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Stationary boundary points for a Laplacian growth problem in higher dimensions

Stephen J. Gardiner and Tomas Sjödin

Abstract

It is known that corners of interior angle less than $\pi/2$ in the boundary of a plane domain are initially stationary for Hele Shaw flow arising from an arbitrary injection point inside the domain. This paper establishes the corresponding result for Laplacian growth of domains in higher dimensions. The problem is treated in terms of evolving families of quadrature domains for subharmonic functions.

1 Introduction

Let $p \in \Omega_0$, where Ω_0 is a bounded domain in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$, and let $\mu_t = \lambda|_{\Omega_0} + t\delta_p$ $(t > 0)$, where λ denotes Lebesgue measure on $\mathbb{R}^N$ and δ_p is the unit measure at p. This paper studies quadrature domains for subharmonic functions with respect to μ_t, by which we mean domains Ω that contain Ω_0 and satisfy

$$\int_\Omega s\, d\lambda \geq \int s\, d\mu_t \quad \text{for all } \lambda\text{-integrable subharmonic functions } s \text{ on } \Omega.$$

It is known (see Sakai [18]) that such domains exist and are unique up to λ-null sets, and that there is a smallest one, which we will denote by Ω_t.

When $N = 2$ the family $\{\Omega_t : t \geq 0\}$ models Hele Shaw flow with initial domain Ω_0 and injection point p. In this case it has been shown (see Sakai [20] and earlier work of King, Lacey and Vázquez [14]) that, if the boundary of the domain Ω_0 has a corner q with interior angle less than $\pi/2$, then this point is (initially) stationary for $\{\Omega_t : t \geq 0\}$; that is, there exists $\varepsilon > 0$ such that $q \in \partial\Omega_t$ when $0 < t < \varepsilon$. Further, corners of angle greater than $\pi/2$ are not stationary, and corners of angle $\pi/2$ may or may not be stationary.

The purpose of this paper is to establish corresponding results in higher dimensions, where the geometry is more complicated and the available tools are more restricted. As in the case of the plane, this models a type of free boundary problem with Laplacian growth where the evolution is driven by a source term. Gustafsson [10] has expounded the connection between

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this problem and the study of fluid flow in a porous medium, as governed by Darcy's law. As we will observe later (in Lemma 14), the notion of a boundary point being initially stationary is independent of the choice of the point p in the domain Ω_0. We will therefore omit reference to p in the statements of our main results. For a conical vertex it turns out that the critical aperture is where the interior half-angle is $\cos^{-1}(1/\sqrt{N})$. Of course, when $N = 2$, this corresponds to a corner of angle $\pi/2$, as discussed above. For a wedge-shaped part of the boundary, the critical aperture remains $\pi/2$. These are simple special cases of the general results we will present below.

Let $B(x, r)$ denote the open ball in $\mathbb{R}^N$ of centre x and radius r, and let $S(x, r) = \partial B(x, r)$, $B = B(0, 1)$ and $S = S(0, 1)$. We will use σ to denote surface area measure (when it exists) on a given surface, and $\widehat{\sigma}$ its normalization to a unit measure. For a function $f : S \to \mathbb{R}$ and $x \in S$ we define $(\nabla_S f)(x)$ and $(\Delta_S f)(x)$ to be $(\nabla f_*)(x)$ and $(\Delta f_*)(x)$ respectively, where f_* is the extension of f from S to $\mathbb{R}^N \backslash \{0\}$ defined by $f_*(y) = f(y/|y|)$. Thus Δ_S is the Laplace-Beltrami operator on S. If ω is a non-empty relatively open subset of S, we define

$$l(\omega) = \inf \left\{ \frac{\int_S |\nabla_S f|^2 \, d\sigma}{\int_S f^2 \, d\sigma} \right\},$$

where the infimum is taken over all Lipschitz functions $f : S \to [0, \infty)$ which vanish on $S \backslash \omega$ but not on all of S. If, further, ω is connected, the quantity $l(\omega)$ is the first eigenvalue of $-\Delta_S$ (see Section 5) and, using u to denote a corresponding eigenfunction, the function $y \longmapsto |y|^\alpha \, u_*(y)$ is harmonic on the conical set $\{rx : x \in \omega, r > 0\}$ if and only if $\alpha(\alpha + N - 2) = l(\omega)$. The characteristic constant $\alpha(\omega)$ of ω is defined to be the non-negative root of this last equation. (If $N = 2$ and ω is an arc of length θ, then $\alpha(\omega) = \pi/\theta$.) It is easy to see that $\alpha(\omega_1) \geq \alpha(\omega_2)$ when $\emptyset \neq \omega_1 \subset \omega_2 \subset S$. For any compact subset L of S we next define

$$\alpha(L) = \sup\{\alpha(\omega) : \omega \text{ is relatively open in } S \text{ and } L \subset \omega\}.$$

These notions are extended to relatively open (or closed) subsets E of $S(0, r)$, for any $r > 0$, by defining $\alpha(E)$ to be $\alpha(\{y/r : y \in E\})$.

Since the plane case has already been extensively investigated we will assume from now on that Ω_0 is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$. For ease of notation we will further assume that $0 \in \partial\Omega_0$ and will investigate when this point is initially stationary for $\{\Omega_t : t \geq 0\}$. Given an increasing continuous function $\phi : (0, \infty) \to (0, 1/2]$ satisfying the doubling condition $\phi(2t) < C\phi(t)$ for some $C > 1$, we define the enlarged domain

$$\Omega(\phi) = \{x \in \mathbb{R}^N \backslash \{0\} : \text{dist}(x, \overline{\Omega}_0) < |x| \, \phi(|x|)\},$$

which also has 0 as a boundary point.

Theorem 1 *Let Ω_0 and ϕ be as above, and let $p_0 \in \Omega_0$. If there is a positive constant C_0 such that*

$$\phi(\rho) \geq C_0 \exp\left\{ \frac{1}{N} \int_\rho^{|p_0|} \frac{2 - \alpha(\Omega(\phi) \cap S(0,t))}{t} dt \right\} \qquad (0 < \rho < |p_0|), \quad (1)$$

then there exists $\varepsilon > 0$ such that $\Omega_t \subset \Omega(\phi)$ when $0 < t < \varepsilon$. In particular, 0 is initially stationary for $\{\Omega_t : t \geq 0\}$.

To any subset E of S we associate the conical set

$$K(E) = \{ry : r > 0, y \in E\}.$$

The complement of a set A in $\mathbb{R}^N$ will be denoted by A^c.

Corollary 2 *Let L be a compact subset of S such that $\alpha(L) > 2$, and suppose there exists $r_0 > 0$ such that $\Omega_0 \cap B(0, r_0) \subset K(L)$. Then 0 is initially stationary for $\{\Omega_t : t \geq 0\}$.*

As will be seen from Theorem 4(a) below we cannot relax the above hypotheses to allow $\alpha(L) = 2$. The next result sheds more light on this critical case.

Theorem 3 *Let ω be a domain relative to S, with Lipschitz boundary, such that $\alpha(\omega) = 2$. Then there is a constant $C(\omega) > 1$ such that 0 is initially stationary for $\{\Omega_t : t \geq 0\}$, where*

$$\Omega_0 = \left\{ x \in B\left(0, e^{-2C(\omega)}\right) : \operatorname{dist}(x, K(\omega)^c) > \frac{C(\omega)\,|x|}{\log(1/|x|)} \right\}. \qquad (2)$$

The denominator in (2) can be replaced by $\log(1/|x|)\log(\log(1/|x|))$ or similar expressions involving further iterated logarithmic factors: the same argument applies. However, part (b) of the next result shows that it cannot be replaced by $(\log(1/|x|))^a$ or $\log(1/|x|)\,(\log(\log(1/|x|)))^a$, where $a > 1$. Thus we have a result which is close to being sharp.

Theorem 4 *Let ω be a domain relative to S with $C^{1,\beta}$ boundary.*

(a) If $\alpha(\omega) \leq 2$, and $K(\omega) \cap B(0, r_0) \subset \Omega_0$ for some $r_0 > 0$, then 0 is not initially stationary for $\{\Omega_t : t \geq 0\}$.

(b) Further, if $\alpha(\omega) = 2$ and $\psi : (0,1] \to (0, 1/2]$ is increasing and satisfies $\int_0^1 t^{-1}\psi(t)dt < \infty$, then 0 is not initially stationary for $\{\Omega_t : t \geq 0\}$, where

$$\Omega_0 = \{x \in B : \operatorname{dist}(x, K(\omega)^c) > |x|\,\psi(|x|)\}.$$

Example 1. Let $0 < \theta_0 < \pi/2$ and consider the truncated cone

$$\Omega_0 = \{(x_1, ..., x_N) \in B(0, 2) : x_N > (\cos \theta_0) |x|\}.$$

Then 0 is initially stationary for Ω_0 if and only if $\theta_0 < \cos^{-1}(1/\sqrt{N})$. To see this, we note that the homogeneous quadratic polynomial given by $u(x) = Nx_N^2 - |x|^2$ is positive and harmonic on the infinite cone about the x_N-axis of half-angle $\cos^{-1}(1/\sqrt{N})$, and vanishes on its boundary. It follows that $u|_S$ is a strictly positive eigenfunction for $-\Delta_S$ on the spherical cap $\Omega_0 \cap S$, and hence (see Section 5) that $\alpha(\Omega_0 \cap S(0, t)) = 2$ $(0 < t < 2)$. We can now appeal to Corollary 2 and Theorem 4(a) to reach the desired conclusion.

Example 2. Let $0 < \theta_0 < \pi$ and consider the truncated wedge

$$\Omega_0 = \{(x_1, ..., x_{N-2}, r \cos \theta, r \sin \theta) : 0 < r < 2, 0 < \theta < \theta_0\}.$$

Then 0 is initially stationary for Ω_0 if and only if $\theta_0 < \pi/2$. This follows by reasoning similar to the previous example, except that the relevant polynomial is now given by $u(x) = x_{N-1}x_N$.

The above results will be established in Sections 4 and 5, following preparatory material in the next two sections. We will employ a range of concepts from potential theory. In particular, we will make crucial use of the technique of partial balayage and the associated notion of localization. Other key tools include a convexity result of Huber, and a Hadamard-type estimate for eigenvalues of the Laplace-Beltrami operator on spherical domains.

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2 Tools from potential theory and partial balayage

The fine topology on $\mathbb{R}^N$ is the coarsest topology for which all subharmonic functions are continuous. A function s on $\mathbb{R}^N$ is called δ-subharmonic if it can be expressed as $s = s_1 - s_2$, where s_1, s_2 are subharmonic functions. If s is δ-subharmonic, then the distributional Laplacian Δs is (locally) a signed measure μ_s. A δ-subharmonic function $s = s_1 - s_2$ will be undefined on the polar set Z where $s_1 = -\infty = s_2$. However, as noted in [8], s has a fine limit $|\mu_s|$-almost everywhere, as well as being finely continuous at all points of Z^c. We assign s this limiting value wherever it exists and, with this convention, reformulate a result of Brezis and Ponce [4] as follows. A short proof of it may be found in [8].

Theorem 5 (Kato's inequality) *If s is a δ-subharmonic function, then* $\Delta s^+ \geq (\Delta s)\big|_{\{s \geq 0\}}$.

We will say that a (positive) measure μ on $\mathbb{R}^N$ $(N \geq 3)$ is carried by a Borel set A if $\mu(A^c) = 0$. The Newtonian potential of a measure μ is given by

$$U\mu(x) = c_N \int |x-y|^{2-N} \, d\mu(y),$$

where the dimensional constant c_N is chosen to yield the distributional identity $-\Delta U\mu = \mu$. If $A \subset \mathbb{R}^N$ and $U\mu \not\equiv \infty$, we define the swept measure $\mu^A = -\Delta \widehat{R}^A_{U\mu}$, where $\widehat{R}^A_v$ denotes the lower semicontinuous regularization of the reduction R^A_v of a positive superharmonic function v relative to A in $\mathbb{R}^N$, given by

$$R^A_v = \inf\{u : u \text{ is positive and superharmonic on } \mathbb{R}^N \text{ and } u \geq v \text{ on } A\}.$$

If V is an open set and $x \in V$, then $\delta_x^{V^c}$ is the harmonic measure for V and x. Later, we will use the fact that $\delta_x^{V^c} \perp \lambda$, and more generally that, if μ is carried by an V, then $\mu^{V^c} \perp \lambda$. (See [3] or, more generally, [12].)

We now recall, without proofs, some basic facts about the notion of partial balayage, which was originally developed by Gustafsson and Sakai (see, for example, [11]). A recent exposition of it may be found in [7], which also contains an application to prove the aforementioned singularity of harmonic measure with respect to λ.

Given a positive measure μ with compact support we define

$$V\mu(x) = \sup\left\{ v(x) : v \text{ is subharmonic and } v \leq U\mu + \frac{|\cdot|^2}{2N} \text{ on } \mathbb{R}^N \right\} - \frac{|x|^2}{2N}$$

and then $\mathcal{B}\mu = -\Delta V\mu$. Thus $V\mu = U(\mathcal{B}\mu)$. A crucial property here is the "structure formula"

$$\mathcal{B}\mu = \lambda|_{\omega(\mu)} + \mu|_{\omega(\mu)^c} \leq \lambda, \quad \text{where} \quad \omega(\mu) = \{V\mu < U\mu\}.$$

The set $\omega(\mu)$ is bounded and open. It will be convenient to define

$$W\mu = U\mu - V\mu,$$

whence $W\mu$ is the smallest lower semicontinuous function w that satisfies $-\Delta w \geq \mu - \lambda$ and $w \geq 0$. It follows from the structure formula that

$$-\Delta W\mu = (\mu - \lambda)|_{\omega(\mu)}. \tag{3}$$

For future reference we assemble below some further useful facts about partial balayage.

Lemma 6 *Let ν, μ and $\mu^{(n)}$ $(n \in \mathbb{N})$ be positive measures with compact support, and let Ω and $\Omega^{(n)}$ $(n \in \mathbb{N})$ be bounded domains in $\mathbb{R}^N$.*
(i) If $\nu \leq \mu$, then $V\nu \leq V\mu$, $W\nu \leq W\mu$ and $\omega(\nu) \subset \omega(\mu)$.

(ii) If $\mu^{(n)} \uparrow \mu$, then $\omega(\mu^{(n)}) \uparrow \omega(\mu)$.
(iii) If $\Omega^{(n)} \uparrow \Omega$, $p \in \Omega^{(1)}$ and $t > 0$, then

$$\bigcup_{n=1}^{\infty} \omega\left(\lambda|_{\Omega^{(n)}} + t\delta_p\right) = \omega\left(\lambda|_{\Omega} + t\delta_p\right).$$

(iv) For any $x \in \mathbb{R}^N$ and $\rho > 0$,

$$\mu(\overline{B}(x,\rho)) > (2\rho)^N \lambda(B) \Longrightarrow B(x,\rho) \subset \omega(\mu). \tag{4}$$

(v) In the case where $\mu = \mu_t = \lambda|_{\Omega_0} + t\delta_p$ we have

$$\Omega_0 \subset \Omega_t = \omega(\mu_t) = \omega(\lambda|_{\Omega_0} + t\delta_p^{\Omega_0^c}) \qquad (t > 0). \tag{5}$$

In particular, $\mathcal{B}\mu_t = \lambda|_{\omega(\mu_t)}$.

Proof. (i) This follows immediately from the above definitions and the characterization of $W\mu$.

(ii) Let $x \in \omega(\mu)$, whence $V\mu(x) < U\mu(x)$. Since $U\mu^{(n)} \uparrow U\mu$, there exists $n \in \mathbb{N}$ such that $U\mu^{(n)}(x) > V\mu(x)$. Hence $U\mu^{(n)}(x) > V\mu^{(n)}(x)$, by part (i), and so $x \in \omega(\mu^{(n)})$. This, together with (i), yields the result.

(iii) This is a special case of part (ii).

(iv) This implication was established in Theorem 2 of Sakai [19].

(v) Let $u = W(\lambda|_{\Omega_0} + t\delta_p^{\Omega_0^c})$ and Z denote the set of irregular boundary points of Ω_0. From the structure formula and the fact that $\delta_p^{\Omega_0^c} \perp \lambda$, we see that $\omega(\lambda|_{\Omega_0} + t\delta_p^{\Omega_0^c})$ carries $\delta_p^{\Omega_0^c}$, and so certainly intersects $\partial\Omega_0$. Thus u, which is non-negative and superharmonic on Ω_0, must be strictly positive on all of Ω_0; that is, $\Omega_0 \subset \omega(\lambda|_{\Omega_0} + t\delta_p^{\Omega_0^c})$. If $u(y) = 0$ for some $y \in Z$, then (by fine continuity and Lemma 7.4.1 of [2]) we can find an open subset V of Ω_0 such that $u(x) \to 0$ as $x \to y$ along V and V^c is thin at y; this is a contradiction, since u would then be a barrier for V at y, yet y is an irregular boundary point of V by the thinness of V^c there. Hence $u > 0$ on $\Omega_0 \cup Z$. Since $u + tU(\delta_p - \delta_p^{\Omega_0^c}) \geq W(\lambda|_{\Omega_0} + t\delta_p)$, and the Dirichlet modification of $W(\lambda|_{\Omega_0} + t\delta_p)$ with respect to Ω_0 majorizes u, we see that $u \leq W(\lambda|_{\Omega_0} + t\delta_p)$ with equality on $(\Omega_0 \cup Z)^c$. Thus $\omega(\mu_t) = \omega(\lambda|_{\Omega_0} + t\delta_p^{\Omega_0^c})$. Further, clearly $\mathcal{B}\mu_t = \lambda|_{\omega(\mu_t)}$.

Finally, if D is a quadrature domain for subharmonic functions with respect to μ_t, then $U\mu_t - U(\lambda|_D) \geq 0$, with equality outside D, so $U\mu_t - U(\lambda|_D) \geq W\mu_t$ and $\omega(\mu_t) \subset D$. It follows that $\Omega_t = \omega(\mu_t)$. ■

3 Localization

We will now develop the notion of localization, which was introduced by Gustafsson and Sakai [11]. If U is an open set in $\mathbb{R}^N$ we denote by $\widetilde{U}$ the union of U with the boundary points of U that are irregular for the Dirichlet problem. Thus $\widetilde{U}$ differs from U by at most a polar set.

Theorem 7 (Localization Theorem) *Let U be an open set and $\mu = \mu_1 + \mu_2$, where μ_1, μ_2 are positive measures with compact support, μ_2 is carried by U, and $U\mu_1$ is continuous on U and everywhere finite. Then there is a measure η, carried by $\partial U \cap \omega(\mu)$ and singular with respect to λ, such that*

(a) $\mathcal{B}(\mu_1 + \eta) \leq \mathcal{B}\mu$;

(b) $\omega(\mu_1 + \eta)\backslash\widetilde{U} = \omega(\mu)\backslash\widetilde{U}$, *and so* $(\mathcal{B}(\mu_1 + \eta))\,|_{U^c} = (\mathcal{B}\mu)\,|_{U^c}$;

(c) $(\mu_2|_R)^{R^c} \leq \eta \leq (\mu_2|_O)^{O^c}$ *on* ∂U, *where* $R = \omega(\mu_1 + \eta) \cap U$ *and* $O = \omega(\mu) \cap U$.

Proof. Let

$$\psi = \begin{cases} W\mu & \text{in } U^c \\ W\mu_1 & \text{in } U \end{cases}$$

and u denote the lower semicontinuous regularization of $\inf \Phi$, where

$$\Phi = \{v \text{ is } \delta\text{-subharmonic} : v \geq \psi \ \text{ and } \ -\Delta v \geq \mu_1 - \lambda\}\,.$$

Since $v \in \Phi$ if and only if

$$v - |\cdot|^2/2N \geq \psi - |\cdot|^2/2N \ \text{ and } \ -\Delta(v - |\cdot|^2/2N) \geq \mu_1,$$

we can use a standard result about infima of locally uniformly lower bounded families of superharmonic functions to see that $u = \inf \Phi$ quasi-everywhere. Further,

$$-\Delta u \geq \mu_1 - \lambda. \tag{6}$$

Since $W\mu \in \Phi$ and $W\mu_1 \leq W\mu$, we see that

$$W\mu_1 \leq u \leq W\mu.$$

The right hand inequality above is an equality everywhere on $\overline{U}^c$ and quasi-everywhere on U^c.

Kato's inequality (Theorem 5), applied to the nonpositive function $s = u - W\mu$, shows that

$$0 \geq (\Delta u - \Delta W\mu)\,|_{\{u - W\mu = 0\}}.$$

Also, by (3), (6) and the fact that $\omega(\mu)^c \subset \{u - W\mu = 0\}$,

$$(\Delta u - \Delta W\mu)\,|_{U^c\backslash\{u - W\mu = 0\}} \leq \mu_2|_{U^c\backslash\{u - W\mu = 0\}} = 0,$$

since $\mu_2(U^c) = 0$. Hence we can define a positive measure η by writing

$$\eta = (\Delta W\mu - \Delta u)|_{U^c}. \tag{7}$$

Since $u = W\mu$ on $\overline{U}^c$, we see that η is carried by ∂U. Further, since $u = W\mu$ quasi-everywhere on ∂U and μ does not charge polar subsets of ∂U, we can solve the Dirichlet problem on U to see that

$$(\Delta W\mu - \Delta u)|_{U^c} = -\left((\Delta W\mu - \Delta u)|_U\right)^{U^c},$$

whence η is singular with respect to λ. (See Section 2.)

It follows from (7) and (3) that

$$\begin{aligned}(-\Delta u)|_{U^c} &= (\mu - \lambda)|_{\{W\mu>0\}\backslash U} + \eta \\ &= (\mu_1 - \lambda)|_{\{W\mu>0\}\backslash U} + \eta \\ &= (\mu_1 - \lambda)|_{\{u>0\}\backslash U} + \eta, \end{aligned} \tag{8}$$

since $\mu_2(U^c) = 0$ and μ_1 does not charge polar sets. Now $u \geq 0$, so

$$(\Delta u)|_{\{u=0\}} \geq 0 \quad \text{and} \quad (\Delta u)|_{\{u=0\}} \perp \lambda \tag{9}$$

by the same arguments as we used above for η (using Kato's inequality and solving the Dirichlet problem on $\{u > 0\}$). Thus, by (7) and (3) again,

$$\begin{aligned} 0 &\leq \eta(\{u=0\}) \\ &= (\Delta W\mu - \Delta u)(\{u=0\}\backslash U) \\ &\leq (\Delta W\mu)(\{u=0\}\backslash U) \\ &\leq \lambda(\{u=0\} \cap \{W\mu>0\} \cap U^c) = 0. \end{aligned} \tag{10}$$

Since $u \leq W\mu$, this shows that η is carried by $\partial U \cap \omega(\mu)$. Further, (10) shows that we can rewrite (8) as

$$(-\Delta u)|_{U^c} = (\mu_1 + \eta - \lambda)|_{\{u>0\}\backslash U}. \tag{11}$$

Also, since $\Delta u \leq \lambda$, by (6), we see from (9) that

$$(\Delta u)|_{\{u=0\}} = 0. \tag{12}$$

By a Poisson integral modification argument and the continuity of $W\mu_1$ on U, we see that

$$-\Delta u = \mu_1 - \lambda \quad \text{on the open set} \quad \{u > W\mu_1\} \cap U. \tag{13}$$

On the other hand, we can apply Kato's inequality to the non-positive function $W\mu_1 - u$ to see that

$$(-\Delta u)|_{\{u=W\mu_1\}} \leq (-\Delta W\mu_1)|_{\{u=W\mu_1\}} = (\mu_1 - \lambda)|_{\{u=W\mu_1\}\cap\omega(\mu_1)},$$

and so, by (6),

$$-\Delta u = \mu_1 - \lambda \quad \text{on} \quad \{u = W\mu_1 > 0\}. \tag{14}$$

Combining (12) – (14), we obtain

$$(-\Delta u)|_U = (\mu_1 - \lambda)|_{\{u>0\}\cap U}, \tag{15}$$

and from (11) we conclude that

$$-\Delta u = (\mu_1 + \eta - \lambda)|_{\{u>0\}}. \tag{16}$$

We now claim that $u = W(\mu_1 + \eta)$. To see this we note that, on $\{u = 0\}$, we have $W\mu_1 = 0$, so $\mu_1 \leq \lambda$ there by the structure formula, and also $\eta = 0$ there by (10). Hence $-\Delta u \geq \mu_1 + \eta - \lambda$, by (16) (which contains (12)), and since $u \geq 0$ we see that

$$u \geq W(\mu_1 + \eta). \tag{17}$$

Let $w = u - W(\mu_1 + \eta)$. By (16), (3) and (17),

$$-\Delta w = (\mu_1 + \eta - \lambda)|_{\{u>0\}\setminus\{W(\mu_1+\eta)>0\}} \leq 0,$$

because $\mu_1 + \eta \leq \lambda$ on $\{W(\mu_1 + \eta) = 0\}$. Hence w is subharmonic. Since it also has compact support, $w \equiv 0$ and the claim is proved.

It follows, by the structure formula and (10), that

$$\mathcal{B}(\mu_1 + \eta) = \lambda|_{\{u>0\}} + (\mu_1 + \eta)|_{\{u=0\}} = \lambda|_{\{u>0\}} + \mu_1|_{\{u=0\}} \leq \lambda$$

and

$$\mathcal{B}\mu = \lambda|_{\{W\mu>0\}} + \mu|_{\{W\mu=0\}}.$$

Since $u \leq W\mu$ and $\mu_1 \leq \mu$, we now see that $\mathcal{B}(\mu_1 + \eta) \leq \mathcal{B}\mu$, so part (a) of the theorem is proved.

On U^c, we know that $\mu = \mu_1$ and $u = W\mu$ quasi-everywhere. Thus $u = W\mu$ on $\widetilde{U}^c$, since U^c is non-thin at each point of $\widetilde{U}^c$. Part (b) now also follows.

It remains to establish (c). We note that $O = \{W\mu > 0\} \cap U$, that $u = W\mu = 0$ on $O^c \cap U$, and that $u = W\mu$ quasi-everywhere on $O^c \cap U^c$ (which equals U^c). Hence $W\mu - u$ vanishes quasi-everywhere on O^c, and so we can solve the Dirichlet problem in O to see that

$$(-\Delta(W\mu - u))^{O^c} = 0.$$

By assumption μ does not charge polar subsets of U^c. Since $u = W(\mu_1 + \eta) \leq W\mu$, we see from part (a) that $U(\mu_1 + \eta) \leq U\mu$, and it follows that Δu also does not charge polar subsets of U^c (see Theorem 1.XI.4(c) of [5]). Thus, by (3), (15), (16), we have

$$\begin{aligned} & (((\mu - \lambda)|_{\{W\mu>0\}} - (\mu_1 - \lambda)|_{\{u>0\}})|_O)^{O^c} \\ = & -\left((\mu - \lambda)|_{\{W\mu>0\}} - (\mu_1 + \eta - \lambda)|_{\{W\mu>0\}}\right)|_{O^c} \\ = & \ \eta|_{U^c\cap\omega(\mu)} = \eta, \end{aligned}$$

in view of (7) and (10). Now

$$\begin{aligned}(\mu-\lambda)|_{\{W\mu>0\}}-(\mu_1-\lambda)|_{\{u>0\}} &= \mu_2|_{\{W\mu>0\}}+(\mu_1-\lambda)|_{\{W\mu>0\}\backslash\{u>0\}}\\ &\leq \mu_2|_{\{W\mu>0\}}\leq\mu_2,\end{aligned}$$

since $\mu_1\leq\lambda$ on $\{u=0\}$. Thus $\eta\leq(\mu_2|_O)^{O^c}$, and the second inequality of part (c) is established.

On R we have $-\Delta W\mu=\mu-\lambda$, and $-\Delta u=\mu_1-\lambda$ by (15), so $-\Delta(W\mu-u)=\mu_2$ there. By the minimum principle, $W\mu-u\geq G_R(\mu_2|_R)$ everywhere, where $G_R\nu$ denotes the Green potential of a measure ν in R and is assigned the value 0 outside R. Also, by Kato's inequality,

$$-\Delta\left(W\mu-u-G_R(\mu_2|_R)\right)\leq 0 \quad\text{on}\quad \partial U\cap\partial R,$$

since the left hand side does not charge polar subsets of U^c. (Recall that $u=W\mu$ quasi-everywhere on ∂U, and $G_R(\mu_2|_R)=0$ quasi-everywhere on ∂R.) Hence

$$(\mu_2|_R)^{R^c}|_{\partial U}=(\Delta G_R(\mu_2|_R))|_{\partial U}\leq\left(\Delta(W\mu-u)\right)|_{\partial U\cap\partial R}\leq\eta,$$

by (7), and the left hand inequality of part (c) also holds on ∂U. ∎

Corollary 8 *Let $p\in\Omega\subset U$, where Ω is a domain and U is open, and let $t>0$. Then*

$$\omega(\lambda|_\Omega+t\delta_p)\backslash\widetilde{U}\subset\omega(\lambda|_\Omega+(t\delta_p)^{U^c}). \tag{18}$$

In particular, if $\omega((t\delta_p)^{U^c})\cap\Omega=\emptyset$, then

$$\omega(\lambda|_\Omega+t\delta_p)\subset\omega((t\delta_p)^{U^c})\cup\widetilde{U}.$$

Proof. We apply the Localization Theorem with $\mu_1=\lambda|_\Omega$ and $\mu_2=t\delta_p$. By part (b) of that result

$$\omega(\lambda|_\Omega+t\delta_p)\backslash\widetilde{U}=\omega(\lambda|_\Omega+\eta)\backslash\widetilde{U},$$

where, by part (c),

$$\eta\leq(t\delta_p)^{O^c}|_{\partial U}\leq(t\delta_p)^{U^c}.$$

Hence (18) holds. In the particular case,

$$W(\lambda|_\Omega+(t\delta_p)^{U^c})=W((t\delta_p)^{U^c}),\ \text{ and so }\ \omega(\lambda|_\Omega+(t\delta_p)^{U^c})=\omega((t\delta_p)^{U^c}),$$

whence

$$\omega(\lambda|_\Omega+t\delta_p)\subset\omega(\lambda|_\Omega+(t\delta_p)^{U^c})\cup\widetilde{U}=\omega((t\delta_p)^{U^c})\cup\widetilde{U}.$$

∎

Lemma 9 *If μ_1, μ_2 are measures with compact support, then*

$$V(\mathcal{B}\mu_1 + \mu_2) = V(\mu_1 + \mu_2), \text{ and so } \mathcal{B}(\mathcal{B}\mu_1 + \mu_2) = \mathcal{B}(\mu_1 + \mu_2).$$

Proof. Let v be an upper semicontinuous function such that $v \leq U(\mu_1+\mu_2)$ and $-\Delta v \leq \lambda$. Then $v \leq U(\mathcal{B}\mu_1 + \mu_2)$ on $\omega(\mu_1)^c$ and

$$-\Delta v \leq \lambda \leq \lambda + \mu_2 = -\Delta U(\mathcal{B}\mu_1 + \mu_2) \text{ on } \omega(\mu_1).$$

Hence $v \leq U(\mathcal{B}\mu_1 + \mu_2)$ everywhere, by the minimum principle, and so $V(\mu_1 + \mu_2) \leq V(\mathcal{B}\mu_1 + \mu_2)$. The reverse inequality is trivial. ■

Lemma 10 *If μ is a non-zero measure with compact support, and $r > 0$ is chosen to satisfy $||\mu|| = r^N \lambda(B)$, then*

$$\omega(\mu) \subset \bigcup_{x \in \mathrm{supp}\mu} \overline{B}(x, r).$$

Proof. Let $\varepsilon > 0$. We can choose a finite covering of suppμ of the form $\{B(x_j, \varepsilon) : j = 1, ..., m\}$, where $x_j \in$ suppμ for each j. Let

$$\mu_j = \mu\Big|_{B(x_j,\varepsilon)\setminus\bigcup_{i=1}^{j-1} B(x_i,\varepsilon)} \text{ and } a_j = \frac{\|\mu_j\|}{\|\mu\|} \quad (j = 1, ..., m).$$

We discard any balls $B(x_j, \varepsilon)$ for which $a_j = 0$, and then renumber the remaining m' balls so that $a_j > 0$ for each $j = 1, ..., m'$. The measure μ is supported by the union of the remaining balls. Thus $\sum \mu_j = \mu$ and $\sum a_j = 1$. Now let

$$v = \sum_{j=1}^{m'} a_j W(a_j^{-1}\mu_j).$$

It follows from Theorem 1 in Sakai [19] that, if μ_0 is a measure with support in $\overline{B}(x_0, r_0)$ and $\rho^N \lambda(B) = \|\mu_0\|$, then $\omega(\mu_0) \subset B(x_0, r_0 + \rho)$. Hence $\omega(a_j^{-1}\mu_j) \subset B(x_j, r + \varepsilon)$ for each j. Since $v \geq 0$ and

$$-\Delta v \geq \sum_{j=1}^{m'} a_j \left(a_j^{-1}\mu_j - \lambda\right) = \mu - \lambda,$$

we see that $v \geq W\mu$. Hence

$$\omega(\mu) \subset \{v > 0\} = \bigcup_{j=1}^{m'} \omega(a_j^{-1}\mu_j) \subset \bigcup_{j=1}^{m'} B(x_j, r + \varepsilon).$$

The result now follows from the compactness of suppμ and the arbitrary nature of ε. ■

Lemma 11 *Suppose that μ is a measure with compact support of the form $\mu = \sum_{j=1}^{\infty} \mu_j$, where each μ_j is a measure, $\mu_j \perp \lambda$ for each j, and there exists $\kappa \in \mathbb{N}$ such that, if $j_1 < j_2 < \ldots < j_\kappa < j_{\kappa+1}$, then*

$$\bigcap_{i=1}^{\kappa+1} \omega(\mu_{j_i}) = \emptyset.$$

Then

$$W(\mu/\kappa) \leq \frac{1}{\kappa} \sum_{j=1}^{\infty} W\mu_j \quad \textit{and} \quad \omega(\mu/\kappa) \subset \bigcup_{j=1}^{\infty} \omega(\mu_j).$$

Proof. Since $\mu_j \perp \lambda$, we see from the structure formula that $-\Delta W\mu_j = \mu_j - \lambda|_{\omega(\mu_j)}$. Hence

$$-\Delta \left(\frac{1}{\kappa} \sum_{j=1}^{\infty} W\mu_j \right) = \frac{1}{\kappa} \sum_{j=1}^{\infty} \left(\mu_j - \lambda|_{\omega(\mu_j)} \right) \geq \frac{1}{\kappa} \sum_{j=1}^{\infty} \mu_j - \lambda = \mu/\kappa - \lambda.$$

The result follows since $\kappa^{-1} \sum W\mu_j \geq 0$. ■

Lemma 12 *Let Ω be a bounded domain in $\mathbb{R}^N$, and let $p_1, p_2 \in \Omega$. If $C > 0$ and K is a compact subset of Ω such that*

$$G_\Omega(p_1, x) \geq C G_\Omega(p_2, x) \quad (x \in \Omega \backslash K),$$

then

$$\omega(\lambda|_\Omega + Ct\delta_{p_2}) \subset \omega(\lambda|_\Omega + t\delta_{p_1}) \qquad (t > 0).$$

Proof. If we extend the function $G_\Omega(\delta_{p_1} - C\delta_{p_2})$ to be zero in Ω^c, and then take its upper semicontinuous regularization, the resulting function is subharmonic on $(K \cup \{p_1\})^c$. Hence $(\delta_{p_1})^{\Omega^c} - (C\delta_{p_2})^{\Omega^c} \geq 0$, which yields the desired result, in view of parts (i) and (v) of Lemma 6. ■

Lemma 13 *Let Ω be a bounded domain in $\mathbb{R}^N$, let $p \in \Omega$ and $r > 0$, and let $r\Omega = \{rx : x \in \Omega\}$. Then*

$$\omega(\lambda|_{r\Omega} + r^N t\delta_{rp}) = r\omega(\lambda|_\Omega + t\delta_p) \qquad (t > 0).$$

Proof. Using a change of variables we see that

$$U(\lambda|_{r\Omega} + r^N t\delta_{rp})(rx) = r^2 U(\lambda|_\Omega + t\delta_p)(x)$$

and

$$U(\lambda|_{r\omega(\lambda|_\Omega + t\delta_p)})(rx) = r^2 U(\lambda|_{\omega(\lambda|_\Omega + t\delta_p)})(x).$$

The result follows, since $U(\lambda|_\Omega + t\delta_p) \geq U(\lambda|_{\omega(\lambda|_\Omega + t\delta_p)})$, with equality precisely on $\omega(\lambda|_\Omega + t\delta_p)^c$. ■

4 Proofs of Theorem 1 and Corollary 2

Lemma 14 *The notion of a boundary point of a bounded domain Ω_0 being initially stationary is independent of the choice of the point p in the definition of Ω_t $(t > 0)$.*

Proof. Let $p_1, p_2 \in \Omega_0$ and let $\Omega_{t,i}$ denote the smallest quadrature domain for subharmonic functions with respect to $\lambda|_{\Omega_0} + t\delta_{p_i}$ $(i = 1, 2)$. By Harnack's inequality there is a positive constant C such that $\mu(1) \leq C\mu(2)$ on $\partial\Omega_0$, where $\mu(i)$ denotes harmonic measure for Ω_0 and p_i. From (5),

$$\Omega_0 \subset \Omega_{t,1} = \omega(\lambda|_{\Omega_0} + t\mu(1)) \subset \omega(\lambda|_{\Omega_0} + Ct\mu(2)) = \Omega_{Ct,2}.$$

Thus, if a boundary point of Ω_0 is initially stationary for $p = p_2$, then the same is true for $p = p_1$. The lemma follows, on reversing the roles of p_1 and p_2. ■

Proof of Theorem 1. Let Ω_0, ϕ and p_0 be as in the statement of the theorem. Then there is a constant $C > 1$ such that $\phi(2t) < C\phi(t)$ for all $t > 0$. We define $\phi_1 = \phi/(4C)$. By Lemma 14 we may assume that $p = p_0$. We denote by u the upper semicontinuous regularization of the function defined to be equal to the Green function $G_{\Omega(\phi_1)}(p, \cdot)$ in $\Omega(\phi_1)$ and 0 elsewhere. Then u is subharmonic on $\mathbb{R}^N \backslash \{p\}$. Let $0 < 2\rho < r_0 < |p|$. A corollary of a result of Huber [13], as noted by Friedland and Hayman (see p.137 of [6]), tells us that

$$2\int_{S(0,r_0)} u^2\, d\widehat{\sigma} \geq \left\{\int_{S(0,\rho)} u^2\, d\widehat{\sigma}\right\} \exp\left\{2\int_\rho^{r_0/2} \frac{A(t)}{t} dt\right\},$$

where $A(t) = \alpha(\Omega(\phi_1) \cap S(0,t))$. If we denote the quantity on the left hand side above by a, then the Cauchy-Schwarz inequality yields

$$\int_{S(0,\rho)} u\, d\widehat{\sigma} \leq \left\{\int_{S(0,\rho)} u^2\, d\widehat{\sigma}\right\}^{1/2} \leq \sqrt{a} \exp\left\{-\int_\rho^{r_0/2} \frac{A(t)}{t} dt\right\}. \tag{19}$$

The Riesz measure μ associated with the subharmonic function u on $\mathbb{R}^N \backslash \{p\}$ coincides with the harmonic measure for $\Omega(\phi_1)$ and p. The hypothesis (1) and the fact that $\Omega(\phi_1) \subset \Omega(\phi)$ together imply that there is a constant $C_1 > 0$ such that

$$\exp\left\{-\int_\rho^{r_0/2} \frac{A(t)}{t} dt\right\} \leq C_1\rho^2,$$

whence

$$u(0) \leq \int_{S(0,\rho)} u\, d\widehat{\sigma} \leq C_1\sqrt{a}\rho^2,$$

in view of (19), and so $u(0)=0$. By Corollary 4.4.4 of [2],

$$\begin{aligned}\int_{S(0,\rho)} u\,d\widehat{\sigma} &= (N-2)\int_0^{\rho} t^{1-N}\mu(B(0,t))\,dt \\ &\geq (N-2)\mu(B(0,\rho/2))\int_{\rho/2}^{\rho} t^{1-N}\,dt \\ &= (2^{N-2}-1)\mu(B(0,\rho/2))\rho^{2-N}. \qquad (20)\end{aligned}$$

From (19) and (20) we see that

$$\begin{aligned}\mu(B(0,\rho/2)) &\leq \sqrt{a}\rho^{N-2}\exp\left\{-\int_{\rho}^{r_0/2}\frac{A(t)}{t}dt\right\} \\ &= C_2\rho^N\exp\left\{\int_{\rho}^{r_0/2}\frac{2-A(t)}{t}dt\right\},\end{aligned}$$

where $C_2=\sqrt{a}(r_0/2)^{-2}$. By (1) we now have

$$\mu(B(0,\rho/2))\leq C_2C_3^{-N}(\rho\phi(\rho))^N \quad (\rho<r_0/2), \qquad (21)$$

where

$$C_3=C_0\exp\left\{\frac{1}{N}\int_{r_0/2}^{|p_0|}\frac{2-A(t))}{t}dt\right\}.$$

We choose $k_0\in\mathbb{N}$ large enough so that $2^{-k_0}<r_0/2$, and then define

$$\varepsilon=\lambda(B)C^{-4N}\min\left\{C_2^{-1}(C_3/16)^N,\left(2^{-k_0-3}\phi(2^{-k_0-2})\right)^N\right\} \qquad (22)$$

and

$$\mu_{k_0}=\mu|_{B(0,2^{-k_0-1})^c}, \qquad \mu_k=\mu|_{B(0,2^{-k})\backslash B(0,2^{-k-1})} \quad (k>k_0).$$

Thus $\mu=\sum_{k=k_0}^{\infty}\mu_k$, since $\mu(\{0\})=0$. By (21) and (22),

$$\begin{aligned}\|\varepsilon\mu_k\| &\leq \varepsilon\mu(B(0,2^{-k})) \\ &\leq \varepsilon C_2C_3^{-N}\left(2^{1-k}\phi(2^{1-k})\right)^N \\ &< \left(C^{-1}2^{-k-3}\phi(2^{-k-2})\right)^N\lambda(B) \quad (k>k_0).\end{aligned}$$

The same inequality also holds when $k=k_0$, by (22) and the fact that $\left\|\mu_{k_0}\right\|\leq 1$. Since $\phi(t)\leq 1$ and $C>1$, it follows from Lemma 10 that

$$\omega(\varepsilon\mu_{k_0})\subset B(0,2^{-k_0-2})^c, \quad \omega(\varepsilon\mu_k)\subset B(0,2^{1-k})\backslash B(0,2^{-k-2}) \quad (k>k_0).$$

More precisely, if $z \in \omega(\varepsilon\mu_k)$, then Lemma 10 shows that there exists $x \in \operatorname{supp}\mu_k$ such that

$$|z - x| < C^{-1}2^{-k-3}\phi(2^{-k-2}). \tag{23}$$

Since $x \in \partial\Omega(\phi_1)$ there exists $y_0 \in \overline{\Omega}_0$ such that

$$|x - y_0| = |x|\,\phi_1(|x|) = \frac{|x|\,\phi(|x|)}{4C} \leq |x - y| \qquad (y \in \overline{\Omega}_0). \tag{24}$$

Since $|z| \geq |x|/2$, we have

$$\begin{aligned} |z - y_0| &\leq |z - x| + |x - y_0| \\ &< \frac{2^{-k-3}\phi(2^{-k-2})}{C} + \frac{|x|\,\phi(|x|)}{4C} \\ &< \frac{|z|}{2}\phi(|z|) + \frac{|z|}{2}\phi(|z|) = |z|\,\phi(|z|). \end{aligned}$$

Hence $\omega(\varepsilon\mu_k) \subset \Omega(\phi)$. Also, for any $y \in \overline{\Omega}_0$, we see from (23) and (24) that

$$\begin{aligned} |z - y| &\geq |y - x| - |z - x| \\ &> \frac{|x|\,\phi(|x|)}{4C} - \frac{2^{-k-3}\phi(2^{-k-2})}{C} \\ &\geq \frac{2^{-k-1}\phi(2^{-k-1})}{4C} - \frac{2^{-k-3}\phi(2^{-k-2})}{C} \geq 0, \end{aligned}$$

so $\omega(\varepsilon\mu_k) \cap \overline{\Omega}_0 = \emptyset$. Thus

$$\omega(\varepsilon\mu_k) \subset \Omega(\phi)\backslash\overline{\Omega}_0 \qquad (k \geq k_0).$$

By Lemma 11, with $\kappa = 3$, we thus see that $\omega((\varepsilon/3)\mu) \subset \Omega(\phi)\backslash\Omega_0$. We can now appeal to (5) and the particular case of Corollary 8, with $\Omega = \Omega_0$ and $U = \Omega(\phi_1)$, to see that

$$\Omega_t = \omega(\mu_t) \subset \omega(t\mu) \cup \widetilde{\Omega(\phi_1)} \subset \Omega(\phi) \qquad (0 < t < \varepsilon/3),$$

which completes the argument. ∎

Proof of Corollary 2. Let L, r_0 and Ω_0 be as in the statement of the corollary, and let $p_0 \in \Omega_0$. Since $\alpha(L) > 2$ we can choose a relatively open subset ω of S such that $L \subset \omega$ and $\alpha(\omega) > 2$. We next choose $\varepsilon \in (0, 1/2]$ such that $\varepsilon < \operatorname{dist}(K(L), S\backslash\omega)$, and define $\phi(t) \equiv \varepsilon$. Then $\Omega(\phi) \cap B(0, r_0/2) \subset K(\omega)$, and so

$$\alpha(\Omega(\phi) \cap S(0,t)) \geq \alpha(\omega) > 2 \quad \text{when} \quad 0 < t < r_0/2.$$

Also, we may arrange that $r_0 \leq |p_0|$. Thus

$$\exp\left\{\frac{1}{N}\int_\rho^{r_0} \frac{2 - \alpha(\Omega(\phi) \cap S(0,t))}{t}dt\right\} \leq (\rho/r_0)^{(\alpha(\omega)-2)/N} \to 0 \qquad (\rho \to 0),$$

and we can clearly choose C_0 to satisfy (1). An application of Theorem 1 now completes the argument. ∎

5 Proof of Theorems 3 and 4

We begin by noting the "integration by parts" formula

$$\int \nabla_S f \cdot \nabla_S g \, d\sigma = \int g\,(-\Delta_S f)\; d\sigma \qquad (f,g \in C^2(S)).$$

This holds because, in the notation of Section 1,

$$\begin{aligned}\int_S \nabla_S f \cdot \nabla_S g \, d\sigma &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B(0,1)\setminus B(0,1-\varepsilon)} \nabla f_* \cdot \nabla g_* \, d\lambda \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B(0,1)\setminus B(0,1-\varepsilon)} \{\nabla\cdot(g_*\nabla f_*) - g_*\Delta f_*\}\, d\lambda \\ &= \int_S g\,(-\Delta_S f)\, d\sigma,\end{aligned}$$

where

$$\int_{B(0,1)\setminus B(0,1-\varepsilon)} \nabla\cdot(g_*\nabla f_*)\; d\lambda = 0$$

by the divergence theorem, since $(\nabla f_*)(x)\cdot x = 0$.

We define the distance function $d_S(x,y)$ between points of S by

$$d_S(x,y) = \arccos(x\cdot y).$$

This metric is equivalent to the usual Euclidean one on S, and a geodesic that connects the points x and y is the minor arc between these two points lying in the intersection of S with the plane that contains the points x, y and the origin.

If we fix y, the function $g(x) = d_S(x,y)$ then satisfies $|(\nabla_S g)(x)| = 1$ for all $x \neq y$. More generally, if ω is an open subset of S with non-empty complement and we define

$$g_\omega(x) = \inf\{d_S(x,z) : z \in S\backslash\omega\} \qquad (x\in S), \tag{25}$$

then

$$|g_\omega(x) - g_\omega(y)| \le d_S(x,y) \quad (x,y\in S). \tag{26}$$

Hence g_ω is Lipschitz continuous, and so differentiable $\sigma-$almost everywhere on S. Also, clearly $g_\omega|_{S\backslash\omega} \equiv 0$. If $x\in\omega$ and γ is a geodesic connecting $x = \gamma(0)$ to a closest point $\gamma(l)$ of $\partial\omega$, parametrized by arc length, then

$$\left|\frac{g_\omega(\gamma(0)) - g_\omega(\gamma(t))}{t}\right| = \frac{d_S(\gamma(0),\gamma(t))}{t} = 1$$

and so $|(\nabla_S g_\omega)(\gamma(0))\cdot\gamma'(0)| = 1$, provided g_ω is differentiable at $\gamma(0)$. In particular,

$$|(\nabla_S g_\omega)(x)| = 1 \quad \text{for } \sigma\text{-almost every } x\in\omega.$$

We will now consider eigenvalues for the operator $-\Delta_S$ on domains $\omega \subset S$ which are not dense in S. By rotational invariance, there is no loss of generality in assuming below that $\overline{\omega} \subset S^*$, where

$$S^* = S \setminus \{(0,0,\ldots,0,1)\}.$$

The stereographic projection $\psi : S^* \to \mathbb{R}^{N-1}$ is given by

$$\psi(x_1, x_2, \ldots, x_N) = \left(\frac{x_1}{1-x_N}, \frac{x_2}{1-x_N}, \ldots, \frac{x_{N-1}}{1-x_N} \right),$$

and

$$\psi^{-1}(y_1, y_2, \ldots, y_{N-1}) = \frac{(2y_1, \ldots, 2y_{N-1}, -1 + y_1^2 + y_2^2 + \ldots + y_{N-1}^2)}{1 + y_1^2 + y_2^2 + \ldots + y_{N-1}^2}.$$

We now define

$$W_0^{1,2}(\omega) = \{u \circ \psi : u \in W_0^{1,2}(\psi(\omega))\},$$

and define weak derivatives on ω in the natural way. Since $\psi(\omega)$ is a bounded subset of $\mathbb{R}^{N-1}$, there are constants $0 < c \leq C < \infty$ such that, for each $x \in \omega$,

$$c|y|^2 \leq y^t D\psi(x)\, (D\psi(x))^t\, y \leq C|y|^2 \quad (y \in \mathbb{R}^N).$$

From this observation the following analogues of the Poincaré inequality (formula (7.44) in [9]), the compactness of the embedding in L^q (Theorem 7.22 in [9]), and Harnack's inequality for operators of the form $\Delta_S + cI$ (see Theorems 8.20 and 8.22 in [9]) are seen to hold for the space $W_0^{1,2}(\omega)$ and the operator Δ_S.

Theorem 15 (Poincaré inequality) *There is a constant $K_2 > 0$, depending on ω, such that*

$$||u||_{L^2(\omega)} \leq K_2 ||\nabla_S u||_{L^2(\omega)} \quad (u \in W_0^{1,2}(\omega)).$$

Theorem 16 (Compactness) *The space $W_0^{1,2}(\omega)$ is compactly embedded in $L^2(\omega)$.*

Theorem 17 (Harnack inequality) *Suppose that $B(y, 4R) \cap S \subset \omega$, where $y \in S$, and let $c \geq 0$. Then every $f \in W_0^{1,2}(\omega)$ which is non-negative and solves $-\Delta_S f = cf$ in ω (in the weak sense) has a continuous representative. Further, there is a constant K_h such that, for any such f,*

$$\sup_{B(y,R)\cap S} f \leq K_h \inf_{B(y,R)\cap S} f.$$

(Alternatively, this last result can be established by noting that, for a suitable choice of α, the function $x \mapsto |x|^\alpha f_*(x)$ is harmonic on $K(\omega)$.)

Recall that

$$l(\omega) = \inf \frac{\int_\omega |\nabla_S g|^2 \, d\sigma}{\int_\omega g^2 \, d\sigma},$$

where the infimum is taken over all non-zero Lipschitz functions $g : S \to [0, \infty)$ with compact support in ω. It follows from the Poincaré inequality that $l(\omega)$ is strictly positive. We can find a sequence (u_n) in $W_0^{1,2}(\omega)$ such that the corresponding sequence $\left(\int_\omega |\nabla_S u_n|^2 d\sigma\right)$ decreases to $l(\omega)$ and $\int u_n^2 d\sigma = 1$ for all n. By the compactness of the embedding of $W_0^{1,2}(\omega)$ in $L^2(\omega)$ there is a subsequence (which we still denote u_n) that converges to some function in $L^2(\omega)$. Since

$$\begin{aligned}
\int_S |\nabla_S(u_n - u_m)|^2 \, d\sigma &= 2\int_S \left(|\nabla_S u_n|^2 + |\nabla_S u_m|^2\right) d\sigma - \int_S |\nabla_S(u_n + u_m)|^2 \, d\sigma \\
&\leq 2\int_S \left(|\nabla_S u_n|^2 + |\nabla_S u_m|^2\right) d\sigma - l(\omega)\int_S (u_n + u_m)^2 \, d\sigma \\
&\to 0 \quad \text{as } m, n \to \infty,
\end{aligned}$$

we see that (u_n) converges in $W_0^{1,2}(\omega)$ to some non-zero weak solution of the equation

$$\int_S |\nabla_S u|^2 \, d\sigma = l(\omega) \int_S u^2 \, d\sigma.$$

We now define the functional

$$I(u) = \int_S |\nabla_S u|^2 \, d\sigma - l(\omega) \int_S u^2 \, d\sigma \qquad (u \in W_0^{1,2}(\omega)).$$

If u is a minimizer of this expression (that is, $I(u) = 0$), then for any smooth function φ on S with compact support in ω we consider the function f given by

$$f(t) = I(u + t\varphi) \qquad (t \in \mathbb{R}).$$

Then

$$f'(0) = 2\left(\int_S \nabla_S u \cdot \nabla_S \varphi \, d\sigma - l(\omega)\int_S u\varphi \, d\sigma\right) = 0,$$

so

$$\int_S \nabla_S u \cdot \nabla_S \varphi \, d\sigma = l(\omega) \int_S u\varphi \, d\sigma$$

for all such φ. Thus, using integration by parts,

$$-\Delta_S u = l(\omega) u$$

in the weak sense. In particular, $l(\omega)$ is the first eigenvalue of $-\Delta_S$ on ω.

We note that $I(|u|) = I(u)$. Hence, if there is a minimizer u, then $|u|$ is also a minimizer. However, a non-negative minimizer u clearly satisfies $-\Delta_S u =$

$l(\omega)u \geq 0$. Hence, if it takes the value zero at some point, then it does so throughout ω, by Harnack's inequality. This excludes the possibility of solutions which change sign in ω. By the connectedness of ω, the eigenspace corresponding to $l(\omega)$ is one-dimensional and there is a strictly positive solution. Further, only this smallest eigenvalue can have a strictly positive associated eigenfunction on ω. To see this we note that, if u' is an eigenfunction associated with a different eigenvalue l', then integration by parts yields

$$l(\omega)\int_\omega uu'\,d\sigma = \int_\omega \nabla_S u \cdot \nabla_S u'\,d\sigma = l'\int_\omega uu'\,d\sigma,$$

and so u' must have variable sign.

We will also need the following Hadamard-type formula for the dependence of the eigenvalues on the domain:

Theorem 18 (Hadamard Formula for l) *Given a domain $\omega \subset S$, where $\overline{\omega} \neq S$ and $\partial\omega$ is Lipschitz, there are positive numbers b, ε such that*

$$l(\omega_\delta) \geq l(\omega) + b\delta \quad (0 < \delta < \varepsilon),$$

where

$$\omega_\delta = \{y \in S : d_S(y, S \setminus \omega) > \delta\}.$$

Proof. Without loss of generality we may assume that $\overline{\omega} \subset S^*$. Let $u_\delta \in W_0^{1,2}(\omega_\delta)$ be a nonnegative function satisfying

$$l(\omega_\delta) = \int_{\omega_\delta} |\nabla_S u_\delta|^2\,d\sigma \quad \text{and} \quad \int_{\omega_\delta} u_\delta^2\,d\sigma = 1.$$

We define $u_\delta = 0$ in ω_δ^c and so can regard u_δ as a function in $W_0^{1,2}(\omega)$. Since $l(\omega_\delta)$ is clearly an increasing function of δ, we see that, for a given $\delta' > 0$,

$$\int_\omega |\nabla_S u_\delta|^2\,d\sigma \leq l(\omega_{\delta'}) \quad (0 < \delta < \delta').$$

Hence the set $\{u_\delta : \delta \in (0, \delta')\}$ is bounded in $W_0^{1,2}(\omega)$. Since $W_0^{1,2}(\omega)$ is compactly embedded in $L^2(\omega)$, it follows that, for every sequence (δ_n) in $(0, \delta')$ satisfying $\delta_n \to 0$, there is a convergent subsequence, which we still denote by (δ_n), such that (u_{δ_n}) converges in $L^2(\omega)$ to some function v satisfying $\int_\omega v^2\,d\sigma = 1$. In particular, we see that there must be numbers $\varepsilon, c > 0$ such that

$$\int_\omega u_\delta\,d\sigma \geq c \quad (0 < \delta < \varepsilon).$$

Let $\phi_\delta = \min\{\delta, g_\omega\}$, where g_ω is defined as in (25). Also, let $\mathcal{H}^{N-2}(E)$ denote the $(N-2)$-dimensional Hausdorff content of a set E. It is easy to see that ϕ_δ is Lipschitz continuous and $\phi_\delta \in W_0^{1,2}(\omega)$. Since $\nabla_S u_\delta = 0$ on $\omega \backslash \omega_\delta$ and $\nabla_S \phi_\delta = 0$ on ω_δ we have, for all $t > 0$,

$$\begin{aligned}
l(\omega) &\leq \frac{\int_\omega |\nabla_S(u_\delta + t\phi_\delta)|^2 \, d\sigma}{\int_\omega (u_\delta + t\phi_\delta)^2 \, d\sigma} \\
&= \frac{\int_\omega |\nabla_S u_\delta|^2 \, d\sigma + t^2 \int_\omega |\nabla_S \phi_\delta|^2 \, d\sigma}{\int_{\omega_\delta} u_\delta^2 \, d\sigma + 2t \int_{\omega_\delta} \phi_\delta u_\delta \, d\sigma + t^2 \int_\omega \phi_\delta^2 \, d\sigma} \\
&= \frac{l(\omega_\delta) + t^2 \sigma(\omega \setminus \omega_\delta)}{1 + 2t\delta \int_{\omega_\delta} u_\delta \, d\sigma + t^2 \int_\omega \phi_\delta^2 \, d\sigma} \\
&= \left(l(\omega_\delta) + t^2 \left(a\delta \mathcal{H}^{N-2}(\partial\omega) + o(\delta)\right)\right) \left(1 - 2t\delta \int_{\omega_\delta} u_\delta \, d\sigma + t^2 o(\delta)\right) \\
&= l(\omega_\delta) - 2t\delta l(\omega_\delta) \int_{\omega_\delta} u_\delta \, d\sigma + at^2 \delta \mathcal{H}^{N-2}(\partial\omega) + (t^2 + t^4) o(\delta) \\
&\leq l(\omega_\delta) - \delta(2tcl(\omega) - at^2 \mathcal{H}^{N-2}(\partial\omega)) + (t^2 + t^4) o(\delta), \qquad (27)
\end{aligned}$$

for a suitable constant $a > 0$. Next, we choose $t = cl(\omega)a^{-1}/\mathcal{H}^{N-2}(\partial\omega)$, whence

$$2tcl(\omega) - at^2 \mathcal{H}^{N-2}(\partial\omega) = c^2 l(\omega)^2 a^{-1}/\mathcal{H}^{N-2}(\partial\omega).$$

Then we may choose $\varepsilon > 0$ so small that the $o(\delta)$ term in (27) satisfies

$$(t^2 + t^4) o(\delta) < \delta c^2 l(\omega)^2 (2a)^{-1}/\mathcal{H}^{N-2}(\partial\omega) \quad (0 < \delta < \varepsilon).$$

The proof is completed by choosing $b = c^2 l(\omega)^2 (2a)^{-1}/\mathcal{H}^{N-2}(\partial\omega)$. ∎

The above theorem remains valid if we replace d_S by the usual Euclidean metric, since these are equivalent metrics on S. This yields the following immediate consequence.

Corollary 19 (Hadamard Formula for $\boldsymbol{\alpha}$) *Given a domain $\omega \subset S$, where $\overline{\omega} \neq S$ and $\partial\omega$ is Lipschitz, there are positive numbers a, ε such that*

$$\alpha(\{y \in S : \mathrm{dist}(y, S \setminus \omega) > \delta\}) \geq \alpha(\omega) + a\delta \quad (0 < \delta < \varepsilon).$$

Proof of Theorem 3. Let ω be as in the statement of the result. By the above formula there are positive numbers a and ε such that

$$\alpha(\{y \in S : \mathrm{dist}(y, S \backslash \omega) > \delta)\} \geq 2 + a\delta \quad (0 < \delta < \varepsilon).$$

Let $C(\omega) = N/a$ and $p_0 \in \Omega_0$, where Ω_0 is given by (2) and

$$\phi(t) = \begin{cases} (\log(1/t))^{-1} & (0 < t < e^{-2}) \\ 1/2 & (t \geq e^{-2}) \end{cases} .$$

Then

$$\begin{aligned}\alpha(\Omega(\phi)\cap S(0,t)) &\geq 2+a\frac{C(\omega)}{\log(1/t)}\\ &= 2+\frac{N}{\log(1/t)} \qquad (0<t<\min\{e^{-2},e^{-C(\omega)/\varepsilon}\}),\end{aligned}$$

so

$$\begin{aligned}&\exp\left\{\frac{1}{N}\int_\rho^{|p_0|}\frac{2-\alpha(\Omega(\phi)\cap S(0,t))}{t}dt\right\}\\ \leq\;& \exp\left\{-\int_\rho^{\min\{e^{-2},e^{-C(\omega)/\varepsilon}\}}\frac{dt}{t\log(1/t)}-\frac{1}{N}\int_{\min\{e^{-2},e^{-C(\omega)/\varepsilon}\}}^{|p_0|}\frac{2}{t}dt\right\}\\ =\;& C\exp\left(\log\left(\log\max\left\{e^2,e^{C(\omega)/\varepsilon}\right\}\right)-\log\left(\log\frac{1}{\rho}\right)\right)\\ =\;& C\max\left\{2,\frac{C(\omega)}{\varepsilon}\right\}\phi(\rho) \qquad (0<\rho<\min\{e^{-2},e^{-C(\omega)/\varepsilon}\}),\end{aligned}$$

and the result now follows from Theorem 1. ■

Proof of Theorem 4. (a) It is enough to consider the case where $\Omega_0 = K(\omega)\cap B(0,r_0)$. By Lemma 14 we may suppose that $p\in\Omega_0\backslash\overline{B}(0,r_0/2)$. We saw above that $l(\omega)$ is the first eigenvalue of $-\Delta_S$ on the spherical domain ω, and that there is an associated eigenfunction which is positive. This function vanishes continuously on the boundary of ω in S. It follows that there is a positive harmonic function h on $K(\omega)$ of the form

$$h(ry)=r^{\alpha(\omega)}h(y) \qquad (y\in S, r>0), \tag{28}$$

where $h(y)\to 0$ at the boundary of ω in S. Since $K(\omega)$ has a Lipschitz boundary, we know from the boundary Harnack principle (see Section 8.7 of [2]) that there is a positive constant C_1 such that

$$G_{\Omega_0}(p,x)\geq C_1h(x) \qquad (x\in K(\omega)\cap B(0,r_0/2)). \tag{29}$$

Further, by the smoothness of the boundary of ω in S, there is a positive constant C_2 such that

$$h(y)\geq C_2\mathrm{dist}(y,S\backslash\omega) \qquad (y\in S).$$

(See, for example, Widman [21].) Since the density of harmonic measure with respect to surface area measure is proportional to the normal derivative of the Green function, it follows from a scaling argument that there is a positive constant C_3 such that the harmonic measure μ for Ω_0 and p satisfies

$$d\mu(x)\geq C_3\,|x|^{\alpha(\omega)-1}\,d\sigma(x) \quad \text{on} \quad \partial\Omega_0\cap B(0,r_0/4). \tag{30}$$

If $\alpha(\omega) < 2$, then there is a positive constant C_4 such that

$$\rho^{-N}\mu(B(0,\rho)) \geq C_4\rho^{\alpha(\omega)-2} \to \infty \quad (\rho \to 0+),$$

and so $0 \in \Omega_t$ for all $t > 0$, in view of (5) and (4).

Now suppose that $\alpha(\omega) = 2$. It follows from (30) that there is a positive constant C_5 such that

$$\mu(B(z,\rho\,|z|)) \geq \frac{C_5}{\rho}(\rho\,|z|)^N \quad (z \in \partial K(\omega); 0 < |z| < r_0/8; 0 < \rho < 1/2),$$

since $\sigma(B(z,\rho\,|z|) \cap \partial\Omega_0)$ is comparable to $(\rho\,|z|)^{N-1}$. Thus, if $\varepsilon > 0$, there exists $\rho > 0$ small enough so that

$$\varepsilon\mu(B(z,\rho\,|z|)) > (2\rho\,|z|)^N\lambda(B) \quad (z \in \partial K(\omega); 0 < |z| < r_0/8).$$

Hence, by (5) and (4) again,

$$\Omega_\varepsilon \supset \omega(\varepsilon\mu) \supset K(\omega_1) \cap B(0, r_0/8),$$

where ω_1 is a domain in S that contains $\overline{\omega}$. By Lemma 9,

$$V(\lambda|_{\Omega_0} + (t+\varepsilon)\delta_p) = V(\mathcal{B}(\lambda|_{\Omega_0} + \varepsilon\delta_p) + t\delta_p) = V\left(\lambda|_{\Omega_\varepsilon} + t\delta_p\right).$$

Noting that $U(\lambda|_{\Omega_0} + (t+\varepsilon)\delta_p) = U\left(\lambda|_{\Omega_\varepsilon} + t\delta_p\right)$ outside Ω_ε, it follows that $\Omega_{t+\varepsilon} = \omega(\lambda|_{\Omega_\varepsilon} + t\delta_p)$, and since $\alpha(\omega_1) < 2$ we see from the previous case that $0 \in \Omega_{t+\varepsilon}$ for all $t > 0$. Since ε can be arbitrarily small, we see that $0 \in \Omega_t$ for all $t > 0$.

(b) The Martin compactification of $K(\omega)$ is homeomorphic to $\overline{K(\omega)} \cup \{\infty\}$, and the Martin function with pole at ∞ is a multiple of the function h in the proof of part (a) above. (See, for example, Kuran [15].) It follows, by the Kelvin transformation, that we have a minimal positive harmonic function h_0 on $K(\omega)$ with pole at 0 of the form

$$h_0(ry) = r^{2-N-\alpha(\omega)}h(y) \quad (y \in S, r > 0).$$

We recall that a set $E \subset K(\omega)$ is said to be minimally thin at 0 with respect to $K(\omega)$ if there is a positive superharmonic function v on $K(\omega)$ such that

$$\inf_E \frac{v}{h_0} > \inf_{K(\omega)} \frac{v}{h_0}.$$

(See Chapter 9 of [2] for an introduction to the notion of minimal thinness.) The hypotheses on ψ imply that

$$\int_{(K(\omega)\backslash\Omega_0)\cap B} |x|^{-N}\, d\lambda(x) < \infty.$$

It follows from this last condition, by Theorems 2 and 3 of [17] and inversion, that $K(\omega)\backslash\Omega_0$ is minimally thin at 0 with respect to $K(\omega)$.

Now let $p \in \Omega_0$. By our assumptions on ψ, there exist $r_0 \in (0,1)$ and $R \in (0, |p|/2)$ such that $B(p, 2R) \subset \Omega_0$ and

$$K(\{x/|x| : x \in B(p, 2R)\}) \cap B(0, r_0) \subset \Omega_0.$$

Let $\omega' = \{x/|x| : x \in B(p, R)\}$. For any sequence $\rho_n \downarrow 0$ the set $\cup_n\{K(\overline{\omega'}) : \rho_n \leq |x| \leq 2\rho_n\}$ is not minimally thin at 0 with respect to $K(\omega)$ (by Theorem 1.1 in [1], or Theorem 2 of [16]). Hence, by Theorem 9.6.2(ii) of [2] and Harnack's inequalities, there are constants $C_1 > 0$ and $r_1 \in (0, r_0/2)$ such that

$$G_{\Omega_0}(p, x) \geq C_1 G_{K(\omega)}(p, x) \quad (x \in K(\overline{\omega'}) \cap B(0, r_1)).$$

Also, in view of (28) and (29), there is a constant $C_2 > 0$ such that

$$G_{K(\omega)}(p, x) \geq C_2 |x|^2 \quad (x \in K(\overline{\omega'}) \cap B(0, r_1)),$$

because $\inf_{y \in \omega'} h(y) > 0$. For $r < r_1/2$ we have

$$\overline{B(rp, rR)} \subset K(\overline{\omega'}) \cap B(0, r_1) \subset \Omega_0$$

and hence

$$\begin{aligned} \frac{1}{r^N} G_{\Omega_0}(p, x) &\geq C_1 C_2 \frac{(|p| - R)^2}{r^{N-2}} \\ &\geq C_1 C_2 R^N |rp - x|^{2-N} \\ &\geq C_3 G_{\Omega_0}(rp, x) \quad (x \in \partial B(rp, rR), r < r_1/2), \end{aligned}$$

where $C_3 = C_1 C_2 c_N^{-1} R^N$. Thus, by the maximum principle,

$$G_{\Omega_0}(p, x) \geq C_3 r^N G_{\Omega_0}(rp, x) \quad (x \in \Omega_0 \backslash B(rp, rR)),$$

and Lemma 12 now yields

$$\omega(\lambda|_{\Omega_0} + t\delta_p) \supset \omega(\lambda|_{\Omega_0} + C_3 r^N t\delta_{rp}) \quad (t > 0). \tag{31}$$

Let

$$\begin{aligned} \omega_n &= \{x \in S : x/n \in \Omega_0\} \\ &= \{x \in S : \operatorname{dist}(x/n, K(\omega)^c) > \psi(1/n)/n\} \\ &= \{x \in S : \operatorname{dist}(x, K(\omega)^c) > \psi(1/n)\} \quad (n \in \mathbb{N}). \end{aligned}$$

Since ψ is increasing and $\int_0^\infty t^{-1}\psi(t)\, dt < \infty$, which implies that $\lim_{t\to 0}\psi(t) = 0$, we see that (ω_n) increases to ω, and so $(K(\omega_n))$ increases to $K(\omega)$. We now fix $t > 0$. By construction $\Omega_0 \subset K(\omega) \cap B$. Since $\alpha(\omega) = 2$, part (a) of the theorem shows that $0 \in \omega(\lambda|_{K(\omega)\cap B} + C_3 t\delta_p)$. Thus, by Lemma 6(iii), there exists $n \in \mathbb{N}$ such that $1/n < r_1/2$ and

$$0 \in \omega(\lambda|_{K(\omega_n)\cap B} + C_3 t\delta_p). \tag{32}$$

(The value of n will depend on t, since $\alpha(\omega_n) > 2$ in general.) Let $r = 1/n$. The definition of ω_n, and the fact that ψ is increasing, together ensure that $K(\omega_n) \cap B(0,r) \subset \Omega_0$. Hence

$$\begin{aligned}\omega(\lambda|_{\Omega_0} + C_3 r^N t\delta_{rp}) &\supset \omega(\lambda|_{K(\omega_n)\cap B(0,r)} + C_3 r^N t\delta_{rp}) \\ &= r\omega(\lambda|_{K(\omega_n)\cap B} + C_3 t\delta_p),\end{aligned}$$

by Lemma 13. Combining this with (31) and (32), we now see that $0 \in \omega(\lambda|_{\Omega_0} + t\delta_p)$. The proof is complete, since t can be arbitrarily small. ■

References

[1] H. Aikawa, "On the minimal thinness in a Lipschitz domain", *Analysis* 5 (1985), 347–382

[2] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer, London, 2001.

[3] J. Bourgain, “On the Hausdorff dimension of harmonic measure in higher dimension”, *Invent. Math.* 87 (1987), 477–483.

[4] H. Brezis and A. C. Ponce, “Kato’s inequality when Δu is a measure”, *C. R. Acad. Sci. Paris, Ser. I* 338 (2004) 599–604.

[5] J. L. Doob, *Classical potential theory and its probabilistic counterpart.* Springer, New York, 1984.

[6] S. Friedland and W. K. Hayman, “Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions”, *Comment. Math. Helv.* 51 (1976), 133–161.

[7] S. J. Gardiner and T. Sjödin, “Partial balayage and the exterior inverse problem of potential theory”, in Bakry, D. (ed.) et al., *Potential theory and stochastics in Albac*, pp.111–123, Bucharest, Theta, 2009.

[8] S. J. Gardiner and T. Sjödin, “Two-phase quadrature domains”, *J. Anal. Math.* 116 (2012), 335–354.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1983.

[10] B. Gustafsson, *Variational Inequality Formulation of a Moving Boundary Problem in Rock Mechanics*, TRITA-MAT-1984-8, Department of Mathematics, KTH Stockholm, 1984.

[11] B. Gustafsson and M. Sakai, "Properties of some balayage operators, with applications to quadrature domains and moving boundary problems", *Nonlinear Anal.* 22 (1994), 1221-1245.

[12] W. Hansen and H. Hueber, "Singularity of harmonic measure for sub-Laplacians", *Bull. Sci. Math. (2)* 112 (1988), 53–64.

[13] A. Huber, "Über Wachstumseigenschaften gewisser Klassen von subharmonischen Funktionen", *Comment. Math. Helv.* 26 (1952), 81–116.

[14] J. R. King, A. A. Lacey and J. L. Vázquez, "Persistence of corners in free boundaries in Hele-Shaw flow", *European J. Appl. Math.* 6 (1995), 455–490.

[15] Ü. Kuran, "On NTA-conical domains", *J. London Math. Soc. (2)* 40 (1989), 467–475.

[16] I. Miyamoto, M. Yanagishita and H. Yoshida, "Beurling-Dahlberg-Sjögren type theorems for minimally thin sets in a cone", *Canad. Math. Bull.* 46 (2003), 252–264.

[17] I. Miyamoto, M. Yanagishita and H. Yoshida, "On harmonic majorization of the Martin function at infinity in a cone", *Czechoslovak Math. J.* 55(130) (2005), 1041–1054.

[18] M. Sakai, "Applications of variational inequalities to the existence theorem on quadrature domains", *Trans. Amer. Math. Soc.* 276 (1983), 267–279.

[19] M. Sakai, "Sharp estimates of the distance from a fixed point to the frontier of a Hele-Shaw flow", *Potential Anal.* 8 (1998), 277–302.

[20] M. Sakai, *Small modifications of quadrature domains.* Mem. Amer. Math. Soc. 206 (2010), no. 969.

[21] K.-O. Widman, "Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations", *Math. Scand.* 21 (1967),17–37.

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