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THE HECKSCHER-OHLIN MODEL AS AN AGGREGATE*

J. Peter Neary

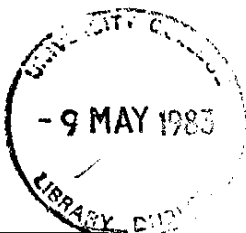
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THE HECKSCHER-OHLIN MODEL AS AN AGGREGATE

A B S T R A C T

This paper extends the results of Neary (1978) where it was shown that, when an appropriate aggregation procedure is adopted, many of the properties of the two-factor, two-commodity Heckscher-Ohlin model continue to hold in a general production model with any numbers of goods and factors. In the present paper it is shown that, if the numbers of goods and factors are the same (though still arbitrarily large) and if joint production and intermediate goods are ruled out, then the parallel between the properties of the standard Heckscher-Ohlin model and the aggregated model is greatly strengthened. In particular, the equalization of factor prices between "similar" free-trading economies and, with some additional restrictions, the "magnification" effects associated with the Rybczynski and Stolper-Samuelson theorems are shown to hold in the aggregated model.



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THE HECKSCHER-OHLIN MODEL AS AN AGGREGATE

"Those who continue to study factor intensities either do not understand the problem or they believe wrongly that factors can somehow be aggregated into two groups, capital and labour, without serious disturbance of the theory".

I.F. Pearce: International Trade, p. 482.

1 Introduction

The opening quotation reflects the widespread view that the Heckscher-Ohlin model of international trade, with its emphasis on sectoral factor intensities, cannot fruitfully be used as a basis for detailed predictions in models of higher dimension than the textbook two-factor, two-commodity case. The purpose of the present paper is not to dispute this viewpoint but rather to suggest that there nevertheless exists a method of aggregation which enables a limited role to be salvaged for the two-by-two Heckscher-Ohlin model as a predictor of comparative statics responses in more general models.

Despite Pearce's admonitions, a number of attempts have been made to generalize the Heckscher-Ohlin model to cases with many commodities and factors.¹ One approach, pursued by Chipman (1969), Kemp and Wegge (1969), Uekawa (1971) and Inada (1971) among others, has attempted to find restrictions on the matrix of input-output coefficients sufficient to ensure that the general model exhibits properties analogous to those of the two-factor, two-commodity model: for example, that an increase in a commodity price should lead to a more-than-proportionate

increase in one factor price and a fall in all other factor prices. Unfortunately, the restrictions which have been found to guarantee outcomes such as this are relatively stringent and difficult to interpret in economic terms. A second approach, initiated by Ethier (1974) and pursued by Kemp and Wan (1976), Diewert and Woodland (1977), Jones and Scheinkman (1977) and Chang (1979), has attempted to establish properties which must hold in the general model without placing any restrictions on the matrix of input-output coefficients. This approach has clarified a number of issues concerning the nature of production relationships in a general model, but the results to which it leads are not very strong (for example, in the absence of joint production, a commodity price increase must lead to a more than proportionate increase in some factor price).

Rather than attempting to pursue these two directions of research, the present paper follows a very different route, which was suggested in Neary (1979). This alternative approach is based on two principles. The first, which is amply illustrated by the papers already mentioned, is that it is fruitless to seek results in many-factor, many-commodity models which are at a level of detail comparable to those available in the two-by-two model. As already noted, this viewpoint has been forcefully argued by Pearce (1970) and, more recently, by Dixit and Norman (1980). The second principle, put forward by Jones (1977), is that even in models of higher dimensions, a "two-by-two" perspective may still prove useful in answering certain limited sets of questions. The approach, based on these two principles, which was introduced in Neary (1979), is to place restrictions not on the technology of a general model but rather on the range of questions which are posed in the con-

text of the model. It was shown there that, provided sufficient restrictions of this kind are imposed, it is possible to aggregate the general model to a form which exhibits many of the properties of the textbook two-by-two model. The objective of the present paper is to investigate whether these properties may be strengthened when the general approach is combined with the imposition of restrictions on the technology of the model itself: in particular, with the assumption that the number of factors and commodities is the same.

The plan of the paper is as follows. Section 2 introduces notation and recaps the general results of Neary (1979). Sections 3 and 4 then apply these results to the special case of equal numbers of goods and factors and note some additional restrictions under which the aggregated model exhibits all the properties of the textbook two-by-two model. These results are illustrated in Section 5 in the context of the model of Gruen and Corden (1970). Section 6 concludes by attempting to put the results of the paper in context while the Appendix gives explicit expressions for some of the equations in the text and also relates our approach to a result of Dixit and Norman's.

2 Aggregation in a General Production Model

The general model is a perfectly competitive one, whose technology can be characterized by the following transformation function:

$$F(v, z) = 0, \quad (2.1)$$

where v is an M -by-1 vector of given factor endowments and z is an N -by-1 vector of endogenous net output levels. Throughout the paper, we assume that there are no obstacles, such as factor-

market or other distortions, impeding the efficient allocation of resources within the economy. In addition, we assume not merely that the aggregate production possibilities set defined implicitly by (2.1) is convex, but that the technology in each individual sector exhibits constant returns to scale. (This is not a very restrictive additional assumption, since increasing returns to scale would be inconsistent with the assumptions we have already made of production efficiency and competitive behaviour, while decreasing returns to scale in any sector can be accommodated by defining a fictitious additional factor which is paid the excess of revenue over costs.) In all other respects the model economy described by (2.1) is an extremely general one: some or all of the outputs may be produced jointly and may be used as intermediate inputs, and no restrictions are placed on the extent to which factors are mobile or immobile between sectors.

Under these assumptions, the economy can be viewed 'as if' it were maximizing, subject to (2.1), a single objective function, namely, national product valued at the commodity price vector s . In order to ensure that all behavioural responses are continuous, we assume in addition that the aggregate production surface defined by the function $F(\cdot)$ contains no "flats" or straight-line segments. This in turn requires that the number of primary factors be at least as great as the number of productive activities (which, if there is no joint production, is the same as the number of commodities produced). For the remainder of the paper we assume that this is the case; provided we interpret our analysis as applying to a small open economy which faces exogenous prices for all commodities, this is not a major restriction.

As is well-known from the work of Samuelson (1953) and others, the assumptions which we have made are sufficient to ensure a number of important properties. In particular, competitive output supplies are non-decreasing functions of prices, since the matrix of commodity price-output responses, $\partial z/\partial s$, is positive semi-definite; competitive factor returns are non-increasing functions of endowments, since the matrix of endowment-factor price responses, $\partial t/\partial v$, is negative semi-definite; and, finally, the cross-responses are "reciprocal", in the sense that the matrices of cross derivatives, $\partial z/\partial v$ and $\partial t/\partial s$, are transposes of one another. Unfortunately, these results are still far short of the level of factor-by-factor and commodity-by-commodity detail which we would require if the textbook two-factor two-commodity model could be generalized to higher dimensions. However, in an earlier paper, Neary (1979), it was suggested that much of the spirit of the two-by-two approach can be salvaged not by relaxing the generality of the model (i.e. by putting more structure on the function $F(\cdot)$), but instead by greatly limiting the range of questions which it is called upon to answer.

To operationalize this approach, we first partition the sets of commodities and factors into two mutually exclusive subsets. For commodities we denote these by vectors x and y , of dimensions n -by-1 and $(N-n)$ -by-1 respectively, and the commodity price vector s is partitioned conformably:

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad s = \begin{bmatrix} p \\ q \end{bmatrix} \quad (2.2)$$

Similarly the endowment vector v is broken into two vectors ℓ and k , of dimensions m -by-1 and $(M-m)$ -by-1 respectively, with a conformable partition of the factor price vector t :

$$v = \begin{bmatrix} \ell \\ k \end{bmatrix} \quad t = \begin{bmatrix} w \\ r \end{bmatrix} . \quad (2.3)$$

So far, these changes are purely a matter of notation. The next step is to assume that exogenous shocks take the form only of changes in the levels of the exogenous variables in each partition. Thus, for commodities, we rule out changes in relative prices within each partition. Formally, this may be written as follows:

$$p = P\pi \quad q = Q\psi , \quad (2.4)$$

where π and ψ are vectors of base-period prices, assumed constant henceforward, and only changes in the scalar price levels P and Q are permitted. This allows us to define Hicksian composite commodities X and Y , which are weighted averages of the outputs of the individual commodities in the corresponding partition, the weights being the base-period prices:²

$$X = \pi'x \quad Y = \psi'y . \quad (2.5)$$

Turning to factors, it is not possible to follow exactly the same procedure as with commodities, since the factor prices w and r are endogenous and so it is not in general permissible to place restrictions on their variability. However, it is possible to carry out an exactly dual procedure by ruling out changes in relative endowments within each partition of the vector of factors:

$$\ell = L\phi \quad k = K\mu , \quad (2.6)$$

where ϕ and μ are vectors of base-period endowments, assumed constant henceforward. This allows us to treat the scalar endowment levels L and K as Leontief composite factors, earning rewards W and R which are weighted averages of the factor prices in the corres-

ponding partition, the weights being the base-period endowments:³

$$W = \phi'w \quad R = \mu'r \quad (2.7)$$

As shown in detail in Neary (1979), these procedures lead to a number of results, of which the most interesting are the following:

1 Factor Intensities and Factor Rewards

The relationship between proportional changes in relative endowment levels and the resulting proportional changes in the relative outputs of composite commodities may be written as follows:

$$\hat{X} - \hat{Y} = \Lambda^{-1} (\hat{L} - \hat{K}) \quad (2.8)$$

where a circumflex denotes a proportional rate of change (e.g. $\hat{X} = d \ln X$ for any scalar X), and Λ is a scalar parameter whose value depends on all the exogenous variables and on the form of the function F . We define commodity X as labour-intensive at the margin if and only if Λ is positive. We can also derive a similar equation linking proportional changes in the relative composite commodity price ratio and the resulting proportional changes in the relative rewards of the composite factors:

$$\hat{W} - \hat{R} = \Theta^{-1} (\hat{P} - \hat{Q}) \quad (2.9)$$

where Θ is another scalar parameter. The key result is that Λ and Θ have the same sign. Thus for example, an increase in the relative price of the composite commodity which is labour-intensive at the margin raises the relative return to the composite factor labour. Our notation has been deliberately chosen to underline the parallel between this result and a well-known property of the textbook two-good, two-factor model, as derived by Jones (1965), among others.

II Quasi-Stolper-Samuelson Theorem

The preceding result is not an exact equivalent of the Stolper-Samuelson theorem, since it does not exhibit what Jones (1965) has called the "magnification" effect, whereby a given change in relative commodity prices gives rise to a greater (i.e. magnified) change in relative factor prices. In fact the magnification effect does not necessarily hold in the general model, but it is possible to derive results analogous to the Stolper-Samuelson theorem for the effects of changes in relative commodity prices on the real rewards of composite factors. Thus, for the real wage, we have the following:

$$\hat{W} - (\alpha_X \hat{P} + \alpha_Y \hat{Q}) = [\theta_K \theta^{-1} + (\theta_X - \alpha_X)] (\hat{P} - \hat{Q}) . \quad (2.10)$$

The parameter α_j is the share of composite commodity j in wage-earners' consumption and so the left-hand side of equation (2.10) is the proportional change, to first order, in the real reward of wage-earners. The parameter θ_j is the share of factor or commodity j in the value of national product. Hence, equation (2.10) states that an increase in the relative price of the composite commodity which is labour-intensive at the margin increases the real wage if wage-earners' tastes do not diverge "too far" from average (so that α_X is close to the share of X in average consumption) and if the relatively labour-intensive commodity is a net export (so that θ and $(\theta_X - \alpha_X)$ have the same sign).

III "As-If" Input-Output Coefficients

Finally, it is possible to define four coefficients A_{ij} ($i=L, K$; $j=X, Y$) which possess a number of properties of the familiar sectoral input-output coefficients a_{ij} ($=v_{ij}/z_j$). In particular, they satisfy

"full-employment" conditions for each composite factor:

$$A_{LX}X + A_{LY}Y = L , \quad (2.11)$$

$$A_{KX}X + A_{KY}Y = K , \quad (2.12)$$

as well as "zero-profit" conditions for each composite sector:

$$A_{LX}W + A_{KX}R = P , \quad (2.13)$$

$$A_{LY}W + A_{KY}R = Q . \quad (2.14)$$

These coefficients measure the concept of "factor intensity at the margin" in exactly the same way as actual input-output coefficients measure the usual concept of factor intensity in the two-factor, two-commodity model: the determinant of the A_{ij} matrix has the same sign as the parameters λ and θ already introduced. Moreover, while the values of the A_{ij} coefficients are altered by changes in the units in which composite factors or commodities are measured, the same is not true of the sign of the determinant of the matrix formed from them. As Pearce (1970, p. 489) has stressed, this invariance of the measure of factor intensity with respect to changes in units of measurement is an essential feature of any satisfactory method of aggregating commodities or factors.

Of course, whereas most statements of the two-factor, two-commodity model begin by presenting equations such as (2.11) to (2.14) expressed in terms of actual input-output coefficients a_{ij} (see, for example, Jones (1965)), these equations are derived here as properties of the model. Hence, while the as-if coefficients A_{ij} have a number of features in common with actual coefficients, they also differ from them in important ways. In particular, they depend not only on the technology of the composite sector to which they refer, but also on technology in the other sector as well as on the economy



wide factor endowments. It is clear therefore that the interpretation of the A_{ij} coefficients is very different from that of actual input-output coefficients a_{ij} . (This issue is pursued below.)

A second drawback of the as-if input-output coefficients is that it is quite possible for one or more of them to be negative in perfectly well-behaved models, which makes their interpretation even more problematic (examples of this are given below). However, this is hardly surprising, since if all the A_{ij} coefficients were positive, then we could begin with equations (2.11) to (2.14) and derive all the strongest properties of the two-good, two-factor model in exactly the same manner as Jones (1965). In particular, the aggregate model would then exhibit the magnification effects associated with both the Rybczynski theorem (an increase in the endowment of one factor causes a more than proportionate increase in the output of the sector which uses it relatively intensively) and the Stolper-Samuelson theorem (an increase in the relative price of one commodity causes a more than proportionate increase in the real reward of the factor used intensively in the production of that commodity).

The next question to which we must turn therefore, which was left unanswered in Neary (1979), is under what circumstances all the A_{ij} coefficients will be non-negative. Ideally, we would like to derive necessary and sufficient conditions for this but the general problem has so far proved intractable. Instead, in the remainder of this paper, we examine this issue in the context of one special model which places considerable restrictions on the underlying technology.

3 Aggregation with Equal Numbers of Goods and Factors

The model to which we now wish to apply the approach outlined in the previous section is sometimes viewed as the "natural" generalization to higher dimensions of the simple two-factor, two-commodity model. Its key feature is that of "evenness": an equal number of factors and commodities is assumed, with each factor used in every sector. In addition, joint production and intermediate inputs are excluded. Despite these considerable limitations, a great deal of work has been done on this model: its central position in trade theory has been stressed by Pearce (1970, especially p. 488) and much of the work mentioned in Section 1 above which has extended the Heckscher-Ohlin model to higher dimensions has concentrated almost exclusively on it (c.f. the papers by Chipman (1969), Kemp and Wegge (1969), Uekawa (1971), Inada (1971) and Ethier (1974)). Many of these authors have been sceptical about the possibility of obtaining strong results for this model, and the findings to be presented below show that this scepticism is justified to a considerable extent. Nevertheless, a detailed examination of this model throws up one interesting case where aggregation is fully justified as well as pointing to the reasons for its failure more generally.

The first step in examining this model is to derive the special form which the matrix of as-if input-output coefficients takes in it. To do this we note that, in the absence of joint production and intermediate inputs, the full-employment conditions for each factor may be written as follows:

$$az = v, \quad (3.1)$$

Because the model is even, the matrix a of actual input-output coefficients is square and so (3.1) consists of n equations in the n unknown output levels, z . If we assume in addition that a is non-singular,⁴ then (3.1) may be solved for outputs as functions of endowments:

$$z = a^{-1}v. \quad (3.2)$$

Before proceeding, we may note that the elements of a and a^{-1} are independent of aggregate factor endowments. This is the key property of the even model, the so-called "local factor-price equalization" property. It may be derived from the competitive profit conditions, dual to (3.1), which require price to equal unit cost in each sector:

$$t'a = s'. \quad (3.3)$$

For this key property to hold, it is necessary that the same N commodities continue to be produced at all times and we assume this henceforward.⁵

We now wish to relate equation (3.2) to the definitions of composite factors and commodities. Firstly, from (2.3) and (2.6), the relationship between the endowments of individual and composite factors is as follows:

$$v = \Gamma V, \quad (3.4)$$

where:

$$V = (L \ K)' \quad (3.5)$$

and

$$\Gamma = \begin{bmatrix} \phi & 0 \\ 0 & \mu \end{bmatrix}. \quad (3.6)$$

Next, from (2.6), the output levels of the composite commodities may

be written as follows:

$$Z = \Sigma' z, \quad (3.7)$$

where:

$$Z = (X \ Y)' \quad (3.8)$$

and:

$$\Sigma = \begin{bmatrix} \pi & 0 \\ 0 & \psi \end{bmatrix}. \quad (3.9)$$

Now, we combine (3.7) with (3.2) and (3.4):

$$Z = \Sigma' a^{-1} v \quad (3.10)$$

$$= \Sigma' a^{-1} \Gamma V. \quad (3.11)$$

By inverting this, we obtain an explicit and particularly simple expression for the as-if input-output coefficients in the even model:

$$AZ = V, \quad (3.12)$$

where:

$$A = (\Sigma' a^{-1} \Gamma)^{-1}. \quad (3.13)$$

It is easily checked that this matrix also satisfies the price equations.

Thus, corresponding to (3.3), we have:

$$T'A = S', \quad (3.14)$$

where:

$$T = (W \ R)' \quad \text{and} \quad S = (P \ Q)'. \quad (3.15)$$

We may observe that, since a is independent of v as we have already noted, the elements of the matrix A do not depend on the levels of the composite factors L and K . This gives our first result: the local factor-price equalization property carries over from

the underlying even (N-by-N) model to the aggregate (two-by-two) model. Although this result is hardly surprising, we state it formally for completeness:

Proposition 1: If two countries, each of which can be represented by an N-by-N model, produce the same N commodities λ are endowed with the same base-period endowment vectors ϕ and μ , and face the same commodity prices p and q ; then, irrespective of the levels of the two countries' endowments of composite factors L and K , the prices of the composite factors will be the same in each country.

with the same technology,

Of course, the A_{ij} coefficients are not independent of base-period endowments, nor are the coefficients for sector X independent of the technology in sector Y . However, all four coefficients are independent of the elasticities of substitution between factors in the N sectors. This may be seen more clearly from the Appendix, where equation (3.13) is written out in full, and some other properties of this matrix are noted.

Having derived (3.13), the remaining task is to find restrictions on the a matrix and/or on the exogenous factor endowments (given by v or ΓV) and the commodity prices (given by s or ΣS) such that all the elements of the two-by-two matrix A are non-negative.⁶ We must also bear in mind the requirements that the parameter values must be such that all outputs and factor prices are non-negative:

$$z = a^{-1}v \geq 0 \quad \text{and} \quad t' = s'a^{-1} \geq 0. \quad (3.16)$$

It might be thought that the restrictions in (3.16) alone are sufficient to rule out negative elements in A , but unfortunately this is not the case as the following numerical example demonstrates.⁷

Example:

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (3.17)$$

The inverse of this matrix of input-output coefficients takes the convenient form:

$$a^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.18)$$

Moreover, the values of the exogenous variables are such that the restrictions in (3.16) are satisfied:

$$z = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} > 0 \quad t = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} > 0. \quad (3.19)$$

(For simplicity we confine attention to the base period, so that $L=K=P=Q=1$.) It may now be checked that if factors 2 and 3 are grouped together and commodities 1 and 2 are grouped together then the resulting A matrix contains one negative element:

$$A = \frac{1}{9} \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}. \quad (3.20)$$

In the light of this example, it is necessary to look for additional conditions which will guarantee that A is non-negative whenever the restrictions (3.16) are met. This search can proceed in a number of

alternative ways: either by imposing conditions on a , or on Γ and Σ , or on both sets of parameters simultaneously.

One set of conditions of the third variety may be stated immediately by adapting the reasoning of Ethier (1983), Section IV. He makes use of the concept of a Minkowski matrix: one which has only positive diagonal elements and negative off-diagonal elements. For such matrices, there is a natural association between goods and factors, and since Ethier assumes that the form of aggregation adopted respects this association he effectively concentrates on what we shall call an even partition of the commodity and factor vectors:

Definition: An even partition of the goods and factors in the even model, is one such that the number of individual commodities included in one of the composite commodities equals the number of individual factors included in one of the composite factors. In other words, not only are M and N equal, but m and n are also equal.

Ethier's results (especially his Propositions 9 and 11) may be restated in our notation and terminology as follows:⁸

Proposition 2 (Ethier): If the matrix a is such that, for some numbering of goods and factors, its inverse is a Minkowski matrix, then for any even partition of the exogenous variables, all four elements of A are non-negative.

This result is of considerable interest, since it provides a direct link between the approach adopted in the present paper and that of the first group of writers mentioned in the Introduction. However, it is also a weak result, since the requirement that the inverse of a be a Minkowski matrix is highly restrictive. In the next section, therefore, we turn to a rather different set of conditions which are also restrictive but which have a more immediate economic interpretation.

4 Aggregation when the a Matrix is Block Triangular

We might expect that aggregation should be easier to justify in cases where certain factors are not used at all in some sectors. In such circumstances, we can always label goods and factors such that the zero entries in the a matrix fall in the lower left-hand corner: i.e., such that the a matrix is upper block triangular. This leads to the next result:

Proposition 3: If a is upper block triangular and if the partition of goods and factors is even, then all the elements of A are non-negative.

Proof: Without loss of generality we choose labels such that the k factors are not used in the x sectors. The a matrix and its inverse are thus given by the following:

$$a = \begin{bmatrix} a_{lx} & a_{ly} \\ 0 & a_{ky} \end{bmatrix} \quad a^{-1} = \begin{bmatrix} a_{lx}^{-1} & -a_{lx}^{-1} & a_{ly} & a_{ky}^{-1} \\ 0 & & a_{ky}^{-1} & \end{bmatrix}. \quad (4.1)$$

Straightforward calculation then gives:

$$A = \begin{bmatrix} (\pi' a_{lx}^{-1} \phi)^{-1} & (\pi' a_{lx}^{-1} \phi)^{-1} \pi' a_{lx}^{-1} a_{ly} a_{ky}^{-1} \mu & (\psi' a_{ky}^{-1} \mu) \\ 0 & & (\psi' a_{ky}^{-1} \mu)^{-1} \end{bmatrix} \quad (4.2)$$

To show that the elements of A are non-negative, note first some of the implications of (3.16) for this model:

$$x = a_{lx}^{-1} \ell - a_{lx}^{-1} a_{ly} a_{ky}^{-1} k > 0, \quad (4.3)$$

$$y = a_{ky}^{-1} k > 0, \quad (4.4)$$

$$w = p' a_{lx}^{-1} > 0. \quad (4.5)$$

From (4.4) and (4.5) it follows immediately that the diagonal elements of A are positive:

$$A_{LX} = (\pi' a_{lx}^{-1} \phi)^{-1} = PL(w'l)^{-1} > 0, \quad (4.6)$$

$$A_{KY} = (\psi' a_{ky}^{-1} \mu)^{-1} = QK(q'y)^{-1} > 0. \quad (4.7)$$

The non-zero off-diagonal term can then be identified by noting that, since k is not used in x :

$$w'l = p'x + w'l_y, \quad (4.8)$$

where $w'l_y$ is the value of payments to the l factors employed in the y sectors. Combining (4.3) and (4.8):

$$A_{LY} = QL(w'l)^{-1} w'l_y(q'y)^{-1}. \quad (4.9)$$

Hence three of the elements of A are strictly positive and the fourth is zero.

QED

In fact, given the conditions specified in this Proposition, the interpretation of the elements of A can be carried further. Suppose the A_{ij} coefficients are manipulated to calculate "as-if" value shares, θ_{ij} . Then, it transpires that the latter equal the actual value shares of the composite factor in the corresponding composite sector, θ_{ij} :

$$\theta_{LX} = WA_{LX}/P = 1 = \theta_{LX} \quad (4.10)$$

$$\theta_{LY} = WA_{LY}/Q = w'l_y(q'y)^{-1} = \theta_{LY} \quad (4.11)$$

$$\theta_{KY} = RA_{KY}/Q = r'k(q'y)^{-1} = \theta_{KY}. \quad (4.12)$$

This shows clearly that aggregation works perfectly under the conditions of Proposition 3.⁹

Unfortunately, this result does not extend to the case where the partition is uneven. Consider, for example, the special case where the a matrix is upper block triangular but the partitioned a

matrix is lower block triangular. In this case, the a matrix can be written as follows, where each of the sub-matrices on the principal diagonal is square (note that this is a matrix generalization of the numerical example given in Section 3):

$$a = \left[\begin{array}{cc|c} a_{lx_1} & a_{lx_2} & 0 \\ 0 & a_{k_1x_2} & a_{k_1y} \\ 0 & 0 & a_{k_2y} \end{array} \right] \quad (4.13)$$

Calculations similar to those used in proving Proposition 3 now yield:

$$A = \left[\begin{array}{cc} (w'l)^{-1} & -(w'l)^{-1} r'k_x (q'y)^{-1} \\ 0 & (q'y)^{-1} \end{array} \right], \quad (4.14)$$

and so one element of A is necessarily negative. Calculating "as-if" value shares as before, we find:

$$\theta = \left[\begin{array}{cc} 1 & -\theta_{KX} \theta_X / \theta_Y \\ 0 & 1 + \theta_{KX} \theta_X / \theta_Y \end{array} \right], \quad (4.15)$$

where θ_X and θ_Y are the shares in national output of the X and Y sectors respectively. None of the four terms in (4.15) equals the corresponding true value share, θ_{ij} . Hence the behaviour of the aggregate model in this case does not correspond to that of the two-by-two Heckscher-Ohlin model.

Attempts to derive explicit results for two other cases have not proved successful: that where both the a matrix and the partitioned a matrix are upper block triangular;¹⁰ and that where the partition is even but the a matrix is not necessarily upper block triangular. Hence, Propositions 2 and 3 are the only conditions as yet available

which guarantee that the strong properties of the textbook two-by-two model carry over to the aggregate model.

5 An Application: The Gruen-Corden Model

Lest it be thought that the results of the last section are unduly disappointing, it should be noted that they have an immediate application to an interesting and well-known model: that of Gruen and Corden (1970), constructed to correspond to the stylized facts of the Australian economy and used by them to demonstrate the possibility that a tariff may paradoxically worsen the terms of trade.

The model distinguishes between two agricultural sectors, producing wool and grain, which do not use capital, and a textiles sector which does not use land, while a third factor, labour, is used in all three sectors. By appropriately ordering the sectors and factors, we can write the a matrix for this model in a form which permits the immediate application of Proposition 3:

$$a = \begin{array}{ccc|c} \text{wool} & \text{grain} & \text{textiles} & \\ \hline \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ a_{23} \\ a_{33} \end{bmatrix} & & \begin{array}{l} \text{land} \\ \text{labour} \\ \text{capital} \end{array} \end{array} \quad (5.1)$$

It is apparent that if the partition indicated by the dashed lines is adopted, all the conditions of Proposition 3 are satisfied. Hence, we may conclude that if the model is aggregated such that wool and grain are combined into a composite commodity, agricultural output (X_A), and if land and labour are combined into a composite factor, then the

resulting model exhibits all the properties of the textbook Heckscher-Ohlin model. This enables us to state some new results for this model.¹¹ For example, by the Rybczynski theorem, an increase in the endowment of capital will lead to an equal proportionate increase in textiles output (the proportionate changes are equal because capital is not used in agriculture) but to an absolute fall in agricultural output. (This must be so, even though it can easily be checked that the output of the relatively land-intensive agricultural good must rise.) In symbols, this may be expressed as follows:

$$\hat{R} > 0 \Rightarrow \hat{X}_T = \hat{K} > 0 > \hat{X}_A. \quad (5.2)$$

Secondly, by the Stolper-Samuelson theorem, an increase in the relative price of textiles must raise the real return to capital and lower the average return to land and labour combined (which we denote by W), both changes being relative to all goods prices:

$$\hat{P}_A > 0 \Rightarrow \hat{R} > \hat{P}_A > 0 > \hat{W}. \quad (5.3)$$

Finally, we may note that, in the light of the results at the end of Section 4, the alternative classification whereby labour and capital are aggregated together (with the same grouping of sectors as before) will not work.

6 Conclusion

The evident difficulty of extending the results of the two-factor, two-commodity Heckscher-Ohlin model to models of higher dimensions has led many economists to conclude, firstly, that small models are useless, and, secondly, that in order to derive predictions in large models on a detailed factor-by-factor and commodity-by-commodity basis, we should rely on computer simulations rather than analytic comparative statics. The present paper and the earlier paper of which it is an application are not at all inconsistent with the second of these conclusions (with which Ivor Pearce is particularly associated). Indeed our starting point has been the futility of seeking detailed predictions in general models without severe and unrealistic restrictions on technology or factor markets. At the same time, we have attempted to dispel the pessimism regarding the relevance of the Heckscher-Ohlin model to economies with many factors and commodities by showing that many of its properties continue to hold when reinterpreted in the context of an appropriate method of aggregation. In particular, we have derived some strong but nonetheless economically relevant conditions under which the aggregated model exhibits all the standard properties of the two-by-two model. However, a complete characterization of the conditions under which this occurs must remain an open question for further research.

Of course, our results are anything but a complete vindication of the two-by-two model, since the requirement of equal numbers of goods and factors is an extremely stringent one and there is no basis to expect it to hold in reality. Our results should rather be seen as showing that, in certain circumstances, the textbook model can

be interpreted as an even model in aggregate form, and also as providing further support for a point made by Jones and Scheinkman (1977): the strongest properties of the Heckscher-Ohlin model are as much consequences of the property of evenness itself as they are of its low dimensionality.

APPENDIX

In this Appendix we derive an explicit expression for the matrix A of as-if input-output coefficients, with no restrictions on the underlying technology except that the model is even. To do this, we first partition the matrix of actual input-output coefficients a in a manner conformable with the partitions of the factor endowment and commodity output vectors:

$$a = \begin{bmatrix} a_{lx} & a_{ly} \\ a_{kx} & a_{ky} \end{bmatrix} \quad (A.1)$$

The notation used in the partitioned a matrix is self-explanatory: a_{lx} , for example is the m -by- n matrix of actual input-output coefficients for the l factors in the x sectors. The inverse of a can now be written in a corresponding form:

$$a^{-1} = \begin{bmatrix} a^{xl} & a^{xk} \\ a^{yl} & a^{yk} \end{bmatrix} \quad (A.2)$$

where the inverse matrix a^{-1} is partitioned conformably with a : e.g., since a_{lx} is of order m -by- n , a^{xl} is of order n -by- m . Substituting from (A.2) into equation (3.13) in the text, we obtain an explicit expression for the as-if input-output coefficients in terms of actual input-output coefficients and base-period prices and endowments only:

$$A = \begin{bmatrix} \underline{A} \psi' a^{yk} \mu & -\underline{A} \psi' a^{yl} \phi \\ -\underline{A} \pi' a^{xk} \mu & \underline{A} \pi' a^{xl} \phi \end{bmatrix} \quad (A.3)$$

where:

$$\underline{A} = \pi' a^{x\ell}_{\phi} \cdot \psi' a^{y\ell}_{\mu} - \pi' a^{xk}_{\mu} \cdot \psi' a^{yk}_{\mu} \quad (A.4)$$

This shows clearly that each A_{ij} depends on all the actual input-output coefficients in the full model.

It is possible to go further in the case of an even partition of the vectors of factors and commodities. The diagonal sub-matrices $a_{\ell x}$ and a_{ky} are now square and so (A.2) may be expressed directly in terms of actual input-output coefficients using the formula for the inverse of a partitioned matrix:

$$a^{-1} = \begin{bmatrix} E & -E a_{kx} a_{ky}^{-1} \\ -a_{ky}^{-1} a_{\ell y} E & a_{ky}^{-1} (I + a_{\ell y} E a_{kx} a_{ky}^{-1}) \end{bmatrix} \quad (A.5)$$

where:

$$E \equiv [a_{\ell x} - a_{kx} a_{ky}^{-1} a_{\ell y}]^{-1} \quad (A.6)$$

Equation (A.6) has also been obtained by Dixit and Norman (op. cit., pp 55-56) for the special case where both m and n equal unity so that E is a scalar; they interpreted the sign of E as indicating whether sector x is relatively more intensive in its use of factor ℓ than the economy as a whole. Hence our results may be viewed as a generalization of that of Dixit and Norman to the case of arbitrary partitions of the factor and commodity vectors in the N -by- N model.

FOOTNOTES

- 1 For recent surveys, see Jones and Neary (1983), Section 2.5, and Ethier (1983). The latter paper also presents many new results, some of which are discussed further in Section 3 below.
- 2 The symbol $'$ denotes transpose. Without loss of generality, we may normalize the base-period prices such that $\pi'e_n = \psi'e_{N-n} = 1$, where e_i is an i -by-1 vector each element of which equals unity.
- 3 As with π and ψ , we may normalize the base-period endowments such that $\phi'e_m = u'e_{M-m} = 1$.
- 4 This of course requires that the matrix a be of full rank, i.e. that the vector of input-output coefficients in any sector should not be a linear combination of those in some other sectors. In order to obtain global results concerning factor-price equalization this possibility must be considered, and concern with it has led to a focus on factor-intensity reversals in the two-by-two model and to a search for restrictions to ensure univalence of the mapping from p to w in the general even model. However, provided we are content with local results, these issues need not detain us here.
- 5 Viewed in this light, the assumption of an equal number of goods and factors is not merely a restriction on technology but also a restriction on the admissible range of variation of the exogenous variables, s and v .
- 6 I am grateful to John Chipman for drawing my attention to the formal resemblance between this problem and that of aggregation (or 'consolidation') in Leontief input-output matrices. See, for example, the papers by Hatanaka (1954), Malinvaud (1956),

Fisher (1962, 1979) and Chipman (1976). Unfortunately, the resemblance does not appear to be sufficiently close to permit the application of theorems from this literature in the present context.

7 I am indebted to T.J. Laffey for suggesting this example.

8 The results of Ethier are in fact considerably stronger than this.

For example, he shows that the inverse of A is a Minkowski matrix if and only if, for any division of the N goods into two groups, a uniform proportional increase in the relative prices of all goods in one group causes the rewards of all factors corresponding to the first group to increase in terms of all goods and the rewards of all factors in the second group to fall in terms of all goods. Clearly, an increase in the real rewards of all individual factors included in a given composite factor is more than sufficient to ensure that the real reward of the composite factor also rises.

9 This proposition may be illustrated by applying an even partition to the numerical example given in Section 3.

10 These conditions are not sufficient to ensure that all the elements of A are non-negative. A counter-example is provided by the numerical example of Section 3 with factors 1 and 2 grouped together and commodities 2 and 3 grouped together.

11 These results are not necessarily difficult to prove by other means, but the advantage of our approach is that they may be stated directly as applications of Proposition 3.

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