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COMPLETELY BOUNDED NORMS OF RIGHT MODULE MAPS

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ABSTRACT. It is well-known that if T is a D_m–D_n bimodule map on the $m \times n$ complex matrices, then T is a Schur multiplier and $\|T\|_{cb} = \|T\|$. If $n = 2$ and T is merely assumed to be a right D_2-module map, then we show that $\|T\|_{cb} = \|T\|$. However, this property fails if $m \geq 2$ and $n \geq 3$. For $m \geq 2$ and $n = 3, 4$ or $n \geq m^2$ we give examples of maps T attaining the supremum

$$C(m, n) = \sup\{\|T\|_{cb} \colon T \text{ a right } D_n\text{-module map on } M_{m,n} \text{ with } \|T\| \leq 1\},$$

we show that $C(m, m^2) = \sqrt{m}$ and succeed in finding sharp results for $C(m, n)$ in certain other cases. As a consequence, if H is an infinite-dimensional Hilbert space and D is a masa in $B(H)$, then there is a bounded right D-module map on $\mathcal{K}(H)$ which is not completely bounded.

Keywords: completely bounded, right module map, matrix numerical range, tracial geometric mean, fidelity

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1. Introduction

Let H be a Hilbert space, let $\mathcal{B}(H)$ be the algebra of bounded linear operators on H, let $\mathcal{K}(H)$ be the ideal of compact operators and let D be a masa in $\mathcal{B}(H)$. If $T \colon \mathcal{K}(H) \to \mathcal{K}(H)$ is a bounded D-bimodule map, then it is well-known that that $\|T\|_{cb} = \|T\|$ (see [11, 8, 9]). While it would certainly be of use to be able to extend this to larger natural classes than D-bimodule maps (generalised Schur multipliers), in the present paper, we consider the effect of relaxing the hypothesis of bimodularity to one-sided modularity over D. While we establish a positive result for dimension 2, we give increasing bounds for higher finite dimensions and a negative answer for the following question [4, Remark 7.10]:

Question 1.1. If H is infinite-dimensional and D is a masa in $\mathcal{B}(H)$, is there a constant $C > 0$ such that $\|T\|_{cb} \leq C\|T\|$ for every bounded, left D-module map $T \colon \mathcal{K}(H) \to \mathcal{K}(H)$?

By symmetry, this question is unchanged if we replace "left" by "right", and this makes our notation marginally neater. So we will focus on right D-module maps.

Of course, if H is finite dimensional, then the answer to this question is yes even if we discard the modularity condition. It then becomes interesting to estimate the optimal constant C. Hence we are led to consider the constants

$$C(m, n) = \sup\{\|T\|_{cb} \colon T \text{ is a right } D_n\text{-module map on } M_{m,n},\ \|T\| \leq 1\}$$

where $M_{m,n}$ is the space of $m \times n$ complex matrices and D_n is the algebra of diagonal $n \times n$ matrices.

The structure of the paper is as follows. We first establish some notation and give some preliminary results in Section 2. In Section 3 we use the second author's work on elementary operators to show that $C(m, 2) = 1$ for every $m \geq 1$. Section 4 contains some technical results comparing the completely bounded norm to the norm arising from the Hilbert-Schmidt norm, and these are used in Section 5 to find some upper bounds for $C(m, n)$. In the next section we construct examples which show that $C(m,n)$ grows with m,n. This leads naturally to a counterexample (in Corollary 6.12) answering Question 1.1, and we are also able to determine the values of $C(m,n)$ in some cases. Finally, in Section 7 we briefly consider similar problems when we restrict attention to special classes of right module maps.

In the last two sections, we pose several unresolved questions about the behaviour of the constants $C(m,n)$.

2. Preliminaries

If X is a vector space, we write $\mathcal{L}(X)$ for the space of linear maps $X \to X$. If $m,n \in \mathbb{N}$, then $M_{m,n}(X)$ is the vector space of $m \times n$ matrices with entries in X. We will write elements of $M_{m,n}(X)$ as $[x_{ij}]_{1\le i\le m,\,1\le j\le n}$ or simply $[x_{ij}]$, where each x_{ij} is in X. If $T \in \mathcal{L}(X)$ and $m,n \in \mathbb{N}$, then the (m,n)-ampliation of T is the map $T_{m,n} \in \mathcal{L}(M_{m,n}(X))$ given by $T_{m,n}[x_{ij}] = [Tx_{ij}]$. We also write $T_n = T_{n,n}$.

Given a norm $\|\cdot\|$ on X, the corresponding operator norm, or simply the norm, of a map $T \in \mathcal{L}(X)$ is

$$\|T\| = \sup\{\|Tx\| \colon x \in X,\ \|x\| \le 1\}.$$

If we are given norms on $M_{m,n}(X)$ for all $m,n \in \mathbb{N}$, then the completely bounded norm of T is

$$\|T\|_{cb} = \sup_{m,n\ge 1} \|T_{m,n}\|.$$

Provided the inclusions of $M_{m,n}(X)$ into $M_{m+1,n}(X)$ and $M_{m,n+1}(X)$ which pad a matrix with an extra row or column of zeros are isometries, we have

$$\|T\| = \|T_1\| \le \|T_2\| \le \|T_3\| \le \cdots \le \|T\|_{cb} = \sup_{n\ge 1} \|T_n\|.$$

For $n \in \mathbb{N}$ we let $\mathbb{C}^n$ denote the Hilbert space of dimension n whose elements are to be thought of as column vectors with n complex entries, with the ℓ^2 norm, and we will also write $\mathbb{C}^\infty$ for $\ell^2(\mathbb{N})$. For $m,n \in \mathbb{N}\cup\{\infty\}$, we write

$$M_{m,n} = \mathcal{B}(\mathbb{C}^n, \mathbb{C}^m) = \{x \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) \colon \|x\| < \infty\}$$

and $M_m = M_{m,m}$. If $s,t \in \mathbb{N}$, then $M_{s,t}(M_{m,n})$ can be naturally identified with the normed vector space $M_{sm,tn}$, and hence inherits the norm from the latter space. Adding a row or column of zeros is then an isometry.

If $v,w \in \mathbb{C}^n$, then vw^* denotes the rank one operator in M_n given by

$$vw^*(x) = \langle x, w\rangle v \quad \text{for } x \in \mathbb{C}^n.$$

For $1 \le i \le n$ (or for $i \ge 1$, if $n = \infty$) we write e_i for the ith standard basis vector in $\mathbb{C}^n$. Then D_n, the diagonal masa of M_n, is the von Neumann algebra generated by the diagonal matrix units $e_ie_i^*$.

Let $n \in \mathbb{N}\cup\{\infty\}$, let $b_1,\dots,b_\ell \in M_n$ and let $b = \begin{bmatrix} b_1 \\ \vdots \\ b_\ell \end{bmatrix}$. For $\xi \in \mathbb{C}^n$, let $Q(b,\xi)$ be the positive semi-definite $\ell \times \ell$ matrix

$$Q(b,\xi) = [\langle b_i\xi, b_j\xi\rangle]_{1\le i,j\le \ell}.$$

We recall the definitions from [12] of *the matrix numerical range of* b,

$$W_{\mathrm{m}}(b) = \{Q(b,\xi) \colon \xi \in \mathbb{C}^n,\ \|\xi\| = 1\}$$

and *the matrix extremal numerical range of* b,

$$W_{\mathrm{m,e}}(b) = \{\beta \in \overline{W_{\mathrm{m}}(b)} \colon \operatorname{trace}(\beta) = \|b\|^2\},$$

(where the norm $\|b\|$ is computed with respect to the norm on $M_{\ell,1}(M_n)$ described above). It is easy to see that $W_{\mathrm{m,e}}(b)$ is the set of elements of the closure of $W_{\mathrm{m}}(b)$

of maximal trace. If $n < \infty$ then $W_{\rm m}(b)$ is a continuous image of the unit sphere of $\mathbb{C}^n$, which is compact. Hence in this case,

$$W_{\rm m,e}(b) = \{Q(b,\xi)\colon \xi \in \mathbb{C}^n,\ \|\xi\| = 1,\ b^*b\xi = \|b\|^2\xi\}.$$

Observe that the vectors ξ appearing in this expression are precisely the unit vectors in the eigenspace of b^*b corresponding to its maximal eigenvalue.

If $a = [a_1\ \dots\ a_\ell]$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_\ell \end{bmatrix}$ for some $a_j \in \mathcal{L}(X)$ and $b_j \in \mathcal{L}(Y)$, then we will write $T = a \odot b$ or say that "a, b represent T" to mean that T is the elementary operator

$$T\colon \mathcal{L}(Y,X) \to \mathcal{L}(Y,X), \quad x \mapsto \sum_{j=1}^{\ell} a_j x b_j.$$

Such a representation of T is far from unique due to bilinearity in (a, b); for example, if $T = a \odot b$, then we also have $T = (a\alpha^{-1}) \odot (\alpha b)$ for any invertible matrix $\alpha \in M_\ell$.

If D is a subring of M_n then $M_{m,n}$ is a right D-module. A *right D-module map* on $M_{m,n}$ is a linear map $T \in \mathcal{L}(M_{m,n})$ such that

$$T(xd) = T(x)d \quad \text{for all } x \in M_{m,n} \text{ and all } d \in D.$$

We write $\mathcal{L}_D(M_{m,n})$ for the set of all right D-module maps on $M_{m,n}$.

Remark 2.1. If $n \in \mathbb{N}$ and T is a bounded right D_n-module map on $M_{m,n}$, then T is an elementary operator of the form $Tx = \sum_{j=1}^n a_j x b_j$ for some $b_j \in D_n$ and $a_j \in M_m$. Indeed, for each j, the map $v \mapsto T(ve_j^*)e_j$ is linear $\mathbb{C}^m \to \mathbb{C}^m$, and it is bounded since T is bounded. Hence there is an operator $a_j \in M_m$ such that $a_j v = T(ve_j^*)e_j$ for $v \in \mathbb{C}^m$. We call the operators a_j the *column operators* of T. Writing $b_j = e_j e_j^*$, we have

$$\sum_{j=1}^n a_j x b_j = \sum_{j=1}^n a_j x e_j e_j^* = \sum_{j=1}^n T(xe_j e_j^*)e_j e_j^* = \sum_{j=1}^n T(x)e_j e_j^* = T(x).$$

We have found a representation $T = a' \odot b'$ where $a' = [a_1\ \dots a_n]$ and $b' = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, and each b_j is diagonal. As discussed in [12, §3], there is a representation $T = a \odot b$ where the entries of a and b are linear combinations of the entries of a' and b', respectively, so that

$$\|T\|_{cb} = \|a\|\,\|b\| = \tfrac{1}{2}(\|a\|^2 + \|b\|^2).$$

Observe that the entries of b are then diagonal, and $\|a\| = \|b\|$ by the arithmetic mean/geometric mean inequality. In [12, Theorem 3.3], the second author shows that a representation $T = a \odot b$ satisfies these equalities if and only if

$$(\bigstar) \qquad \operatorname{conv} W_{\rm m,e}(a^*) \cap \operatorname{conv} W_{\rm m,e}(b) \neq \emptyset$$

where $\operatorname{conv} S$ denotes the convex hull of a subset S of a vector space.

If $n = \infty$, so that T is a bounded right D_∞-module map on $\mathcal{B}(H, \mathbb{C}^m)$ where $H = \ell^2(\mathbb{N})$, then the same argument gives $Tx = \sum_{j=1}^\infty a_j x b_j$ where the operators $a_j \in M_m$ are given by $a_j v = T(ve_j^*)e_j$ and $b_j = e_j e_j^* \in \mathcal{B}(H)$, and the series converges in the strong operator topology.

The relevance of the following lemma to our problem is plain in light of Remark 2.1, and condition $(\bigstar)$ in particular.

Lemma 2.2. *Let $n \in \mathbb{N}$, let $\ell \in \mathbb{N}$ and let $b_1, \dots, b_\ell \in D_n$. If $b = \begin{bmatrix} b_1 \\ \vdots \\ b_\ell \end{bmatrix}$, then*

$$W_{\mathrm{m,e}}(b) = \operatorname{conv}\{Q(b, e_p) \colon 1 \le p \le n,\ b^* b e_p = \|b\|^2 e_p\}.$$

In particular, $W_{\mathrm{m,e}}(b)$ is convex.

Proof. The matrix $b^*b = \sum_{j=1}^{\ell} b_j^* b_j$ is positive semi-definite and diagonal with largest eigenvalue $\|b\|^2$. Let r be the dimension of the corresponding eigenspace. Permuting $e_1, \dots, e_n$ if necessary, we have $b^*b = \|b\|^2(I_r \oplus d)$ for some positive semi-definite $d \in D_{n-r}$ with $\|d\| < 1$. So $Q(b, \xi) \in W_{\mathrm{m,e}}(b)$ if and only if $\xi = \sum_{p=1}^{r} \xi_p e_p$ for some $\xi_p \in \mathbb{C}$ such that $\sum_{p=1}^{r} |\xi_p|^2 = 1$. Each b_j is diagonal so the vectors e_q are eigenvectors, hence

$$\begin{aligned} Q(b, \xi) &= \Big[\sum_{p,q=1}^{r} \langle b_i \xi_p e_p, b_j \xi_q e_q \rangle \Big] \\ &= \Big[\sum_{p=1}^{r} \langle b_i \xi_p e_p, b_j \xi_p e_p \rangle \Big] \\ &= \sum_{p=1}^{r} |\xi_p|^2 Q(b, e_p). \end{aligned}$$

□

The following argument is essentially contained in any of [8, 9, 11].

Lemma 2.3. *Let H, K be Hilbert spaces, let X be a subspace of $\mathcal{B}(H, K)$, and let $A \subseteq \mathcal{B}(H)$ be a right norming set for X, meaning that $xa \in X$ for all $x \in X$ and $a \in A$, and for every $n \ge 1$ and every $z \in M_n(X)$, we have*

$$\|z\|_{M_n(X)} = \sup\{\|zb\|_{M_{n,1}(X)} \colon b \in M_{n,1}(A),\ \|b\| \le 1\}.$$

If $T \colon X \to X$ is a bounded, linear map such that $T(xa) = T(x)a$ for all $x \in X$, $a \in A$, then $\|T_n\| = \|T_{n,1}\|$ for all $n \ge 1$.

Proof. The inequality $\|T_{n,1}\| \le \|T_n\|$ is clear. On the other hand, if $z \in M_n(X)$ and $b \in M_{n,1}(A)$, then

$$T_n(z)b = \Big[\sum_j T(z_{ij}) b_j \Big]_i = \Big[\sum_j T(z_{ij} b_j) \Big]_i = T_{n,1}(zb)$$

and $\|zb\| \le \|z\|_{M_n(X)} \|b\|_{M_{n,1}(A)}$. Since A is a right norming set for X,

$$\begin{aligned} \|T_n\| &= \sup\{\|T_n(z)\|_{M_n(X)} \colon z \in M_n(X),\ \|z\| \le 1\} \\ &= \sup\{\|T_{n,1}(zb)\|_{M_{n,1}(X)} \colon b \in M_{n,1}(A),\ \|b\| \le 1,\ z \in M_n(X),\ \|z\| \le 1\} \\ &\le \|T_{n,1}\|. \end{aligned}$$

□

As shown in [8, 11], the set D_n of diagonal matrices in M_n is a right norming set for $M_{m,n}$. Thus we immediately obtain:

Proposition 2.4. *If $m, n \in \mathbb{N} \cup \{\infty\}$ and T is a right D_n-module map on $M_{m,n}$, then $\|T\|_{cb} = \sup_{k \ge 1} \|T_{k,1}\|$.* □

Remark 2.5. If $m = 1$ or $n = 1$ (that is, if the matrices on which our maps act have either one row, or one column) then $\|T\|_{cb} = \|T\|$ for every $T \in \mathcal{L}(M_{m,n})$. For if $n = 1$, then $M_{m,n} = \mathbb{C}^m$, and every linear map $T \colon \mathbb{C}^m \to \mathbb{C}^m$ may be written as $Tx = ax$ for $a \in M_m$. Hence $\|T\| = \|a\|$. Moreover $T_k \colon M_k(\mathbb{C}^m) \to M_k(\mathbb{C}^m)$ is given by left multiplication by a block diagonal matrix $a^{(k)}$ (with k copies of a on the diagonal), and $\|T_k\| = \|a^{(k)}\| = \|a\| = \|T\|$, so $\|T\|_{cb} = \|T\|$. If $m = 1$, we can apply a similar argument with right multiplication or use the previous case on the map $T^* \colon M_{n,m} \to M_{n,m}$ given by $T^*(x) = T(x^*)^*$.

3. Two columns

We now show that, surprisingly, the conclusion $\|T\|_{cb} = \|T\|$ of Remark 2.5 persists for right D_2-module maps on $M_{m,2}$.

Lemma 3.1. *If X is a set of positive semi-definite 2×2 matrices with trace 1 and there is a rank one projection $p \in \operatorname{conv} X$, then $p \in X$.*

Proof. Conjugating by a suitable unitary matrix, we may assume that $p = e_1e_1^*$. Now p is a convex combination of some $\alpha_1,\dots,\alpha_k \in X$ and each α_j is positive semi-definite. Since the $(2,2)$ entry of p is zero, the $(2,2)$ entry of each α_j is zero, which implies that the off-diagonal entries of each α_j are also zero. Since $\operatorname{trace}\alpha_j = 1$, we have $\alpha_j = p$ for all j. □

Theorem 3.2. *If $m \in \mathbb{N}$ and $T\colon M_{m,2} \to M_{m,2}$ is a right D_2-module map, then $\|T\|_{cb} = \|T\|$.*

Proof. Suppose $\|T\|_{cb} = 1$. By Remark 2.1, $Tx = a_1xb_1 + a_2xb_2$ for some $a_1, a_2 \in M_m$ and $b_1, b_2 \in D_2$ such that $\|a\| = \|b\| = 1$ where $a = [a_1\ a_2]$ and $b = \left[\begin{smallmatrix} b_1 \\ b_2\end{smallmatrix}\right]$. By Lemma 2.2, $W_{\mathrm{m,e}}(b)$ is convex, so by [12, Theorem 3.3], $W_{\mathrm{m,e}}(b)$ intersects the convex hull of $W_{\mathrm{m,e}}(a^*)$. By [12, Proposition 3.1], it suffices to show that $W_{\mathrm{m,e}}(a^*) \cap W_{\mathrm{m,e}}(b) \neq \emptyset$.

Observe that $b^*b = b_1^*b_1 + b_2^*b_2$ is a 2×2 diagonal positive semi-definite matrix of norm 1, so its 1-eigenspace $E_1(b^*b)$ has dimension 1 or 2.

If $\dim E_1(b^*b) = 1$, then $b^*b = \left[\begin{smallmatrix} 1 & 0 \\ 0 & t\end{smallmatrix}\right]$ or $b^*b = \left[\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right]$ for some $t \in [0,1)$. If $b^*b = \left[\begin{smallmatrix} 1 & 0 \\ 0 & t\end{smallmatrix}\right]$, then

$$W_{\mathrm{m,e}}(b) = \{Q(b, ze_1)\colon z \in \mathbb{T}\} = \{Q(b, e_1)\} = \{e_1e_1^*\}.$$

Since $e_1e_1^*$ is a rank one projection in $\operatorname{conv} W_{\mathrm{m,e}}(a^*)$, we have $e_1e_1^* \in W_{\mathrm{m,e}}(a^*)$ by Lemma 3.1. Hence $W_{\mathrm{m,e}}(a^*) \cap W_{\mathrm{m,e}}(b) \neq \emptyset$. Similarly, if $b^*b = \left[\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right]$ then $e_2e_2^* \in W_{\mathrm{m,e}}(a^*) \cap W_{\mathrm{m,e}}(b) \neq \emptyset$.

Now suppose that $\dim E_1(b^*b) = 2$. Then $b^*b = I_2$, so if we write $\beta_i = Q(b, e_i)$ for $i = 1,2$, then Lemma 2.2 shows that $W_{\mathrm{m,e}}(b) = \operatorname{conv}\{\beta_1, \beta_2\}$. For $i = 1,2$, let us write $b_i = \left[\begin{smallmatrix} b_{i1} & 0 \\ 0 & b_{i2}\end{smallmatrix}\right]$ and let $v_i = \left[\begin{smallmatrix} b_{1i} \\ b_{2i}\end{smallmatrix}\right]$. A simple calculation reveals that $\|v_i\| = 1$ and $\beta_i = v_iv_i^*$. If $\beta_1 = \beta_2$, then this rank one projection is in $W_{\mathrm{m,e}}(a^*)$ by Lemma 3.1, so $W_{\mathrm{m,e}}(a^*) \cap W_{\mathrm{m,e}}(b) \neq \emptyset$. So we may assume that $\beta_1 \neq \beta_2$, so that $W_{\mathrm{m,e}}(b)$ is the proper closed line segment joining β_1 and β_2.

For $t \in \mathbb{R}$, let $\beta(t) = t\beta_1 + (1-t)\beta_2$ and consider the closed convex set

$$S = \{t \in \mathbb{R}\colon \beta(t) \in \operatorname{conv} W_{\mathrm{m,e}}(a^*)\}.$$

Now β_1 and β_2 are distinct and $\|\beta_i\|_2 = 1$, where $\|\cdot\|_2$ is the Hilbert-Schmidt norm on M_2. Moreover, $(M_2, \|\cdot\|_2)$ is strictly convex, and its closed unit ball contains $W_{\mathrm{m,e}}(a^*)$ since the trace-class norm of every matrix in $W_{\mathrm{m,e}}(a^*)$ is 1, which dominates its Hilbert-Schmidt norm. Hence $S \subseteq [0,1]$, say $S = [s_1, s_2]$ where $0 \le s_1 \le s_2 \le 1$, and $\beta(s_1)$ and $\beta(s_2)$ are in the boundary of $\operatorname{conv} W_{\mathrm{m,e}}(a^*)$, and are the extreme points of $\operatorname{conv} W_{\mathrm{m,e}}(a^*) \cap W_{\mathrm{m,e}}(b)$.

Given a hermitian 2×2 matrix α with trace 1, say $\alpha = \left[\begin{smallmatrix} a & b \\ \bar{b} & 1-a\end{smallmatrix}\right]$, let us write

$$\theta(\alpha) = (a, \operatorname{Re} b, \operatorname{Im} b) \in \mathbb{R}^3.$$

Observe that the map θ defined on this convex set of matrices is injective and respects convex combinations. Consider

$$e = \theta(\beta(s_1)), \quad L = \theta(W_{\mathrm{m,e}}(b)), \quad W = \theta(W_{\mathrm{m,e}}(a^*)), \quad C = \operatorname{conv} W.$$

By construction, e is an extreme point of $C \cap L$ which lies in the boundary of C. Let Π be a supporting hyperplane for C through e, so that

$$e \in \Pi = \{x \in \mathbb{R}^3 \colon \langle x, \eta\rangle = r\}$$

for some non-zero vector $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ and some $r \in \mathbb{R}$, chosen so that

$$C \subseteq \Pi^+ = \{x \in \mathbb{R}^3 \colon \langle x, \eta\rangle \geq r\}.$$

Since $e \in C = \operatorname{conv} W$ and $e \in \Pi$ we have $e \in \operatorname{conv}(\Pi \cap W)$; for otherwise, e would be a proper convex combination of points in W involving at least one $x \in W$ with $\langle x, \eta\rangle > r$, hence $\langle e, \eta\rangle > r$ so $e \notin \Pi$, a contradiction.

We have

$$W = \{\theta(Q(a^*, \xi)) \colon \xi \in E_1(aa^*),\ \|\xi\| = 1\}.$$

Since $e \in \operatorname{conv}(\Pi \cap W)$ and Π is an affine 2-dimensional space, Carathéodory's theorem [1] shows that $e \in \operatorname{conv}\{w_1, w_2, w_3\}$ for some $w_1, w_2, w_3 \in \Pi \cap W$. Choose unit vectors ξ_1, ξ_2, ξ_3 in $E_1(aa^*)$ so that $w_j = \theta(Q(a^*, \xi_j))$. Let $F = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$ and let

$$W' = \{\theta(Q(a^*, \xi)) \colon \xi \in F,\ \|\xi\| = 1\}.$$

By construction, $e \in \operatorname{conv} W'$. We now wish to show that W' is convex. Let p be the orthogonal projection $\mathbb{C}^m \to F$ and consider the three self-adjoint operators $h_1, h_2, h_3 \in \mathcal{B}(F)$ given by

$$h_1 = pa_1a_1^*|_F, \quad h_2 = p\operatorname{Re}(a_2a_1^*)|_F, \quad h_3 = p\operatorname{Im}(a_2a_1^*)|_F.$$

Observe that

$$W' = \{(\langle h_1\xi, \xi\rangle, \langle h_2\xi, \xi\rangle, \langle h_3\xi, \xi\rangle) \colon \xi \in F,\ \|\xi\| = 1\}$$

is the joint numerical range of h_j, $j = 1, 2, 3$. Moreover, $W' \subseteq C \subseteq \Pi^+$, so if we write

$$h = \eta_1h_1 + \eta_2h_2 + \eta_3h_3 - rI_F \in \mathcal{B}(F),$$

then $h \geq 0$ and $h\xi_j = 0$ for $j = 1, 2, 3$, so $h = 0$. Choose $j \in \{1, 2, 3\}$ with $\eta_j \neq 0$. Since $h = 0$, the set W' is affinely equivalent to the joint numerical range of the pair of hermitian operators $\{\eta_kh_k \colon k \neq j\}$, which is convex by the Toeplitz–Hausdorff theorem [2]. Hence W' is convex, so $e \in W'$ and

$$\beta(s_1) = \theta^{-1}(e) \in \theta^{-1}(W') \cap W_{\mathrm{m,e}}(b) \subseteq W_{\mathrm{m,e}}(a^*) \cap W_{\mathrm{m,e}}(b) \neq \emptyset. \qquad \square$$

The case $m = \infty$ is now more or less immediate.

Corollary 3.3. *If $T \colon M_{\infty,2} \to M_{\infty,2}$ is a right D_2-module map, then $\|T\| = \|T\|_{cb}$.*

Proof. Otherwise, there is a counterexample T with $1 = \|T\| < \|T\|_{cb}$. Recall that $M_{\infty,2} = \mathcal{B}(\mathbb{C}^2, H)$ where $H = \ell^2(\mathbb{N})$. By Proposition 2.4, there is some $k > 1$ and some $x \in M_{k,1}(M_{\infty,2}) = \mathcal{B}(\mathbb{C}^2, H \otimes \mathbb{C}^k)$ with $\|x\| < 1 < \|T_{k,1}(x)\|$. Given $m \in \mathbb{N}$, let $p_m \in \mathcal{B}(H)$ be the orthogonal projection onto the linear span of $\{e_i \colon 1 \leq i \leq m\}$ and consider $q_m = p_m \otimes I_k$. Every operator in $\mathcal{B}(\mathbb{C}^2, H \otimes \mathbb{C}^k)$ has rank at most 2, so is compact, and $T_{k,1}$ is bounded (in fact, $\|T_{k,1}\| \leq k$). Hence there is $m \in \mathbb{N}$ such that $\|q_mT_{k,1}(q_mx)\| > 1$. Let us identify $M_{m,2}$ with the subspace $p_m(M_{\infty,2})$ of $M_{\infty,2}$, and consider $S \colon M_{m,2} \to M_{m,2}$, $y \mapsto p_mT(y)$. This is a right D_2-module map and

$$\|S\| \leq \|T\| = 1 < \|q_mT_{k,1}(q_mx)\| = \|S_{k,1}(q_mx)\| \leq \|S\|_{cb},$$

contradicting Theorem 3.2. $\square$

4. CB norms and Hilbert-Schmidt norms

Given n, m, let $\mathcal{L}(M_{m,n})$ be the set of linear maps $M_{m,n} \to M_{m,n}$. For a map $T \in \mathcal{L}(M_{m,n})$, we continue to write

$$\|T\| = \sup\{\|Tx\| \colon x \in M_{m,n},\ \|x\| = 1\}$$

for the operator norm of T with respect to the operator norm $\|\cdot\|$ on $M_{m,n}$, and we will also consider the quantity

$$|||T||| = \sup\{\|Tx\|_2 \colon x \in M_{m,n},\ \|x\|_2 = 1\},$$

that is, the operator norm of T with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$ on $M_{m,n}$. Note that if $n = \infty$ or $m = \infty$, then all of these "norms" may take the value ∞.

For $T \in \mathcal{L}(M_{m,n})$, let $T^* \in \mathcal{L}(M_{n,m})$ be the map given by

$$T^*(x) = T(x^*)^*, \quad x \in M_{n,m}.$$

Clearly, $\|T^*\| = \|T\|$ and $|||T^*||| = |||T|||$.

Remark 4.1. The norm $|||\cdot|||$ behaves particularly nicely when we take ampliations: if $T \in \mathcal{L}(M_{m,n})$ and $s, t \in \mathbb{N}$, then viewing $T_{s,t}$ as a map on $M_{ms,nt}$, we have $|||T_{s,t}||| = |||T|||$. Indeed, the inequality $|||T||| \le |||T_{s,t}|||$ is trivial, and

$$|||T_{s,t}||| = \sup \|[Tx_{ij}]\|_2 = \sup \sqrt{\sum_{\substack{1\le i\le s\\ 1\le j\le t}} \|Tx_{ij}\|_2^2} \le |||T|||\,,$$

where the suprema are taken over those $x_{ij} \in M_{m,n}$ (for $1 \le i \le s$ and $1 \le j \le t$) so that $[x_{ij}] \in M_{ms,nt}$ has $\|[x_{ij}]\|_2 = 1$.

Below, we show that in many cases, $|||\cdot|||$ is comparable with the operator norm for the right module maps T under consideration. This allows us to estimate $\|T\|$ and $\|T\|_{cb}$, and these estimates are used to find some upper bounds for $\|T\|_{cb}/\|T\|$.

Proposition 4.2. *Let $m, n \in \mathbb{N} \cup \{\infty\}$. If $T \colon M_{m,n} \to M_{m,n}$ is a right D_n-module map with column operators $\{a_j \colon 1 \le j < n+1\}$, then*

$$|||T||| = \sup_j \|a_j\| \le \|T\|.$$

Proof. Recall that the column operators $a_j \in \mathcal{L}(\mathbb{C}^m)$ of T were defined in Remark 2.1. Suppose, for convenience of notation, that $n < \infty$. Let $a \in \mathcal{L}((\mathbb{C}^m)^n)$ be the diagonal direct sum of $a_1, \ldots, a_n$, so that $a(\xi_1, \ldots, \xi_n) = (a_1\xi_1, \ldots, a_n\xi_n)$ for $\xi_1, \ldots, \xi_n \in \mathbb{C}^m$. Then $\|a\| = \max_j \|a_j\|$. Since T is a right D_n-module map, we have

$$\begin{aligned}
|||T||| &= \sup\{\|Tx\|_2 \colon x \in M_{m,n},\ \|x\|_2 \le 1\} \\
&= \sup\Big\{\sqrt{\sum_{j=1}^n \|T(x)e_j\|^2} \colon x \in M_{m,n},\ \|x\|_2 \le 1\Big\} \\
&= \sup\Big\{\sqrt{\sum_{j=1}^n \|T(xe_je_j^*)e_j\|^2} \colon x \in M_{m,n},\ \|x\|_2 \le 1\Big\} \\
&= \sup\Big\{\sqrt{\sum_{j=1}^n \|a_j(xe_j)\|^2} \colon x \in M_{m,n},\ \sum_{j=1}^n \|xe_j\|^2 \le 1\Big\} \\
&= \sup\Big\{\sqrt{\sum_{j=1}^n \|a_j(\xi_j)\|^2} \colon \xi_j \in \mathbb{C}^m,\ \sum_{j=1}^n \|\xi_j\|^2 \le 1\Big\} \\
&= \sup\{\|a\xi\| \colon \xi \in (\mathbb{C}^m)^n,\ \|\xi\| \le 1\} = \|a\| = \max_j \|a_j\|.
\end{aligned}$$

Moreover, if $\eta \in \mathbb{C}^m$ and $1 \le j \le n$ then $\|\eta e_j^*\| = \|\eta\|$, so

$$\begin{aligned}
\|T\| &= \sup\{\|Tx\| \colon x \in M_{m,n},\ \|x\| \le 1\} \\
&\ge \sup\{\|T(\eta e_j^*)\| \colon \eta \in \mathbb{C}^m,\ \|\eta\| \le 1\} = \sup\{\|a_j\eta\| \colon \eta \in \mathbb{C}^m,\ \|\eta\| \le 1\} = \|a_j\|,
\end{aligned}$$

so $\|T\| \ge \max_j \|a_j\|$.

If $n = \infty$ then the proof is similar. □

The following lemma will be used to obtain a useful inequality in the other direction in Proposition 4.4 below.

Lemma 4.3. *Let $m, n \in \mathbb{N} \cup \{\infty\}$ with $k = \min\{m, n\} < \infty$. If $T \colon M_{m,n} \to M_{m,n}$ is a linear map, then $\|T\| \le \sqrt{k}\, |||T|||$.*

Proof. Suppose $k = m \le n$ and $|||T||| = 1$. For $x \in M_{m,n}$, let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ be the eigenvalues of xx^*. We have

$$\|x\|^2 = \lambda_1 \le \|x\|_2^2 = \sum_{j=1}^k \lambda_j \le k\lambda_1 = k\|x\|^2,$$

so $\|x\| \le \|x\|_2 \le \sqrt{k}\|x\|$. Hence $\|T(x)\| \le \|T(x)\|_2 \le \|x\|_2 \le \sqrt{k}\|x\|$, so $\|T\| \le \sqrt{k}$.

If $m > n$, consider the map $T^* \in \mathcal{L}(M_{n,m})$. □

If $c_1 \in M_{m,n}$ and $k \in \mathbb{N}$, then we write $c_1^{(k)}$ for the block-diagonal operator in $M_k(M_{m,n})$ with k copies of c_1 running down the diagonal. Similarly, if $c = \begin{bmatrix} c_1 \\ \vdots \\ c_\ell \end{bmatrix}$ where $c_1, \dots, c_\ell \in M_{m,n}$, then $c^{(k)} = \begin{bmatrix} c_1^{(k)} \\ \vdots \\ c_\ell^{(k)} \end{bmatrix}$. The utility of this notation is revealed by observing that if $T = a \odot b$ and $s, t \in \mathbb{N}$, then $T_{s,t} = a^{(s)} \odot b^{(t)}$.

Proposition 4.4. *Let $\ell, n \in \mathbb{N}$, let $k = \min\{\ell, n\}$ and let $K = \min\{\ell^2, n\}$. If $T \colon M_{\ell,n} \to M_{\ell,n}$ is a right D_n-module map, then*

$$\|T\|_{cb} = \|T_{k,1}\| \le \sqrt{K}\, |||T|||\,.$$

In particular, $\|T\|_{cb} = \|T_{n,1}\| \le \sqrt{n}\, |||T|||$.

Proof. By Remark 2.1, T is an elementary operator, and there are matrices $a_1, \dots, a_n \in M_\ell$ and $b_1, \dots, b_n \in D_n$ such that $Tx = \sum_{j=1}^n a_j x b_j$ and

$$\|T\|_{cb} = \tfrac{1}{2}(\|a\|^2 + \|b\|^2) \quad \text{where} \quad a = [a_1 \ \dots \ a_n] \text{ and } b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

By [12, Theorem 3.3] we have $\|T\|_{cb} = \|T_n\|$.

By Lemma 2.2, $W_{\mathrm{m,e}}(b)$ is convex. By [12, Proposition 2.4], the set $W_{\mathrm{m,e}}((a^*)^{(\ell)})$ is convex, so intersects $W_{\mathrm{m,e}}(b^{(\ell)})$, so $\|T\|_{cb} = \|T_\ell\|$. Hence

$$\|T_k\| = \min\{\|T_\ell\|, \|T_n\|\} = \|T\|_{cb}.$$

By Lemma 2.3, $\|T\|_{cb} = \|T_{k,1}\|$. By Remark 4.1, the map $T_{k,1} \in \mathcal{L}_{D_n}(M_{k\ell,n})$ satisfies $|||T_{k,1}||| = |||T|||$, hence $\|T_{k,1}\| \le \sqrt{K}\, |||T|||$ by Lemma 4.3. □

5. Upper bounds for $C(m,n)$

For $n, m \in \mathbb{N} \cup \{\infty\}$, recall that

$$C(m,n) = \sup\{\|T\|_{cb} \colon T \in \mathcal{L}_{D_n}(M_{m,n}),\ \|T\| \le 1\}.$$

We have $C(m,1) = C(1,n) = C(m,2) = 1$ by Remark 2.5 and Theorem 3.2.

We will now give some upper bounds for $C(m,n)$.

Proposition 5.1. *If $m \le m'$ and $n \le n'$ then $C(m,n) \le C(m',n')$. In other words, C is an increasing function for the product order.*

Proof. Given $T \in \mathcal{L}_{D_n}(M_{m,n})$ with $\|T\| = 1$, let $T' \in \mathcal{L}_{D_{n'}}(M_{m',n'})$ be the map

$$T'(x) = \begin{bmatrix} T(qxp) & 0_{m\times(n'-n)} \\ 0_{(m'-m)\times n} & 0_{(m'-m)\times(n'-n)} \end{bmatrix}, \quad x \in M_{m',n'}$$

where

$$q = [I_m \ 0_{m\times(m'-m)}] \in M_{m,m'} \quad \text{and} \quad p = \begin{bmatrix} I_n \\ 0_{(n'-n)\times n} \end{bmatrix} \in M_{n',n}.$$

It is easy to see that $\|T'\| = \|T\| = 1$ and $\|T\|_{cb} = \|T'\|_{cb}$. Hence $C(m,n) \le C(m',n')$. □

Proposition 5.2. *If $m,n,s,t \ge 1$ then $C(m,n)C(s,t) \le C(ms,nt)$.*

Proof. Suppose that $C(m,n) > \alpha$ and $C(s,t) > \beta$. There are $T \in \mathcal{L}_{D_n}(M_{m,n})$ and $S \in \mathcal{L}_{D_t}(M_{s,t})$ with $\|T\|, \|S\| < 1$ and $\|T\|_{cb} > \alpha$ and $\|S\|_{cb} > \beta$. Consider the tensor product map $T \otimes S \in \mathcal{L}(M_{m,n} \otimes M_{s,t})$, which is defined on elementary tensors by $T \otimes S(x \otimes y) = T(x) \otimes S(y)$. This is a right $D_n \otimes D_t$ module map, and identifying $M_{m,n} \otimes M_{s,t}$ isometrically with $M_{ms,nt}$ in the usual way, we have $D_n \otimes D_t = D_{nt}$ and it follows that $T \otimes S \in \mathcal{L}_{D_{nt}}(M_{ms,nt})$ with $\|T \otimes S\| = \|T\|\,\|S\| < 1$ and $\|T \otimes S\|_{cb} = \|T\|_{cb}\|S\|_{cb} > \alpha\beta$. So $C(ms,nt) > \alpha\beta$ whenever $C(m,n) > \alpha$ and $C(s,t) > \beta$, hence $C(ms,nt) \ge C(m,n)C(s,t)$. □

Lemma 5.3. *Let $n \in \mathbb{N}$ with $n \ge 2$. If $y \in M_{m,n}$ and $\|y(e_ie_i^* + e_je_j^*)\| \le 1$ for $1 \le i < j \le n$, then $\|y\| \le \sqrt{n/2}$.*

Proof. Let $p_{ij} = e_ie_i^* + e_je_j^*$ for $1 \le i < j \le n$. Since each p_{ij} is a projection, we have $\|yp_{ij}y^*\| = \|(yp_{ij})(yp_{ij})^*\| = \|yp_{ij}\|^2 \le 1$. Moreover,

$$\sum_{1\le i<j\le n} p_{ij} = (n-1)I_n.$$

Hence

$$\|y\| = \sqrt{\|yy^*\|} = \sqrt{\frac{1}{n-1}\left\|\sum_{1\le i<j\le n} yp_{ij}y^*\right\|} \le \sqrt{\frac{1}{n-1}\sum_{1\le i<j\le n}\|yp_{ij}y^*\|}$$
$$\le \sqrt{\frac{1}{n-1}\binom{n}{2}} = \sqrt{\frac{n}{2}}. \qquad \square$$

The following simple estimate applies to arbitrary linear maps between operator spaces, and is analogous to the well-known bound $\|T_n\| \le n\|T\|$ ([8, Exercise 3.11], due to Smith).

Lemma 5.4. *Let $k, m, n \in \mathbb{N}$. For any $T \in \mathcal{L}(M_{m,n})$, we have*

$$\|T_{k,1}\| \le \sqrt{k}\|T\|.$$

Proof. There is $x \in M_{k,1}(M_{m,n})$, say $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$ (where $x_j \in M_{m,n}$ for $1 \le j \le k$), with $\|x\| = 1$ and $\|T_{k,1}(x)\| = \|T_{k,1}\|$. Clearly we can write $T_{k,1}(x) = \begin{bmatrix} Tx_1 \\ Tx_2 \\ \vdots \\ Tx_k \end{bmatrix}$, and since $\|x_j\| \le \|x\| \le 1$ for $1 \le j \le n$, we have

$$\begin{aligned}
\|T_{k,1}\| &= \|T_{k,1}(x)\| \\
&= \sqrt{\|T_{k,1}(x)^*T_{k,1}(x)\|} \\
&= \sqrt{\left\|\sum_{j=1}^k T(x_j)^*T(x_j)\right\|} \\
&\le \sqrt{\sum_{j=1}^k \|T(x_j)^*T(x_j)\|} \\
&= \sqrt{\sum_{j=1}^k \|T(x_j)\|^2} \\
&\le \|T\|\sqrt{\sum_{j=1}^k \|x_j\|^2} \\
&\le \sqrt{k}\|T\|. \qquad \square
\end{aligned}$$

Theorem 5.5. *If $m, n \in \mathbb{N}$ and $n \ge 2$ then $C(m,n) \le \sqrt{\min\{m, n/2\}}$.*

Proof. Let $T \in \mathcal{L}_{D_n}(M_{m,n})$ with $\|T\| = 1$ and $\|T\|_{cb} = C(m,n)$. By Proposition 4.4 and Lemma 5.4, we have

$$C(m,n) = \|T\|_{cb} = \|T_{m,1}\| \le \sqrt{m}\|T\| = \sqrt{m}$$

so it only remains to show that $C(m,n) \le \sqrt{n/2}$.

Since $\|T\|_{cb} = \|T_{n,1}\|$ by Proposition 4.4, there is $x \in M_{m,1}(M_{m,n})$ with $\|x\| = 1$ such that $y = T_{m,1}(x)$ has $\|y\| = C(m,n)$. Let $a_1, \ldots, a_n$ be the column operators of T, so that

$$T = [a_1 \ \ldots \ a_n] \odot \begin{bmatrix} e_1 e_1^* \\ \vdots \\ e_n e_n^* \end{bmatrix}.$$

Given i, j with $1 \le i < j \le n$, consider

$$S = [a_i \ a_j] \odot \begin{bmatrix} e_1 e_1^* \\ e_2 e_2^* \end{bmatrix} \in \mathcal{L}_{D_2}(M_{m,2}).$$

Clearly, $\|S\| \le \|T\|$. If $w = [xe_i \ xe_j] \in M_{m^2,2}$, then $\|w\| \le \|x\| = 1$. Moreover, $y(e_i e_i^* + e_j e_j^*) \in M_{m^2,n}$ can be recovered from $S_{m,1}(w) \in M_{m^2,2}$ by padding with $n-2$ columns of zeros, so $\|S_{m,1}(w)\| = \|y(e_i e_i^* + e_j e_j^*)\|$. Since $\|S\| = \|S\|_{cb}$ by Theorem 3.2, we have

$$1 = \|T\| \ge \|S\| = \|S\|_{cb} \ge \|S_{m,1}(w)\| = \|y(e_i e_i^* + e_j e_j^*)\|.$$

By Lemma 5.3,

$$C(m,n) = \|y\| \le \sqrt{n/2}. \qquad \square$$

Fix $n \in \mathbb{N}$. The sequence $C(2,n), C(3,n), C(4,n), \ldots$ is increasing by Proposition 5.1. We will now show that it is eventually constant.

In [13, Theorem 1.3], the second author establishes an exact formula for the norm of an elementary operator T, which we now recall. If $\ell \in \mathbb{N}$ and X, Y are positive semi-definite elements of M_ℓ, then the *tracial geometric mean* of X and Y is

$$\operatorname{tgm}(X,Y) = \|\sqrt{X}\,\sqrt{Y}\|_1 = \operatorname{trace}\sqrt{\sqrt{X} Y \sqrt{X}}$$

where $\|\cdot\|_1$ denotes the trace-class norm on M_ℓ. If T is an elementary operator on M_m which is represented by $a \in M_{1,\ell}(M_m)$ and $b \in M_{\ell,1}(M_m)$, then the formula is:

$$(\dagger) \qquad \|T\| = \sup\{\operatorname{tgm}(X,Y) \colon X \in W_{\mathrm{m}}(a^*),\ Y \in W_{\mathrm{m}}(b)\}.$$

In fact, a generalisation of this formula is shown to hold for elementary operators on any C*-algebra A.

We need to show that (†) holds in the rectangular case, too. If T is an elementary operator on $M_{m,n}$ with $n > m$ which is represented by $a \in M_{1,\ell}(M_m)$ and $b \in M_{\ell,1}(M_n)$, consider the map

$$\tilde{T} \colon M_n \to M_n, \qquad x \mapsto \begin{bmatrix} T(px) \\ 0_{(n-m)\times n} \end{bmatrix},$$

where $p \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^m)$ is the orthogonal projection onto the linear span of $\{e_1, \ldots, e_m\}$, which is viewed simultaneously as $\mathbb{C}^m$ and as a subspace of $\mathbb{C}^n$. That is, $\tilde{T}$ is "T applied to the first m rows, and zero on the remaining rows". Clearly, $\|T\| = \|\tilde{T}\|$. If $a = [a_1 \ \ldots \ a_\ell]$ and $\tilde{a} = [\tilde{a}_1 \ \ldots \ \tilde{a}_\ell]$ where $\tilde{a}_j = \left[\begin{smallmatrix} a_j & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M_n$ is "a_j padded with $n-m$ zero rows and columns", then $\tilde{T}$ is represented by $\tilde{a}, b$, and $W_{\mathrm{m}}(\tilde{a}^*) = \{rX \colon r \in [0,1],\ X \in W_{\mathrm{m}}(a^*)\}$. So, since $\operatorname{tgm}(rX, Y) = \sqrt{r}\operatorname{tgm}(X,Y)$ for $r \in [0,1]$,

$$\|T\| = \|\tilde{T}\| = \sup\{\operatorname{tgm}(X,Y) \colon X \in W_{\mathrm{m}}(a^*),\ Y \in W_{\mathrm{m}}(b)\}.$$

If T is an elementary operator on $M_{m,n}$ with $n < m$, then (†) still holds, as may be seen by considering T^*.

Remark 5.6. The tracial geometric mean (or, sometimes, its square) is called *fidelity* in quantum information theory [3, 6, 7, 14], where it is interpreted as a measure of the closeness of two quantum states (positive semi-definite trace-class operators with trace 1).

Theorem 5.7. *If $1 \le n < m \le \infty$, then $C(m,n) = C(n,n)$.*

Proof. Suppose first that $m < \infty$. The supremum $C(m,n)$ is then attained, so there is $T \in \mathcal{L}_{D_n}(M_{m,n})$ with $\|T\| = 1$ and $\|T\|_{cb} = C(m,n)$. By Proposition 4.4, $\|T\|_{cb} = \|T_{n,1}\|$. By (†), there are unit vectors $\xi_1, \dots, \xi_n$ in $\mathbb{C}^m$ and $\eta \in \mathbb{C}^n$, and $r_1, \dots, r_n \in [0,1]$ with $\sum_{j=1}^n r_j^2 = 1$ such that the vector $\xi = \begin{bmatrix} r_1\xi_1 \\ \vdots \\ r_n\xi_n \end{bmatrix}$ satisfies

$$\|T\|_{cb} = \operatorname{tgm}\big(Q((a^*)^{(n)}, \xi), Q(b,\eta)\big).$$

Let K be an n-dimensional subspace of $\mathbb{C}^m$ containing $\xi_1, \dots, \xi_n$, and let us identify K with $\mathbb{C}^n$. Then writing p for the orthogonal projection of $\mathbb{C}^m$ onto K, let $\tilde{a}_j = pa_j|_K$, let $\tilde{a} = [\tilde{a}_1 \ \dots \ \tilde{a}_n]$ and let $\tilde{T}$ be the elementary operator on M_n represented by $\tilde{a}, b$. By our choice of K, we have $\|\tilde{T}\|_{cb} = \|T\|_{cb}$ and $Q(\tilde{a}^*, \xi) = Q(a, \xi)$ for $\xi \in K$, so

$$\begin{aligned} \|\tilde{T}\| &= \sup\{\operatorname{tgm}\big(Q(\tilde{a}^*, \xi), Q(b,\eta)\big) : \xi \in K,\ \eta \in \mathbb{C}^n, \|\xi\|, \|\eta\| = 1\} \\ &\le \sup\{\operatorname{tgm}\big(Q(a^*, \xi), Q(b,\eta)\big) : \xi \in \mathbb{C}^m,\ \eta \in \mathbb{C}^n, \|\xi\|, \|\eta\| = 1\} \\ &= \|T\|. \end{aligned}$$

Since $\tilde{T}$ is a right D_n-module map, we have

$$C(n,n) \ge \frac{\|\tilde{T}\|_{cb}}{\|\tilde{T}\|} \ge \frac{\|T\|_{cb}}{\|T\|} = C(m,n) \ge C(n,n)$$

and hence $C(n,n) = C(m,n)$.

The case $m = \infty$ now follows by the argument of Corollary 3.3. □

Remark 5.8. This reduces Theorem 3.2 to the 2×2 case, but does not appear to greatly simplify the proof.

6. More than two columns

We now give some examples which establish non-trivial lower bounds for $C(m,n)$ when $n \ge 3$. The matrix extremal numerical range of an ℓ-tuple $[a_1 \ \dots \ a_\ell]^*$ is closely connected to the joint numerical range of the operators $a_j a_i^*$ for $1 \le i < j \le \ell$. Moreover, the joint numerical range of three matrices (even three hermitian matrices) need not be convex, and an explicit example of this phenomenon is given in [5, Example 1.1].

Let

$$a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad a_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is easy to see that the joint numerical range of the operators $a_j a_i^*$ for $1 \le i < j \le 3$ is affinely equivalent to a 2-sphere, so is not convex. Our first example is the map whose column operators are a_1, a_2, a_3.

Example 6.1. *The map $T \colon M_{2,3} \to M_{2,3}$,*

$$T \colon \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \mapsto \begin{bmatrix} a & c & f \\ b & -d & e \end{bmatrix}$$

is a right D_3-module map with

$$\sqrt{2} = \|T\| < \|T_{2,1}\| = \|T\|_{cb} = \sqrt{3}.$$

So $C(2,3) = \sqrt{3/2}$.

Proof. T is a right D_3-module map and a Hilbert-Schmidt isometry. By Lemma 4.3, $\|T\| \le \sqrt{2}$, and we have equality since $\|T[\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}]\| = \sqrt{2}$.

By Proposition 4.4, $\|T\|_{cb} \le \sqrt{3}$, and if

$$x = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix} \quad \text{then} \quad T_{2,1}(x) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}$$

and $\|x\| = 1$ while $\|T_{2,1}(x)\| = \sqrt{3}$. So $\|T\|_{cb} = \sqrt{3} = \|T_{2,1}\|$. Hence $C(2,3) \ge \sqrt{3/2}$, and we have equality by Theorem 5.5. □

Extending the previous example by one column yields:

Example 6.2. *The map* $T\colon M_{2,4} \to M_{2,4}$,

$$T\colon \begin{bmatrix}a & c & e & g\\ b & d & f & h\end{bmatrix} \mapsto \begin{bmatrix}a & c & f & h\\ b & -d & e & -g\end{bmatrix}$$

is a right D_4*-module map with*

$$\sqrt{2} = \|T\| < \|T_{2,1}\| = \|T\|_{cb} = 2.$$

So $C(2,4) = \sqrt{2}$.

Proof. T is a right D_4-module map and a Hilbert-Schmidt isometry. By Lemma 4.3, $\|T\| \le \sqrt{2}$, and we have equality since $\|T[\begin{smallmatrix}1&0&0&0\\0&0&1&0\end{smallmatrix}]\| = \sqrt{2}$. By Proposition 4.4, $\|T\|_{cb} \le 2$, and if

$$x = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\\ 1 & -1 & 0 & 0\end{bmatrix} \quad \text{then} \quad T_{2,1}(x) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}$$

and $\|x\| = 1$ while $\|T_{2,1}(x)\| = 2$. So $\|T\|_{cb} = 2 = \|T_{2,1}\|$. Hence $C(2,4) \ge \sqrt{2}$, and Theorem 5.5 gives the reverse inequality. □

Example 6.2 may be generalised as follows:

Theorem 6.3. *For each* $m \in \mathbb{N}$ *with* $m > 1$*, there is a right* D_{m^2}*-module map* $T \in \mathcal{L}(M_{m,m^2})$ *with* $\sqrt{m} = \|T\| < \|T_{m,1}\| = \|T\|_{cb} = m$*. Hence* $C(m,m^2) = \sqrt{m}$.

Proof. We have $C(m,m^2) \le \sqrt{m}$ by Theorem 5.5.

Let $\rho = e^{2\pi i/m}$ and consider the $m \times m$ matrices

$$g = \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \rho & \cdots & 0\\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \rho^{m-1}\end{bmatrix}, \qquad h = \begin{bmatrix}0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}$$

so that h is the matrix for the m-cycle permutation $\alpha = (1\ 2\ \dots\ m)$.

For $1 \le j \le m^2$, let $0 \le r < m$ and $1 \le s \le m$ with $j = mr + s$, and define

$$a_j = g^{-(s-1)}h^{-r}.$$

Take T to be the right D_{m^2}-module map with column operators a_j ($1 \le j \le m^2$). Since each a_j is unitary, T is a Hilbert-Schmidt isometry and so $\|T\| \le \sqrt{m}$ by Lemma 4.3. (By Proposition 4.4, $\|T\|_{cb} \le m$, but we will not actually need that.)

For $1 \leq i \leq m$ and $1 \leq j \leq m^2$, let $v_j^i \in \mathbb{C}^m$ be the vector

$$v_j^i = \rho^{(i-1)(s-1)} e_{\alpha^r(i)} \quad \text{where } j = mr + s \text{ with } 0 \leq r < m,\ 1 \leq s \leq m$$

and define $x^i \in M_{m,m^2}$ by

$$x^i = \sum_{\substack{j=mr+s \\ 0 \leq r < m \\ 1 \leq s \leq m}} v_j^i e_j^*$$

Observe that $a_j v_j^i = e_i$ for every j. Hence, $Tx^i = e_i w^*$ where $w = \sum_{j=1}^{m^2} e_j \in \mathbb{C}^{m^2}$, and so $\|Tx^i\| = \|e_i\| \, \|w\| = m$.

If

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{bmatrix} \in M_{m,1}(M_{m,m^2}) \quad \text{then} \quad T_{m,1}(x) = \begin{bmatrix} e_1 w^* \\ e_2 w^* \\ \vdots \\ e_m w^* \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} w^*,$$

so $\|T_{m,1}(x)\| = \|[e_1^* \ \dots \ e_m^*]\| \, \|w\| = m^{3/2}$. On the other hand, a calculation shows that the rows of x are mutually orthogonal and have norm $\sqrt{m}$, and so $\|x\| = \sqrt{m}$. Hence $m \leq \|T_{m,1}\| = \|T\|_{cb}$ and $C(m, m^2) \geq \|T\|_{cb}/\|T\| \geq m/\sqrt{m} = \sqrt{m}$. □

Remark 6.4. By Lemma 5.4,

$$\|T_{m-1,1}\| \leq \sqrt{m-1}\|T\| < \|T_{m,1}\| = \|T\|_{cb} = \sqrt{m}\|T\| = m\,\|\!|T|\!\|$$

for the D_{m^2}-module maps T on M_{m,m^2} constructed in this proof. Thus the estimates of Proposition 4.4 and Theorem 5.5 are sharp, at least for $n = m^2$.

Corollary 6.5. *If $m, n \in \mathbb{N}$ with $n \geq m^2$ then $C(m,n) = \sqrt{m}$.*

Proof. Since $C(m,n)$ is increasing in n, this is an immediate consequence of Theorems 5.5 and 6.3. □

Theorem 6.6. *If $m, n \in \mathbb{N}$ with $2 \leq n \leq m^2$, then $C(m,n) \geq \sqrt{n/m}$.*

Proof. Let $T \in \mathcal{L}_{D_{m^2}}(M_{m,m^2})$ and $x \in M_{m^2}$ be as in the proof of Theorem 6.3. Let $a_1, \dots, a_{m^2}$ be the column operators of T and consider the map $S \in \mathcal{L}_{D_n}(M_{m,n})$ whose column operators are $a_1, \dots, a_n$. Also, let $x_j = xe_j \in \mathbb{C}^{m^2}$ be the jth column of x and let $y = [x_1 \ \dots \ x_n] \in M_{m^2,n}$. By following the earlier argument, it is not hard to see that $S_{m,1}(y)$ is the matrix in $M_{m^2,n}$ whose columns are the first n columns of $T_{m,1}(x)$, and hence that $\|S_{m,1}(y)\| = \sqrt{n}$. Since $\|y\| \leq \|x\| = 1$, we have

$$\|S\|_{cb} \geq \|S_{m,1}\| \geq \|S_{m,1}(y)\| = \sqrt{n}.$$

Clearly $\|S\| \leq \|T\| \leq \sqrt{m}$. Hence

$$C(m,n) \geq \frac{\|S\|_{cb}}{\|S\|} \geq \sqrt{\frac{n}{m}}.$$ □

Remark 6.7. For $(m,n) = (2,3)$, the operator S in this proof was considered in Example 6.1, and we have equality in the bounds $\|S\|_{cb} \geq \sqrt{n}$ and $\|S\| \leq \sqrt{m}$ in this case.

We now summarise the best bounds we have obtained for $C(m,n)$.

Corollary 6.8. *Let $m, n \in \mathbb{N} \cup \{\infty\}$.*

(i) If $m = 1$ or $n \in \{1, 2\}$ then $C(m,n) = 1$.
(ii) If $m \geq n$ then $C(m,n) = C(n,n)$.
(iii) If $n \geq m^2$ then $C(m,n) = \sqrt{m}$.
(iv) If $2 \leq n \leq m^2$ then $\sqrt{\max\{\lfloor\sqrt{n}\rfloor, n/\lceil\sqrt{n}\rceil\}} \leq C(m,n) \leq \sqrt{\min\{m, n/2\}}$.

Proof. Statements (i)–(iii) and the upper bound in (iv) have been discussed already, in Remark 2.5, Theorem 3.2, Corollary 6.5 and Theorem 5.5.

Suppose that $2 \le n \le m^2$. Let $k = \lfloor \sqrt{n} \rfloor$. Then $m \ge \sqrt{n} \ge k$ and $n \ge k^2$, and C is increasing by Proposition 5.1, so $C(m,n) \ge C(k,k^2) = \sqrt{k}$ by Theorem 6.3. Similarly, if we write $\ell = \lceil \sqrt{n} \rceil$ then $m \ge \ell$ so $C(m,n) \ge C(\ell,n) \ge \sqrt{n/\ell}$ by Theorem 6.6. □

Question 6.9. If $m \le 2$ or $n \le 4$ then these bounds yield exact values of $C(m,n)$, but we have been unable to find the exact values of $C(m,n)$ in many other cases. In particular, what is $C(3,5)$?

Remark 6.10. It seems improbable that the lower bounds we have obtained could be sharp in general. In particular, it would seem surprising if $C(6,6)$ turned out to be no larger than $C(2,4) = \sqrt{2}$.

Question 6.11. Is $C(n,n)$ strictly increasing in n for $n \ge 2$?

We now answer Question 1.1 in the negative. Recall that a masa in $\mathcal{B}(H)$ is said to be *discrete* if it is generated by its minimal projections. Moreover [10], if H is separable and a masa D in $B(H)$ is not discrete, then D is unitarily equivalent to the direct sum of a discrete masa and a diffuse masa, namely the masa $L^\infty[0,1]$ in $B(L^2[0,1])$.

Corollary 6.12. *If H is an infinite-dimensional Hilbert space and D is a masa in $\mathcal{B}(H)$, then there is a bounded right D-module map $T\colon \mathcal{K}(H) \to \mathcal{K}(H)$ which is not completely bounded.*

Proof. By restricting attention to a separable subspace if necessary, we may assume that H is separable.

First suppose that D is discrete. By considering the minimal projections in D, we may identify H with $\bigoplus_{m\ge1} H_m$ where $H_m = \mathbb{C}^{m^2}$ for $m \ge 1$, in such a way that the minimal projections of D are identified with the coordinate projections of H_m.

Let p_m be the orthogonal projection in $D \subseteq \mathcal{B}(H)$ with range H_m. By Theorem 6.3, there is a right D_{m^2}-module map $T_{(m)}\colon \mathcal{B}(H_m) \to \mathcal{B}(H_m)$ with $\|T_{(m)}\| = 1$ and $\|T_{(m)}\|_{cb} \ge \sqrt{m}$. The map $T\colon \mathcal{K}(H) \to \mathcal{K}(H)$, $x \mapsto \sum^{\oplus}_{m\ge1} T_{(m)}(p_m x p_m)$ has $\|T\| = 1$ and $\|T\|_{cb} \ge \sup_{m\ge1} \|T_{(m)}\|_{cb} = \infty$.

If D is not discrete, then D is unitarily equivalent to the direct sum of a discrete masa in $B(H_0)$ for some Hilbert space H_0 (possibly zero) and a diffuse masa. If $D_1 \subseteq B(H_1)$ and $D_2 \subseteq B(H_2)$ are two diffuse masas where H_1 and H_2 are separable Hilbert spaces, then D_1 and D_2 are unitarily equivalent [10, Lemma 2.3.6]. Restricting attention to the diffuse part, we may assume that $H = L^2[0,1] \otimes \ell^2(\mathbb{N})$ and $D = L^\infty[0,1] \mathbin{\overline{\otimes}} \ell^\infty(\mathbb{N})$.

Let ι be the identity map on $\mathcal{K}(L^2[0,1])$ and let θ be the contraction on $\mathcal{K}(\ell^2(\mathbb{N}))$ given by applying the construction of first part of the proof to the masa $\ell^\infty(\mathbb{N})$ in $\mathcal{B}(\ell^2(\mathbb{N}))$. Under the natural identification $\mathcal{K}(H) = \mathcal{K}(L^2[0,1]) \otimes \mathcal{K}(\ell^2(\mathbb{N}))$, the mapping $\Theta = \iota \otimes \theta$ is then a contractive right D-module map on $K(H)$ which is not completely bounded. □

Remark 6.13. Under the same hypotheses, using weakly convergent sums in place of the norm convergent sums in this construction provides a bounded right D-module map $\mathcal{B}(H) \to \mathcal{B}(H)$ which is not completely bounded.

Remark 6.14. If $T\colon M_{m,n} \to M_{m,n}$ is a right D_n-module map with $1 = \|T\| < \|T\|_{cb}$, then just as in Proposition 5.2, the kth tensor power of T, that is, the map $T^{\otimes k}\colon M_{m^k,n^k} \to M_{m^k,n^k}$ is a right D_{n^k}-module map with $\|T^{\otimes k}\| = 1$ and $\|T^{\otimes k}\|_{cb} = \|T\|_{cb}^k \to \infty$ as $k \to \infty$. Thus Example 6.1 may be used in place of Theorem 6.3 to establish Corollary 6.12.

7. Subsets of the right module maps

For $m, n \in \mathbb{N}$, let $\mathcal{S}(m,n)$ be a subset of $\mathcal{L}(M_{m,n})$ containing a nonzero mapping and let

$$C_{\mathcal{S}}(m,n) = \sup\Big\{ \frac{\|T\|_{cb}}{\|T\|} \colon T \in \mathcal{S}(m,n),\ T \neq 0 \Big\}.$$

Above, we have considered the case $\mathcal{S}(m,n) = \mathcal{L}_{D_n}(M_{m,n})$ and have shown that the corresponding function $C = C_{\mathcal{S}}$ can take values larger than 1. On the other hand, if $\mathcal{S}$ is the class of Schur multipliers, then $C_{\mathcal{S}}$ is identically 1. It seems natural to ask for which classes of operators $\mathcal{S}$ we still have $C_{\mathcal{S}}(m,n) > 1$ for some m, n. Of course, if $\mathcal{S}(m,n) \subseteq \mathcal{L}_{D_n}(M_{m,n})$ then $1 \le C_{\mathcal{S}}(m,n) \le C(m,n)$.

Let $m, n \in \mathbb{N}$. Given $\alpha \in S_m$, let u_α be the corresponding permutation unitary satisfying $u_\alpha(e_i) = e_{\alpha(i)}$ for $1 \le i \le m$. Let

$$\mathcal{P}(m,n) = \left\{ [u_{\alpha_1}\ \dots\ u_{\alpha_n}] \odot \begin{bmatrix} e_1 e_1^* \\ \vdots \\ e_n e_n^* \end{bmatrix} \colon \alpha_j \in S_m \right\} \subseteq \mathcal{L}_{D_n}(M_{m,n}).$$

This is a natural class of right D_n-module maps in which to seek maps with larger cb norm than norm. Indeed, if we drop the right modularity requirement, then the classic example of such a map is the transpose of a square matrix, which is a carefully chosen permutation of the matrix entries; $\mathcal{P}$ is precisely the set of right D_n-module maps which are permutations of the matrix entries. We initially looked for examples in this class, and having had no luck, were eventually led to Examples 6.1 and 6.2, and so to Theorem 6.3. Since we concentrated on the $2 \times n$ and the 3×3 cases, it is nice to be able to offer the following explanation for this initial failure.

Proposition 7.1. $C_{\mathcal{P}}(2,n) = C_{\mathcal{P}}(3,3) = 1$.

Proof. By Theorem 3.2, $C_{\mathcal{P}}(2,n) \le C(2,n) = 1$, so $C_{\mathcal{P}}(2,n) = 1$. Alternatively, since S_2 is abelian, this is an immediate consequence of [12, Remark 2.5].

Now consider $u \odot e \in \mathcal{P}(3,3)$ where $u = [u_1\ u_2\ u_3]$ and $u_j = u_{\alpha_j}$ for some $\alpha_j \in S_3$. Observe that if u_0 is a unitary matrix in M_m then the norms and completely bounded norms of $u \odot e$ and $u_0 u \odot e$ coincide. So, taking $u_0 = u_1^{-1}$, we may assume that α_1 is the identity permutation. Similarly, conjugating each α_j by some $\alpha_0 \in S_3$ will not change the norm or completely bounded norm of the corresponding elementary operator. Hence up to symmetry there are three cases to consider:

(1) $\alpha_2 = (1\ 2\ 3)$ and $\alpha_3 = (1\ 3\ 2) = \alpha_2^{-1}$;
(2) $\alpha_2 = (1\ 2)$ and $\alpha_3 = (1\ 2\ 3)$; and
(3) $\alpha_2 = (1\ 2)$ and $\alpha_3 = (1\ 3)$.

In the first case, the unitaries all commute and hence $\|T\| = \|T\|_{cb}$ by [12, Remark 2.5]. In both of the latter two cases,

$$U = \{u_j u_i^* \colon 1 \le i < j \le 3\} = \{u_{(1\ 2)}, u_{(1\ 2\ 3)}, u_{(1\ 3)}\}$$

and the joint numerical range of these three unitaries contains zero, since for every $u \in U$ we have $\langle u e_1, e_1 \rangle = 0$. Hence $W_{\mathrm{m,e}}(\frac{1}{\sqrt{3}} u^*)$ contains a positive semidefinite diagonal 3×3 matrix of trace 1, and Lemma 2.2 shows that $W_{\mathrm{m,e}}(e)$ consists of all such matrices. Hence

$$W_{\mathrm{m,e}}(\tfrac{1}{\sqrt{3}} u^*) \cap W_{\mathrm{m,e}}(e) \neq \emptyset$$

and so $T = u \odot e$ has $\|T\|_{cb} = \|T\|$. $\square$

However, a more persistent search reveals that $C_{\mathcal{P}}$ is not constant.

Example 7.2. *If*

$$T = [u_{(1)}\ u_{(1\ 2)}\ u_{(1\ 3)}\ u_{(2\ 3)}] \odot \begin{bmatrix} e_1e_1^* \\ e_2e_2^* \\ e_3e_3^* \\ e_4e_4^* \end{bmatrix} \in \mathcal{L}_{D_4}(M_{3,4}),$$

then $\|T\| = \sqrt{3}$ *and* $\|T_{2,1}\| > 1.0775\sqrt{3}$. *Hence*

$$C_{\mathcal{P}}(3,4) \geq \frac{\|T_{2,1}\|}{\|T\|} > 1.0775.$$

Proof. We have $\|T\| \leq \sqrt{3}\,|||T||| = \sqrt{3}$ by Lemma 4.3, and the lower bound is given by considering the norm one matrix $\left[\begin{smallmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{smallmatrix}\right]$. Let

$$x = \begin{bmatrix} -2/3 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 1/3 & 0 & 0 & 2/3 \\ -2/3 & 0 & 0 & -1/3 \end{bmatrix}.$$

Observe that $\|x\| = 1$, since we can reorder the rows and columns to recognise it as the direct sum of two 3×2 matrices with orthonormal columns. Now

$$T_{2,1}(x) = \begin{bmatrix} -2/3 & 1 & 1/\sqrt{2} & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & -1/3 \\ -2/3 & 0 & 1/\sqrt{2} & 2/3 \end{bmatrix},$$

and a computation with Mathematica reveals that $\|T_{2,1}(x)\|^2$ is the largest root of $18x^3 - 72x^2 + 33x - 2 = 0$, and hence that $\|T_{2,1}(x)\| > 1.0775\sqrt{3}$. □

Remark 7.3. Numerical estimates obtained from a GNU Octave program using the tracial geometric mean formula (†) give an improved lower bound for $\|T_{2,1}\|$ for the operator T in the preceding example of $1.13\|T\|$.

Corollary 7.4. $C_{\mathcal{P}}(\infty, \infty) = \infty$.

Proof. Let T be the map of Example 7.2. Considering the tensor powers $T^{\otimes k}$, we see that $T^{\otimes k} \in \mathcal{P}(3^k, 4^k)$ and

$$C_{\mathcal{P}}(\infty,\infty) \geq \sup_{k\geq 1} \frac{\|T^{\otimes k}\|_{cb}}{\|T^{\otimes k}\|} = \sup_{k \geq 1} \left(\frac{\|T\|_{cb}}{\|T\|}\right)^k = \infty.$$ □

Question 7.5. If $\min\{m, n\} < \infty$, is it ever true that $1 < C_{\mathcal{P}}(m, n) = C(m, n)$?

Finally, we pose a question about the class of module maps whose column operators are unitary:

$$\mathcal{U}(m,n) = \left\{[u_1\ \dots u_n] \odot \begin{bmatrix} e_1e_1^* \\ \vdots \\ e_ne_n^* \end{bmatrix} : u_j \in \mathcal{U}(M_m),\ 1 \leq j \leq n\right\} \subseteq \mathcal{L}_{D_n}(M_{m,n})$$

where $\mathcal{U}(M_m)$ is the set of unitary operators in M_m. The examples constructed in Theorem 6.3 are in $\mathcal{U}(m, m^2)$, so $C(m, m^2) = C_{\mathcal{U}}(m, m^2)$ for all $m \geq 1$.

Question 7.6. Is $C(m, n) = C_{\mathcal{U}}(m, n)$ for all $m, n \geq 1$?

References

[1] Peter M. Gruber, *Convex and discrete geometry*, Springer 2007.

[2] Karl E. Gustafson and Duggirala K. M. Rao, *Numerical range*, Springer 1997.

[3] Richard Jozsa, *Fidelity for mixed quantum states*, J. Mod. Optics **41** (1994), 2315–2323.

[4] Kate Juschenko, Rupert H. Levene, Ivan G. Todorov and Ljudmila Turowska, *Compactness properties of operator multipliers*, J. Funct. Anal. **256** (2009), 3772–3805.

[5] Chi-Kwong Li and Yiu-Tung Poon, *Convexity of the joint numerical range*, SIAM J. Matrix Anal. Appl. **21** (2000), 668–678.

[6] Jarosław A. Miszczak, Zbigniew Puchała, Paweł Horodecki, Armin Ulhmann and Karol Życzkowski *Sub- and super-fidelity as bounds for quantum fidelity*, Quantum Inf. Comput. **9** (2009), 103–130.

[7] Michael A. Nielsen and Isaac L. Chuang, *Quantum computation and quantum information*, CUP 2000.

[8] Vern I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. CUP 2002.

[9] Florian Pop, Allan M. Sinclair and Roger R. Smith, *Norming C^*-algebras by C^*-subalgebras*, J. Funct. Anal. **175** (2000), 168–196.

[10] Allan M. Sinclair and Roger R. Smith, *Finite von Neumann algebras and masas*, LMS Lect. Notes 351, CUP 2008.

[11] Roger R. Smith, *Completely bounded module maps and the Haagerup tensor product*, J. Funct. Anal. **102** (1991), 156–175.

[12] Richard M. Timoney, *Computing the norms of elementary operators*, Illinois J. Math. **47** (2003), 1207–1226.

[13] Richard M. Timoney, *Some formulae for norms of elementary operators*, J. Operator Theory **57** (2007), 121–145.

[14] John Watrous, *Distinguishing quantum operations having few Kraus operators*, Quantum Inf. Comput. **8** (2008), 819–833.

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