform operation, [with a kernel $\exp(+j2\pi vx)$]. We account for the sign change by flipping the direction of the axis in the spatial domain, see Fig. 6.1. Noting that

$$\begin{aligned}FT\left\{\sin\left(2\pi k_{0}x\right)\right\}(k) &= \frac{j}{2}\left[\delta(k+k_{0})-\delta(k-k_{0})\right],\\ FT\left\{\cos\left(2\pi k_{0}x\right)\right\}(k) &= \frac{1}{2}\left[\delta(k+k_{0})+\delta(k-k_{0})\right],\end{aligned}$$

then

$$
\begin{aligned}
I_{\mathrm{FT}}^{n}(v) &= \delta(x+n\Delta)\left\{\sum_{m=1}^{N-n} a_m a_{m+n}\cos(\phi_m-\phi_{m+n}) - j\sum_{m=1}^{N-n} a_m a_{m+n}\sin(\phi_m-\phi_{m+n})\right\} \\
&+ \delta(x-n\Delta)\left\{\sum_{m=1}^{N-n} a_m a_{m+n}\cos(\phi_m-\phi_{m+n}) + j\sum_{m=1}^{N-n} a_m a_{m+n}\sin(\phi_m-\phi_{m+n})\right\}.
\end{aligned} \tag{8}
$$

We see from Eq. (8) that the complex numbers (arising from the summation of sines and cosines) multiplying the $\delta(x-n\Delta)$ and $\delta(x+n\Delta)$ components are complex conjugates of each other. Hence we see that

$$
Re\left\{I_{\mathrm{FT}}^{n}(n\Delta)\right\} = \sum_{m=1}^{N-n} a_m a_{m+n}\cos(\phi_m-\phi_{m+n}), \tag{9}
$$

and that

$$
Im\left\{I_{\mathrm{FT}}^{n}(n\Delta)\right\} = -\sum_{m=1}^{N-n} a_m a_{m+n}\sin(\phi_m-\phi_{m+n}). \tag{10}
$$

We also remember that the iterator n above spans the range: $1 \leq n \leq N-1$. Examining Eq. (9) and Eq. (10) we recognize that the only unknowns are the values for the phase parameters, ϕ_n. Both the initial amplitudes a_n are given and the LHS of each of equation is found by calculating the Fourier transform of the intensity distribution that we measure in the Fourier plane. We also note that the larger the value of n is, the lower the number of terms from Eq. (9) and Eq. (10) that contribute to both $Im\left\{I_{\mathrm{FT}}^{n}(n\Delta)\right\}$ and $Re\left\{I_{\mathrm{FT}}^{n}(n\Delta)\right\}$. When we set about trying to find ϕ_n from Eq. (9) and Eq. (10), we should start by first setting $n = N-1$, in which case the equations reduce to the following:

$$
Re\left\{I_{\mathrm{FT}}^{N-1}\left[(N-1)\Delta\right]\right\} = a_1 a_N \cos(\phi_1-\phi_N), \tag{11}
$$

and that

$$
Im\left\{I_{\mathrm{FT}}^{N-1}\left[(N-1)\Delta\right]\right\} = -a_1 a_N \sin(\phi_1-\phi_N). \tag{12}
$$

If we arbitrarily set $\phi_1 = 0$, then we can calculate ϕ_N from the following

$$
\phi_N = \arctan\left[\frac{\sin(\phi_N)}{\cos(\phi_N)}\right], \tag{13}
$$

which means that we now have values for both ϕ_1 and ϕ_N. Now setting $n = N-2$, Eq. (9) and Eq. (10) will reduce to the following:

$$
Re\left\{I_{\mathrm{FT}}^{N-2}\left[(N-2)\Delta\right]\right\} = a_1 a_{N-1}\cos(\phi_1-\phi_{N-1}) + a_2 a_N \cos(\phi_2-\phi_N), \tag{14}
$$

and that

$$
Im\left\{I_{\mathrm{FT}}^{N-2}\left[(N-1)\Delta\right]\right\} = -a_1 a_{N-1}\sin(\phi_1-\phi_{N-1}) - a_2 a_N \sin(\phi_2-\phi_N). \tag{15}
$$

This is where the difficulties with PR begin in earnest. This pair of coupled equations has eight different solution for (ϕ_2, ϕ_{N-2}), which are all valid. However only one of the solution pairs corresponds to the actual correct physical result, another solution is its complex conjugate. If we randomly choose from these solutions we will have a probability of 1/8 that we have chosen the physically correct answer.

Having decided on a particular choice of (ϕ_2, ϕ_{N-2}), we can then set $n = N-3$ and repeat the process. We have found from analysis that each step leads to equations that have the form of Eq. (14) and Eq. (15), with two unknown values and each with a set of eight solution pairs. Hence at each step we have a $1/8$ chance of guessing the physically correct solution, and for N point sources we will need to make approximately $N/2$ guesses at each step of the algorithm. Hence the chances that we correctly guess the correct phase for each point source is $(1/8)^{N/2}$ - a vanishingly small probability. In the experimental work described at the beginning of this chapter $N \approx 4$ million. Hence there are very fundamental limits on our ability to recover the physically correct phase values from intensity measurements made in an imaging and Fourier plane.

3. CONCLUSION

In iterative PR algorithms an initial guess is made as to the correct phase solution. The iterative algorithm is run which behaves like a gradient descent optimization process. For a given set of intensity measurements these algorithms do indeed find a set of phase values that minimize the error between the measured intensity values and the calculated intensities. However since there is such a large number of possible solutions to this problem, these optimized phase values are very unlikely to be the physically correct solution.

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