HARMONIC DIVISORS AND RATIONALITY OF ZEROS OF JACOBI POLYNOMIALS

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Abstract. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n with parameters α, β. The main result of the paper states the following: If $b \neq 1, 3$ and c are non-zero relatively prime natural numbers then $P_n^{(k+(d-3)/2, k+(d-3)/2)}\left(\sqrt{b/c}\right) \neq 0$ for all natural numbers d, n and $k \in \mathbb{N}_0$. Moreover, under the above assumption, the polynomial $Q(x) = \frac{b}{c}\left(x_1^2 + ... + x_{d-1}^2\right) + \left(\frac{b}{c} - 1\right) x_d^2$ is not a harmonic divisor, and the Dirichlet problem for the cone $\{Q(x) < 0\}$ has polynomial harmonic solutions for polynomial data functions.

1. Introduction

A polynomial $Q(x)$ is called a *harmonic divisor* if there exists a polynomial $p(x) \neq 0$ such that the product $Q(x)p(x)$ is harmonic, i.e. that

$$\Delta(Q(x)p(x)) = 0 \text{ for all } x \in \mathbb{R}^d,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator in the euclidean space $\mathbb{R}^d$. The notion of a harmonic divisor arises naturally in the investigation of stationary sets for the wave and heat equation [1],[2], and the injectivity of the spherical Radon transform [3]. In the study of the Cauchy problem in the category of formal power series it is often necessary to assume that a given polynomial $Q(x)$ is *not* a harmonic divisor, see [15], [16], [17], [18].

Let $\gamma \in (0,1)$. In this paper we are interested in the Dirichlet problem for the closed cone

$$\Omega_\gamma := \left\{(x_1, ..., x_d) \in \mathbb{R}^d : x_d \geq 0 \text{ and } \gamma^2\left(x_1^2 + ... + x_{d-1}^2\right) \leq \left(1 - \gamma^2\right) x_d^2\right\}. \quad (1)$$

Using some standard arguments we shall see that the Dirichlet problem for polynomial data functions has unique harmonic polynomial solutions provided that the quadratic homogeneous polynomial

$$Q_\gamma(x_1, ..., x_d) = \gamma^2\left(x_1^2 + ... + x_{d-1}^2\right) + \left(\gamma^2 - 1\right) x_d^2 \quad (2)$$

is not a harmonic divisor.

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Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers $n = 1, 2, 3, ...$ and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. D. Armitage has shown in [6] that Q_γ is not a harmonic divisor if and only if

$$C_{m-k}^{k+(d-2)/2}(\gamma) \neq 0 \tag{3}$$

for all $m \in \mathbb{N}_0$ and for all $k \in \{0, ..., m\}$. Here $C_n^\lambda(x)$ is the Gegenbauer polynomial (or ultraspherical polynomial) of degree n and parameter λ. Using the fact that Gegenbauer polynomials are expressible by Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (see Section 2) the condition (3) is equivalent to

$$P_n^{(k+(d-3)/2,k+(d-3)/2)}(\gamma) \neq 0 \text{ for all } k, n \in \mathbb{N}_0. \tag{4}$$

Since Jacobi polynomials have rational coefficients it is clear that (4) is satisfied for transcendental numbers γ. The question arises whether one may find rather simple numbers γ, say rational numbers, such that (4) holds. In this paper we shall prove that

$$P_n^{(k+(d-3)/2,k+(d-3)/2)}\left(\sqrt{b/c}\right) \neq 0 \text{ for all } k, n \in \mathbb{N}_0 \tag{5}$$

for all relatively prime natural numbers b, c with $b \neq 1, 3$. Our method of proof relies on simple divisibility arguments and an old result of Legendre about the divisibility properties of binomial coefficients.

The paper is organized as follows. In Section 2 we shall recall some standard identities for Jacobi polynomials which will be essential for our arguments. Section 3 contains the main result which will be derived from a more general theorem for Jacobi polynomials $P_n^{(\alpha,\beta)}$ where the parameters α, β are integers or half-integers.

In Section 4 we apply our results to Chebyshev polynomials providing a new proof of the following fact proven by D. H. Lehmer in [27]: Let k be an integer and $m \in \mathbb{N}_0$. If there exist a natural number c and $b \in \mathbb{N}_0$ such that

$$x_{k,m} := \cos \frac{k\pi}{m+1} = \sqrt{b/c}$$

then $x_{k,m}$ is equal to one of the numbers $0, 1, 1/2, 1/\sqrt{2}, 3/\sqrt{2}$.

In Section 5 we give applications to the Dirichlet problem as explained above.

2. Jacobi polynomials

Let us recall that the Pochhammer symbol $(\alpha)_k$ for a complex number α and $k \in \mathbb{N}_0$ is defined by

$$(\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1)$$

with the convention that $(\alpha)_0 = 1$. The Gegenbauer polynomial $C_n^\lambda(x)$ can be expressed through Jacobi polynomials by the formula (see [5, p. 302])

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda+(1/2))_n} P_n^{(\lambda-(1/2),\lambda-(1/2))}(x),$$

where the Jacobi polynomial $P^{(\alpha,\beta)}(x)$ for complex parameters α and β is defined by

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k,$$

see [5, p. 99]. For our purposes the following formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \left(\frac{1+x}{2}\right)^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(-n-\beta)_k}{(\alpha+1)_k} \left(\frac{x-1}{x+1}\right)^k,$$

is very convenient, see [5, p. 117]. Using that

$$\frac{(-n)_k}{k!} = \frac{(-1)^k}{k!} n(n-1)\ldots(n-(k-1)) = (-1)^k \binom{n}{k}$$

and $(-1)^k(-n-\beta)_k = (n+\beta+1-k)_k$ one obtains the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \left(\frac{1+x}{2}\right)^n Q_n^{(\alpha,\beta)}\left(\frac{x-1}{x+1}\right) \tag{6}$$

where we define the polynomial $Q_n^{(\alpha,\beta)}(y)$ by

$$Q_n^{(\alpha,\beta)}(y) = \sum_{k=0}^{n} \frac{(n+\beta+1-k)_k}{(\alpha+1)_k} \binom{n}{k} y^k. \tag{7}$$

Clearly (6) implies that

$$P_n^{(\alpha,\beta)}\left(2x^2-1\right) = \frac{(\alpha+1)_n}{n!} x^{2n} Q_n^{(\alpha,\beta)}\left(\frac{x^2-1}{x^2}\right). \tag{8}$$

We recall from [5, p. 117] that

$$P_{2n}^{(\alpha,\alpha)}(x) = \frac{\Gamma(2n+\alpha+1)\, n!}{\Gamma(n+\alpha+1)(2n+1)!} P_n^{(\alpha,-1/2)}\left(2x^2-1\right). \tag{9}$$

Taking the parameter β equal to $-1/2$ in formula (8) one obtains from (9) the formula

$$P_{2n}^{(\alpha,\alpha)}(x) = \frac{\Gamma(2n+\alpha+1)(\alpha+1)_n}{\Gamma(n+\alpha+1)(2n+1)!} x^{2n} Q_n^{(\alpha,-1/2)}\left(\frac{x^2-1}{x^2}\right). \tag{10}$$

For $x = \sqrt{b/c}$ this means that

$$P_{2n}^{(\alpha,\alpha)}\left(\sqrt{b/c}\right) = \frac{\Gamma(2n+\alpha+1)(\alpha+1)_n}{\Gamma(n+\alpha+1)(2n+1)!} \frac{b^n}{c^n} Q_n^{(\alpha,-1/2)}\left(\frac{b-c}{b}\right). \tag{11}$$

Similarly we have (see [5, p. 117])

$$P_{2n+1}^{(\alpha,\alpha)}(x) = \frac{\Gamma(2n+\alpha+2)\, n!}{\Gamma(n+\alpha+1)(2n+1)!} \cdot x \cdot P_n^{(\alpha,1/2)}\left(2x^2-1\right)$$

and

$$P_{2n+1}^{(\alpha,\alpha)}(x)=\frac{\Gamma(2n+\alpha+2)(\alpha+1)_n}{\Gamma(n+\alpha+1)(2n+1)!}x^{2n+1}Q_n^{(\alpha,1/2)}\left(\frac{x^2-1}{x^2}\right). \tag{12}$$

Thus

$$P_{2n+1}^{(\alpha,\alpha)}\left(\sqrt{b/c}\right)=\frac{\Gamma(2n+\alpha+2)(\alpha+1)_n}{\Gamma(n+\alpha+1)(2n+1)!}x^{2n+1}Q_n^{(\alpha,1/2)}\left(\frac{b-c}{b}\right). \tag{13}$$

In the next section we shall analyse the polynomial $Q_n^{(\alpha,\beta)}(x)$.

3. The main result

At first let us introduce some definitions and notations: for an integer $a\neq 0$ and a prime number p (so by definition $p\geq 2$) define $v_p(a)$ as the largest number $m\in\mathbb{N}_0$ such that p^m divides a, and define $v_p(0)=\infty$. Thus, $v_p(a)$ is the multiplicity of the prime factor p occurring in the prime decomposition of a. For a rational number $r=\frac{a}{b}$ one defines $v_p(r):=v_p(a)-v_p(b)$.

Let n be a natural number and p be a prime number. Let us write its p-adic decomposition by $n=n_tp^t+n_{t-1}p^{t-1}+...+n_1p+n_0$ where $n_0,...,n_t\in\{0,1,...,p-1\}$. The sum of the p-digits of n is defined by $\sigma_p(n)=n_0+...+n_t$. A beautiful result due to Legendre says that

$$v_p(n!)=\frac{n-\sigma_p(n)}{p-1},$$

see e.g. [40]. Since the sum $n_0+...+n_t$ is positive for $n\geq 1$ we conclude that

Lemma 1. *For any prime number p and any natural number n one has*

$$v_p(n!)\leq\frac{n-1}{p-1}.$$

The following simple lemma will be our main tool. For convenience of the reader we include the proof although it might be part of mathematical folklore.

Lemma 2. *Let $Q_n(x)=\sum_{k=0}^n a_kx^k$ be a polynomial with rational coefficients and $a_n\neq 0$ and $a_0\neq 0$. Let b and c be non-zero integers and let p be a prime number dividing c and not b. Assume that*

$$v_p\left(c^k\frac{a_{n-k}}{a_n}\right)\geq 1 \tag{14}$$

for $k=1,...,n$. Then $Q_n\left(\frac{b}{c}\right)\neq 0$.

Proof. We write $Q_n(x)=\sum_{k=0}^n a_{n-k}x^{n-k}$ and obtain

$$\frac{c^n}{a_n}Q_n\left(\frac{b}{c}\right)=b^n+\sum_{k=1}^n b^{n-k}c^k\frac{a_{n-k}}{a_n}. \tag{15}$$

Note that in the sum in (15), each term has p-adic valuation ≥ 1. On the other hand, b^n is not divisible by p. Hence $Q_n\left(\frac{b}{c}\right)$ can not be zero and we actually have proved that

$$v_p\left(Q_n\left(\frac{b}{c}\right)\right) = v_p\left(\frac{a_n}{c^n}\right). \tag{16}$$

□

Remark 3. *Let D_n be the least natural number such that $D_n a_{n-k}/a_n$ is an integer for all $k = 1, ..., n$. Multiplying (15) with D_n shows that $D_n \frac{c^n}{a_n} Q_n\left(\frac{b}{c}\right)$ is a non-zero integer and therefore the following inequality holds:*

$$\left|Q_n\left(\frac{b}{c}\right)\right| \geq \frac{|a_n|}{|c^n|} \cdot \frac{1}{D_n}. \tag{17}$$

We shall need the following elementary lemma. The proof is included for convenience of the reader:

Lemma 4. *If m is a natural number and $k \in \mathbb{N}_0$ then*

$$2^{2k-1} \cdot \left(m - \frac{1}{2}\right)_k = \frac{(2m+2k-3)!\,(m-1)!}{(m+k-2)!\,(2m-2)!}. \tag{18}$$

Proof. For $k \geq 1$ the term $2^{2k-1} \cdot (m - 1/2)_k$ is equal to

$$2^{k-1}\,(2m-1)\,(2m+1)\,...\,(2m+2k-3)\,.$$

Clearly this is equal to

$$2^{k-1}\frac{(2m-1)\,(2m)\,(2m+1)\,...\,(2m+2k-4)\,(2m+2k-3)}{(2m)\,(2m+2)\,....\,(2m+2k-4)}$$

and from this one obtains the right hand side of (18). For $k = 0$ one easily checks that (18) holds as well. □

Now we will state the main result of the paper and it is convenient to recall formula (7):

$$Q_n^{(\alpha,\beta)}(y) = \sum_{k=0}^{n} \frac{(n+\beta+1-k)_k}{(\alpha+1)_k}\binom{n}{k} y^k. \tag{19}$$

Theorem 5. *Let $n \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{N}_0$ and $\delta \in \{0, 1\}$. Then*

$$Q_n^{\left(-\frac{\delta}{2}+\alpha, -\frac{1}{2}+\beta\right)}\left(\frac{b}{c}\right) \neq 0 \tag{20}$$

for all non-zero relatively prime integers b and c if either (i) 2 divides c or (ii) there exists a prime number $p \geq \beta + 3$ dividing c and but not $2\beta + 1$, or (iii) there exists a prime number $p > (\beta + 3)/2$ such that p^2 divides c.

Proof. 1. Replace β in (19) by $-\frac{1}{2}+\beta$. Lemma 4 (put $m:=n+\beta-k+1\geq 1$) yields

$$(n+1/2+\beta-k)_k=\frac{1}{2^{2k-1}}\frac{(2n+2\beta-1)!\,(n+\beta-k)!}{(n+\beta-1)!\,(2n+2\beta-2k)!}.$$

2. In the first case suppose that $\delta=0$. Since $\alpha\in\mathbb{N}_0$ we have $(\alpha+1)_k=(\alpha+k)!/\alpha!$. Thus the k-th coefficient of the polynomial $Q_n^{(\alpha,-1/2+\beta)}(y)$ is given by

$$a_k:=\binom{n}{k}\frac{\alpha!}{(\alpha+k)!}\frac{1}{2^{2k-1}}\frac{(2n+2\beta-1)!\,(n+\beta-k)!}{(n+\beta-1)!\,(2n+2\beta-2k)!}. \tag{21}$$

Then

$$\frac{a_{n-k}}{a_n}=2^{2k}\binom{n}{k}\frac{(\alpha+n)!}{(\alpha+n-k)!}\frac{(\beta+k)!}{\beta!}\frac{(2\beta)!}{(2\beta+2k)!}.$$

Note that

$$2^k\frac{(\beta+k)!}{\beta!}\frac{(2\beta)!}{(2\beta+2k)!}=2^k\frac{(\beta+1)\ldots(\beta+k)}{(2\beta+1)\ldots.(2\beta+2k)}=\frac{1}{T_k(\beta)} \tag{22}$$

where

$$T_k(\beta):=(2\beta+1)(2\beta+3)\ldots(2\beta+2k-1). \tag{23}$$

Thus

$$\frac{a_{n-k}}{a_n}=2^k\binom{n}{k}\frac{(\alpha+n)!}{(\alpha+n-k)!}\frac{1}{T_k(\beta)}.$$

3. In the second case we have $\delta=1$, so the first parameter in (19) is equal to $-1/2+\alpha$. By formula (18) applied to $m=\alpha+1$ we obtain

$$(\alpha+1)_k=\left(m-\frac{1}{2}\right)_k=\frac{1}{2^{2k-1}}\frac{(2\alpha+2k-1)!\alpha!}{(\alpha+k-1)!\,(2\alpha)!}.$$

Thus the k-th coefficient of $Q_n^{(-1/2+\alpha,-1/2+\beta)}(x)$ is equal to

$$a_k=\binom{n}{k}\frac{(2n+2\beta-1)!}{(n+\beta-1)!}\frac{(2\alpha)!\,(\alpha+k-1)!\,(n+\beta-k)!}{\alpha!\,(2\alpha+2k-1)!\,(2n+2\beta-2k)!}. \tag{24}$$

Hence

$$\frac{a_{n-k}}{a_n}=\binom{n}{k}\frac{(n-k+\alpha-1)!}{(n+\alpha-1)!}\frac{(2n+2\alpha-1)!}{(2n-2k+2\alpha-1)!}\frac{(\beta+k)!\,(2\beta)!}{(2\beta+2k)!\beta!}. \tag{25}$$

Since

$$\begin{aligned}f_k&:=\frac{(n-k+\alpha-1)!}{(n+\alpha-1)!}\frac{(2n+2\alpha-1)!}{(2n-2k+2\alpha-1)!}\\&=\frac{(2n-2k+2\alpha)(2n-2k+2\alpha+1)\ldots(2n+2\alpha-1)}{(n-k+\alpha)\ldots(n+\alpha-1)}\end{aligned}$$

it is easy to see that $f_k = 2^k g_k$ with

$$g_k := (2n - 2k + \alpha + 1)(2n - 2k + \alpha + 3) \dots (2n + 2\alpha - 1).$$

Thus using (22) we obtain the following formula for the case $\delta = 1$:

$$\frac{a_{n-k}}{a_n} = \binom{n}{k} g_k \frac{1}{T_k(\beta)}.$$

4. Let now p be a prime number dividing c. In both cases, δ equal to 0 or 1, the natural number $T_k(\beta)$ is a denominator of a_{n-k}/a_k. We shall show that condition (14) in Lemma 2, namely

$$v_p\left(c^k \frac{a_{n-k}}{a_n}\right) \geq v_p\left(\frac{c^k}{T_k(\beta)}\right) \geq 1 \text{ for } k = 1, \dots, n, \tag{26}$$

is satisfied under the assumptions of the theorem, and therefore the proof will be finished.

If $p = 2$ we see that $v_2(T_k(\beta)) = 0$ for $k = 1, \dots, n$ since $T_k(\beta)$ is a product of odd numbers, so (26) is satisfied.

Assume now that $p \geq \beta + 3$. Then it is easy to see that the inequality

$$\frac{2\beta + 2k - 2}{p - 1} \leq k - 1 \tag{27}$$

holds for all $k = 3, \dots, n$. Indeed, (27) says that the function $f(k) = (k-1)(p-1) - (2\beta + 2k - 2)$ is non-negative for $k = 3, \dots, n$. Since f is a linear map, we have only to check that $f(3) \geq 0$, so $2(p-1) - 2\beta - 4 \geq 0$, which is obviously true since $p \geq \beta + 3$. By Lemma 1 we have

$$v_p(T_k(\beta)) \leq v_p((2\beta + 2k - 1)!) \leq \frac{2\beta + 2k - 2}{p - 1} \tag{28}$$

and by (27) we infer $v_p(T_k(\beta)) \leq k - 1$ that for $k = 3, \dots, n$, so (26) is satisfied for $k = 3, \dots, n$. We consider now the cases $k = 1, 2$. By assumption we know that

$$v_p(T_1(\beta)) = v_p(2\beta + 1) = 0. \tag{29}$$

Thus (14) holds for $k = 1$. Moreover, (29) implies

$$v_p(T_2(\beta)) = v_p((2\beta + 1)(2\beta + 3)) = v_p(2\beta + 3).$$

Suppose that $v_p(2\beta + 3) \geq 2$: then $2\beta + 3 \geq p^2 \geq (\beta + 3)^2 = \beta^2 + 6\beta + 9$ which is obviously nonsense. Thus $v_p(T_2(\beta)) \leq 1$ and $v_p(c^2/T_2(\beta)) \geq 1$. Hence (26) holds for all $k = 1, \dots, n$ and the result follows.

5. Now assume that p^2 divides c. If p is an integer $> (\beta + 3)/2$ then clearly

$$p \geq \frac{2\beta + 7}{4} = \frac{\beta + 3}{2} + \frac{1}{4}.$$

We have to show that (26) holds for all $k = 1, ..., n$. Note that by Lemma 1

$$v_p\left(\frac{c^k}{T_k(\beta)}\right) \geq 2k - v_p(T_k(\beta)) \geq 2k - \frac{2\beta + 2k - 2}{p-1}.$$

We conclude that $v_p\left(c^k/T_k(\beta)\right) \geq 1$ for $k = 3, ..., n$ since $h(k) := (2k-1)(p-1) - 2\beta - 2k + 2 \geq 0$ for $k = 3, ..., n$. The latter is true since $h(k) \geq h(3) = 5(p-1) - 2\beta - 4$ and by our assumption $p \geq (2\beta + 7)/4$. Now we check that $v_p\left(c^k/T_k(\beta)\right) \geq 1$ for $k = 1, 2$. Suppose that $v_p(2\beta + 1) \geq 2$ or $v_p(2\beta + 3) \geq 2$: then $p^2 \leq 2\beta + 3$ and our assumption $(2\beta + 7)/4 \leq p$ yields

$$4\beta^2 + 28\beta + 49 = (2\beta + 7)^2 \leq 16p^2 \leq 32\beta + 48.$$

Hence $(2\beta - 1)^2 \leq 0$, a contradiction since β is an integer. Thus $v_p(2\beta + 1) \leq 1$ and $v_p(2\beta + 3) \leq 1$ and therefore

$$v_p\left(\frac{c}{T_1(\beta)}\right) \geq 2 - 1 \geq 1 \text{ and } v_p\left(\frac{c^2}{T_2(\beta)}\right) \geq 4 - 2 \geq 2 \geq 1.$$

The proof is complete. □

Let us consider the case $n = 1$. From (19) we infer that $Q_1^{(\alpha,\beta)}(x) = 1 + \frac{\beta+1}{\alpha+1}x$, and specializing to our case of half-integers we obtain

$$Q_1^{\left(-\delta/2+\alpha,-\frac{1}{2}+\beta\right)}(x) = 1 + \frac{2\beta + 1}{2\alpha + 2 - \delta}x.$$

Thus $x_{1,\alpha,\beta,\delta} := -(2\alpha + 2 - \delta)/(2\beta + 1)$ is a rational zero. This already shows that the assumption that the prime number p does not divide $2\beta + 1$ in (ii) of Theorem 5 can not be omitted. In Section 4 we shall see similar examples where the degree n may be arbitrarily high.

Note that Theorem 5 does not give any information if the denominator c is equal to 1. Indeed, in this case we may have integer zeros, e.g. for $\beta = 1$ and $\delta = 0$ and $\alpha = 5$ we have

$$Q_4^{\left(5,\frac{1}{2}\right)}(x) = \frac{1}{256}(x + 4)\left(5x^3 + 100x^2 + 176x + 64\right).$$

Now we are going to prove the main result announced in the introduction:

Theorem 6. *Let d be a natural number and let b and c be relatively prime natural numbers. If m is even and $b \neq 1$ then*

$$P_m^{(k+(d-3)/2,k+(d-3)/2)}\left(\sqrt{\frac{b}{c}}\right) \neq 0 \text{ for all } k, m \in \mathbb{N}_0.$$

If m is odd and $b \neq 1, 3$ then the same conclusion holds.

Proof. Assume that m is even, say $m = 2n$. For $x = \sqrt{b/c}$ use the identity (11), namely

$$P_{2n}^{(\alpha,\alpha)}\left(\sqrt{b/c}\right) = \frac{\Gamma\left(2n+\alpha+1\right)\left(\alpha+1\right)_n}{\Gamma\left(n+\alpha+1\right)\left(2n+1\right)!}\frac{b^n}{c^n}Q_n^{(\alpha,-1/2)}\left(\frac{b-c}{b}\right).$$

Clearly $b-c$ and b are relatively prime. Since $b \neq 1$ there exists a prime number $p \geq 2$ dividing b. Theorem 5 for the case $\beta = 0$ shows that $Q_n^{(\alpha,-1/2)}\left(\frac{b-c}{b}\right) \neq 0$. For $m = 2n+1$ we use (13). Since $b \neq 1, 3$ there exists either a prime number $p \neq 3$ dividing b, or 3^2 divides b. Theorem 5 for the case $\beta = 1$ finishes the proof. □

In Theorem 5 it is assumed that the prime number p divides the denominator c. We are now turning to a criterion where the prime number p divides the nominator. In the case $\delta = 1$ we may deduce a result by using a symmetry property of the polynomials $Q_n^{(\alpha,\beta)}(y)$:

Proposition 7. *Let α, β be complex numbers. Then for any $y \neq 0$*

$$Q_n^{(\alpha,\beta)}(y) = \frac{(\beta+1)_n}{(\alpha+1)_n}\cdot y^n Q_n^{(\beta,\alpha)}\left(\frac{1}{y}\right).$$

Proof. One may derive this result directly from the definition. Alternatively, one may use the well known fact that $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ and use formula (6). Then the substitution $y = (x-1)/(x+1)$ finishes the proof. □

Theorem 8. *Let $n \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}_0$. Then*

$$Q_n^{(-1/2+\alpha,-1/2+\beta)}\left(\frac{b}{c}\right) \neq 0$$

for all non-zero relatively prime integers b and c if either (i) 2 divides b or (ii) there exists a prime number $p \geq \alpha+3$ dividing b but not $2\alpha+1$,or (iii) there exists a prime number $p > (\beta+3)/2$ such that p^2 divides b.

Proof. By Proposition 7 there exists a non-zero rational number $r_n(\alpha,\beta)$ such that

$$Q_n^{(-1/2+\alpha,-1/2+\beta)}(b/c) = r_n(\alpha,\beta)\frac{b^n}{c^n}Q_n^{(-1/2+\beta,-1/2+\alpha)}\left(\frac{c}{b}\right). \tag{30}$$

Now apply Theorem 5 for the case $\delta = 1$ to the right hand side of (30). □

Let us recall that the Jacobi polynomials $P_n^{(0,0)}(x)$ coincide with the Legendre polynomials. It is still an unsolved question whether the Legendre polynomials are irreducible over the rationals, see [23], [24], [30], [40] and [41]. H. Ille has shown that $P_n^{(0,0)}(x)$ has no quadratic factors which implies that $P_n^{(0,0)}\left(\sqrt{b/c}\right) \neq 0$ for all $n, b, c \in \mathbb{N}$ (even for the case $b = 1, 3$). In passing we note that recent research is devoted to the study of irreducibility of the Laguerre polynomials $L_n^{\alpha}(x)$ initiated by I. Schur, see [20], [22], [36], and for a family of Jacobi polynomials see [12]. For general questions about irreducibility of polynomial with rational coefficients we refer to [28], [31] and [38].

4. Applications to Chebyshev polynomials

Note that $Q_n^{(\alpha,\beta)}(x) > 0$ for all $x > 0$ whenever α, β are real numbers $\geq -1/2$. Let us take in Theorem 5 and 8 the parameters α and β equal to zero. Then we infer that

$$Q_n^{(-1/2,-1/2)}\left(\frac{b}{c}\right) \neq 0 \text{ for all } \frac{b}{c} \neq -1. \tag{31}$$

Taking α and β equal to 1 we infer that

$$Q_n^{(1/2,1/2)}(b/c) \neq 0 \text{ for all } \frac{b}{c} \neq -1, -3, -1/3. \tag{32}$$

Next we shall show that indeed

$$Q_{3m-1}^{(1/2,1/2)}(-1/3) = 0 \text{ and } Q_{3m-1}^{(1/2,1/2)}(-3) = 0 \text{ and } Q_{2m-1}^{(1/2,1/2)}(-1) = 0 \tag{33}$$

for all natural numbers m; in particular one can not omit in Theorem 5 the condition that the prime number p does not divide $3 = 2\beta + 1$ (with $\beta = 1$). For the proof of (33) we use that the relationship of the polynomial $P_n^{(1/2,1/2)}(x)$ to the Chebyshev polynomial $U_n(x)$ of the second kind, namely

$$P_n^{(1/2,1/2)}(x) = \frac{(2n+2)!}{2^{n+1}\left[(n+1)!\right]^2} U_n(x) = \frac{(2n+2)!}{2^{n+1}\left[(n+1)!\right]^2} \frac{\sin(n+1)\theta}{\sin\theta}, \tag{34}$$

where $x = \cos\theta$, cf. [5, p. 241], and

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \text{ and } \cos\left(\frac{\pi}{2}\right) = 0.$$

If we set $\theta = \pi/3$ then $x = \cos\theta = 1/2$ and $P_{3m-1}^{(1/2,1/2)}(1/2) = 0$ by (34). Using (6) we infer that $Q_{3m-1}^{(1/2,1/2)}(-1/3) = 0$. The cases $\theta = 2\pi/3$ and $\theta = \pi/2$ are similar.

Now we use Theorem 5 to derive the following result (see [27] and [39]):

Theorem 9. *The number $x := \cos\frac{k\pi}{m+1}$ is rational if and only if x is equal to one of the numbers $0, \pm 1, \pm 1/2$.*

Proof. We may assume that $m > 0$ and we put $\theta = k\pi/(m+1)$. By (34), $P_m^{(1/2,1/2)}(x) = 0$ for $x = \cos\theta$. Assume that $x \neq 0, \pm 1$ and $x = b/c$. Then $b - c \neq 0$ and $b \neq 0$ and by (6)

$$0 = P_m^{(1/2,1/2)}(b/c) = d_m Q_m^{(1/2,1/2)}\left(\frac{b-c}{b+c}\right)$$

for some non-zero constant d_m. By (32) we conclude that $\frac{b-c}{b+c} \in \{-1, -3, -1/3\}$. It follows that either $b - c = -(b+c)$ which implies $b = 0$, or $b - c = -3b - 3c$, so $4b = -2c$, so $b/c = -1/2$, or $3(b-c) = -b - c$ which implies that $4b = 2c$, so $b/c = 1/2$. $\square$

Theorem 9 is a special case of the following result due to D.H. Lehmer [27]: Let $n > 2$ and k and n relatively prime. Then $2\cos(2\pi k/n)$ is an algebraic integer of degree $\varphi(n)/2$ where φ is Euler's φ-function, see also [32, Theorem 3.9]. For example, we have

$$\cos(\pi/4) = 1/\sqrt{2} \text{ and} \cos(\pi/6) = \sqrt{3}/2.$$

The question when $\cos(2\pi k/d)$ is the square root of a positive rational number was discussed by J. L. Varona in [39] using recurrence relations, see also [4, Chapter I]. We shall give here an alternative proof based on Theorem 9.

Theorem 10. *Let k be an integer and $m \in \mathbb{N}_0$. Suppose that there exist natural numbers b, c such that*

$$\cos\frac{k\pi}{m+1} = \sqrt{b/c}.$$

Then $\cos(k\pi/(m+1))$ *is equal to one of the numbers* $0, 1, 1/2, 1/\sqrt{2}, \sqrt{3}/2$.

Proof. This is a simple consequence of Theorem 9 using that $2\cos^2\alpha - 1 = \cos(2\alpha)$. Thus, if $\cos(k\pi/(m+1))$ is a square root of a rational number, then $\cos(2k\pi/(m+1))$ is a rational number and by Theorem 9 is one of $0, \pm 1, \pm 1/2$. □

5. Applications to the Dirichlet problem

Let $G \subset \mathbb{R}^d$ be a domain and ∂G the boundary of G. We say that the Dirichlet problem is solvable if for each continuous function f on ∂G there exists a continuous function u defined on the closure of G such that u is harmonic in G and $f(\xi) = u(\xi)$ for all $\xi \in \partial G$.

It is well known that the Dirichlet problem can be solved explicitly if G is a ball or an ellipsoid, see [7]. An elegant proof of this fact was presented in [25] (see also [8] and [9]), which can be extended to domains defined by quadratic polynomials in the following way:

Theorem 11. *Let $Q(x)$ be a polynomial of degree ≤ 2. If Q is* not *a harmonic divisor then for each polynomial $f(x)$ of degree $\leq m$ there exists a* harmonic *polynomial u of degree $\leq m$ such that*

$$u(\xi) = f(\xi) \text{ for all } \xi \in Q^{-1}\{0\} := \left\{x \in \mathbb{R}^d : Q(x) = 0\right\} \tag{35}$$

Proof. Let $\mathcal{P}(\mathbb{R}^d)$ be the set of all polynomials in the variables $x_1, ..., x_d$. The so-called Fischer operator $F_Q : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is defined by

$$F_Q(p) := \Delta(Pq) \text{ for all } q \in \mathcal{P}(\mathbb{R}^d).$$

The fact that $Q(x)$ is *not* a harmonic divisor is equivalent to the injectivity of F_Q. Since $Q(x)$ is a polynomial of degree ≤ 2 the Fischer operator F_Q maps the space of all polynomials of degree $\leq m$ into itself. Therefore injectivity of F_Q implies the bijectivity of F_Q. To find the solution u of the Dirichlet problem one defines

$$u = f - Q \cdot F_Q^{-1}(\Delta(f)).$$

Then u obviously satisfies (35) and u is harmonic since $\Delta u = \Delta f - F_Q \circ F_Q^{-1}(\Delta f) = 0$. □

Theorem 12. *Let $\gamma := \sqrt{b/c} < 1$ with relatively prime natural numbers b, c with $b \neq 1, 3$. Let Ω_γ be the cone defined in (1). Then for each polynomial f of degree $\leq m$ there exists a harmonic polynomial u of degree $\leq m$ such that $f(\xi) = u(\xi)$ for all $\xi \in \partial\Omega_\gamma$.*

Proof. The assumptions of Theorem 12 imply that Q_γ is not a harmonic divisor. By Theorem 11 there exists a harmonic polynomial u such that $u(\xi) = f(\xi)$ for all $\xi \in Q_\gamma^{-1}(0)$. Since $\partial\Omega_\gamma \subset Q_\gamma^{-1}(0)$ the proof is complete. □

For more applications of the Fischer operator we refer to [35] and [37]. For a discussion of polynomial solutions in the Dirichlet problem (Khavinson-Shapiro conjecture) we refer to [10], [11], [13], [14], [19], [26], [29], [34].

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