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# A NOTE ON 4-RANK DENSITIES

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ABSTRACT. For certain real quadratic number fields, we prove density results concerning 4-ranks of tame kernels. We also discuss a relationship between 4-ranks of tame kernels and 4-class ranks of narrow ideal class groups. Additionally, we give a product formula for a local Hilbert symbol.

## 1. INTRODUCTION

Let  $F$  be a real quadratic number field and  $\mathcal{O}_F$  its ring of integers. In [4], the authors gave an algorithm for computing the 4-rank of the tame kernel  $K_2(\mathcal{O}_F)$ . The idea of the algorithm is to consider matrices with Hilbert symbols as entries and compute matrix ranks over  $\mathbb{F}_2$ . Recently, the author used these matrices to obtain “density results” concerning the 4-rank of tame kernels, see [6], [7].

In this note, we consider the 4-rank of  $K_2(\mathcal{O})$  for the real quadratic number fields  $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$  for primes  $p_1 \equiv p_2 \equiv p_3 \equiv 1 \pmod{8}$ . We will see that

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 0, 1, 2, \text{ or } 3.$$

For squarefree, odd integers  $d$ , consider the set

$$X = \{d : d = p_1 p_2 p_3, p_i \equiv 1 \pmod{8}\}$$

for distinct primes  $p_i$ .

Using GP/PARI [1], we computed the following: For  $50881 \leq d < 2 \times 10^7$ , there are 7257  $d$ 's in  $X$ . Among them, there are 2121  $d$ 's (29.23%) yielding 4-rank 0, 3977  $d$ 's (54.80%) yielding 4-rank 1, 1086  $d$ 's (14.96%) yielding 4-rank 2, and 73  $d$ 's (1.01%) yielding 4-rank 3. In fact, we prove

**Theorem 1.1.** *For the fields  $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$ , 4-rank 0, 1, 2, and 3 appear with natural density  $\frac{1}{4}$ ,  $\frac{17}{32}$ ,  $\frac{13}{64}$ , and  $\frac{1}{64}$  respectively in  $X$ .*

In the appendix we point out a beautiful result which may not be well known. It is a product formula from [11] for a certain local Hilbert symbol. This product formula both simplifies numerical computations and is a generalization of Propositions 4.6 and 4.4 in [2] and [7], respectively.

## 2. MATRICES

Hurrelbrink and Kolster [4] generalize Qin's approach in [8], [9] and obtain 4-rank results by computing  $\mathbb{F}_2$ -ranks of certain matrices of local Hilbert symbols. Specifically, let  $F = \mathbb{Q}(\sqrt{d})$ ,  $d > 1$  and squarefree. Let  $p_1, p_2, \dots, p_t$  denote the

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odd primes dividing  $d$ . Recall 2 is a norm from  $F$  if and only if all  $p_i$ 's are  $\equiv \pm 1 \pmod 8$ . If so, then  $d$  is a norm from  $\mathbb{Q}(\sqrt{2})$ , thus

$$d = u^2 - 2w^2$$

for  $u, w \in \mathbb{Z}$ . Now consider the matrix:

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\ (d, -1)_2 & (d, -1)_{p_1} & \cdots & (d, -1)_{p_t} \end{pmatrix}.$$

If 2 is not a norm from  $F$ , set  $v = 2$ . Otherwise, set  $v = u + w$ . Replacing the 1's by 0's and the  $-1$ 's by 1's, we calculate the matrix rank over  $\mathbb{F}_2$ . From [4],

**Lemma 2.1.** *Let  $F = \mathbb{Q}(\sqrt{d})$ ,  $d > 0$  and squarefree. Then*

$$4\text{-rank } K_2(\mathcal{O}_F) = t - \text{rk}(M_{F/\mathbb{Q}}) + a' - a$$

where

$$a = \begin{cases} 0 & \text{if 2 is a norm from } F \\ 1 & \text{otherwise} \end{cases}$$

and

$$a' = \begin{cases} 0 & \text{if both -1 and 2 are norms from } F \\ 1 & \text{if exactly one of -1 or 2 is a norm from } F \\ 2 & \text{if none of -1 or 2 are norms from } F. \end{cases}$$

Recall that our case is  $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$  for primes  $p_1 \equiv p_2 \equiv p_3 \equiv 1 \pmod 8$ . In this case  $a = a'$  and we may delete the last row of  $M_{F/\mathbb{Q}}$  without changing its rank (see discussions preceding Proposition 5.13 and Lemma 5.14 in [4]). Also note that  $v$  is an  $p_1$ -adic unit and hence

$$(-p_1 p_2 p_3, v)_{p_1} = (p_1, v)_{p_1} = \left(\frac{v}{p_1}\right).$$

Similarly,  $(-p_1 p_2 p_3, v)_{p_2} = \left(\frac{v}{p_2}\right)$  and  $(-p_1 p_2 p_3, v)_{p_3} = \left(\frac{v}{p_3}\right)$ . From Lemma 2.1 we have

$$4\text{-rank } K_2(\mathcal{O}_F) = 3 - \text{rk}(M_{F/\mathbb{Q}})$$

and the matrix  $M_{F/\mathbb{Q}}$  is of the form

$$\begin{pmatrix} 1 & \left(\frac{p_2}{p_1}\right)\left(\frac{p_3}{p_1}\right) & \left(\frac{p_1}{p_2}\right) & \left(\frac{p_1}{p_3}\right) \\ 1 & \left(\frac{p_2}{p_1}\right) & \left(\frac{p_1}{p_2}\right)\left(\frac{p_3}{p_2}\right) & \left(\frac{p_2}{p_3}\right) \\ (-d, u+w)_2 & \left(\frac{v}{p_1}\right) & \left(\frac{v}{p_2}\right) & \left(\frac{v}{p_3}\right) \end{pmatrix}.$$

Let us now prove Theorem 1.1.

*Proof.* The idea in [6] and [7] is to first consider an appropriate normal extension  $N$  of  $\mathbb{Q}$  and then relate the splitting of the primes  $p_i$  in  $N$  to their representation by certain quadratic forms. The next step is classifying 4-rank values in terms of values of the symbols  $(-d, v)_2, \left(\frac{v}{p_i}\right)$ . The values of these symbols are then characterized in terms of  $p_i$  satisfying the alluded to quadratic forms. We then associate Artin symbols to the primes  $p_i$  and apply the Chebotarev density theorem. In what follows, we classify the 4-rank values in terms of the symbols  $(-d, v)_2, \left(\frac{v}{p_i}\right)$  and in parenthesis give the relevant densities in  $X$  obtained by using the above machinery. Let us consider the following four cases (see Table III in [9]).

Case 1: Suppose  $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = \left(\frac{p_3}{p_2}\right) = 1$ . Then we immediately have that

- 4-rank  $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 3 \iff \text{rank}(M_{F/\mathbb{Q}}) = 0 \iff (-d, v)_2 = 1$   
and  $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = 1 \left(\frac{1}{64}\right)$
- 4-rank  $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 2 \iff \text{rank}(M_{F/\mathbb{Q}}) = 1 \iff (-d, v)_2 = -1$   
or  $(-d, v)_2 = 1$  and  $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = -1$  and  $\left(\frac{v}{p_3}\right) = 1$  or  $(-d, v)_2 = 1$  and  
 $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_3}\right) = -1$  and  $\left(\frac{v}{p_2}\right) = 1$  or  $(-d, v)_2 = 1$  and  $\left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = -1$   
and  $\left(\frac{v}{p_1}\right) = 1 \left(\frac{7}{64}\right)$ .

Case 2: Suppose  $\left(\frac{p_3}{p_2}\right) = \left(\frac{p_3}{p_1}\right) = 1, \left(\frac{p_2}{p_1}\right) = -1$ . Thus

- 4-rank  $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 2 \iff \text{rank}(M_{F/\mathbb{Q}}) = 1 \iff (-d, v)_2 = 1$   
and  $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = 1$  or  $(-d, v)_2 = 1$  and  $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_2}\right) = -1 \left(\frac{3}{32}\right)$
- 4-rank  $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 1 \iff \text{rank}(M_{F/\mathbb{Q}}) = 2 \iff (-d, v)_2 = -1$   
or  $(-d, v)_2 = 1$  and  $\left(\frac{v}{p_1}\right) = \left(\frac{v}{p_3}\right) = -1$  and  $\left(\frac{v}{p_2}\right) = 1$  or  $(-d, v)_2 = 1$   
and  $\left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = -1$  and  $\left(\frac{v}{p_1}\right) = 1 \left(\frac{9}{32}\right)$ .

Case 3: Suppose  $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = -1, \left(\frac{p_3}{p_2}\right) = 1$ . Thus

- 4-rank  $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 1 \iff \text{rank}(M_{F/\mathbb{Q}}) = 2 \iff (-d, v)_2 = 1$   
 $\left(\frac{3}{16}\right)$ .
- 4-rank  $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1 p_2 p_3})}) = 0 \iff \text{rank}(M_{F/\mathbb{Q}}) = 3 \iff (-d, v)_2 = -1$   
 $\left(\frac{3}{16}\right)$

Case 4: Suppose  $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = \left(\frac{p_3}{p_2}\right) = -1$ . Then



calculated using Lemmas 5.3 and 5.4 in [4]. In this appendix we provide a product formula which allows one to calculate  $(-d, u+w)_{p_k}$  using 2 factors of  $d$  at a time.

Let  $d$  be a squarefree integer and assume that all odd prime divisors of  $d$  are  $\equiv \pm 1 \pmod{8}$ . Then  $d$  is a norm from  $F = \mathbb{Q}(\sqrt{2})$  and we have the representation

$$d = u^2 - 2w^2$$

with  $u > 0$ . Let  $l$  be any odd prime dividing  $d$ . Note that  $l$  does not divide  $u+w$  and so

**Remark 3.1.**  $(-d, u+w)_l = (l, u+w)_l = \left(\frac{u+w}{l}\right)$ .

Recall that any odd prime divisor  $l$  of  $d$  is  $\equiv \pm 1 \pmod{8}$ . We fix  $x$  and  $y$  according to the representation:

$$(-1)^{\frac{l-1}{2}} l = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(x + y\sqrt{2}) = x^2 - 2y^2$$

with  $x \equiv 1 \pmod{4}$ ,  $x, y > 0$ . Observe that mod 8,  $x$  is odd. Also we can arrange for  $x \equiv 1 \pmod{4}$  by multiplying  $x + y\sqrt{2}$  by  $(1 + \sqrt{2})^2$ .

For  $l \equiv 1 \pmod{8}$ , we have  $l = x^2 - 2y^2$  and so  $\left(\frac{l}{y}\right) = 1$ . Thus  $\left(\frac{y}{l}\right) = 1$ . For  $l \equiv 7 \pmod{8}$ ,

$$1 = \left(\frac{-l}{y}\right) = \left(\frac{-1}{y}\right) \left(\frac{l}{y}\right) = (-1)^{\frac{y-1}{2}} (-1)^{\frac{y-1}{2}} \left(\frac{y}{l}\right) = \left(\frac{y}{l}\right).$$

Now let  $r$  be an integer not divisible by  $l$  which can be represented as a norm from  $\mathbb{Q}(\sqrt{2})$ . Denote by  $\pi_r = s + t\sqrt{2}$  an element such that  $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\pi_r) = r$  with  $s, t > 0$ . Now let  $u_r$  and  $w_r$  be such that

$$u_r + w_r\sqrt{2} = (1 + \sqrt{2})(x + y\sqrt{2})(s + t\sqrt{2}).$$

By the choice of  $x, y, s, t$ , we have  $u_r > 0$ . Note that

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) = -(-1)^{\frac{l-1}{2}} lr.$$

Now fix  $\mathfrak{l} = \langle x - y\sqrt{2} \rangle$  a prime ideal above  $l$  in  $\mathbb{Q}(\sqrt{2})$ . As  $l$  splits in  $\mathbb{Q}(\sqrt{2})$ ,  $\mathbb{Z}[\sqrt{2}]/\mathfrak{l} \cong \mathbb{Z}/l\mathbb{Z}$ . This allows us to work mod  $\mathfrak{l}$  as opposed to mod  $l$ . From the above,  $u_r + w_r = 2xs + 3tx + 3sy + 4yt$ . Modulo  $\mathfrak{l}$ , we have

$$\begin{aligned} u_r + w_r &\equiv 2sy\sqrt{2} + 3ty\sqrt{2} + 3sy + 4yt \\ &\equiv y(3 + 2\sqrt{2})(s + t\sqrt{2}). \end{aligned}$$

As  $\left(\frac{y}{l}\right) = 1$ ,  $\left(\frac{u_r + w_r}{l}\right) = \left(\frac{y}{l}\right) \left(\frac{\pi_r}{l}\right) = \left(\frac{\pi_r}{l}\right)$  where

$$\left(\frac{\pi_r}{l}\right) = \begin{cases} 1 & \text{if } x^2 \equiv \pi_r \pmod{\mathfrak{l}} \text{ is solvable} \\ -1 & \text{otherwise.} \end{cases}$$

In the case  $r = \prod_i^{t-1} p_i$  where  $p_i \equiv \pm 1 \pmod{8}$ , we obtain for each  $p_i$  an element  $\pi_i \in \mathbb{Q}(\sqrt{2})$  of norm  $(-1)^{\frac{p_i-1}{2}} p_i$ . Let  $c$  be the number of primes dividing  $r$  which are congruent to 7 modulo 8. Then we have (up to squares of units)  $\pi_r = (1 + \sqrt{2})^c \prod_i^{t-1} \pi_i$ . This yields

$$\left(\frac{u_r + w_r}{l}\right) = \left(\frac{(1 + \sqrt{2})^c}{l}\right) \prod_i^{t-1} \left(\frac{\pi_i}{l}\right),$$

and so

$$\left(\frac{u_r+w_r}{l}\right) = \left(\frac{u_{-1}+w_{-1}}{l}\right)^c \prod_i^{t-1} \left(\frac{u_{p_i}+w_{p_i}}{l}\right).$$

As  $-1$  and  $2$  are also norms from  $\mathbb{Q}(\sqrt{2})$ , we can include  $r$ 's having factors  $-1$  or  $\pm 2$ . Thus for  $r = (-1)^n (2)^m \prod_i^{t-1} p_i$  with  $m, n = 0, 1$ , and each  $p_i \equiv \pm 1 \pmod{8}$  and  $l \neq p_i$  for any  $i$ , we have

$$\textbf{Remark 3.2.} \quad \left(\frac{u_r+w_r}{l}\right) = \left(\frac{u_{-1}+w_{-1}}{l}\right)^{n+c} \left(\frac{u_2+w_2}{l}\right)^m \prod_i^{t-1} \left(\frac{u_{p_i}+w_{p_i}}{l}\right).$$

Setting  $r = \frac{d}{l}$ , we have  $-(-1)^{\frac{l-1}{2}} d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r \sqrt{2})$ . So for any prime  $l \equiv 7 \pmod{8}$ , we have  $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r \sqrt{2}) = d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u + w \sqrt{2})$ . Then, up to squares,  $\left(\frac{u_r+w_r}{l}\right) = \left(\frac{u+w}{l}\right)$ . For prime divisors  $l \equiv 1 \pmod{8}$ , we have  $-d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r \sqrt{2})$  and so we include  $\left(\frac{u_{-1}+w_{-1}}{l}\right)$ . To summarize,

$$\textbf{Remark 3.3.} \quad (-d, u+w)_l = \begin{cases} \left(\frac{u_r+w_r}{l}\right) & \text{if } l \equiv 7 \pmod{8} \\ \left(\frac{u_{-1}+w_{-1}}{l}\right) \left(\frac{u_r+w_r}{l}\right) & \text{if } l \equiv 1 \pmod{8}. \end{cases}$$

We may now reduce to the following  $d = rl$ :  $d = -l$ ,  $d = 2l$ , and  $d = pl$ , i.e. calculate the symbols  $\left(\frac{u_{-1}+w_{-1}}{l}\right)$ ,  $\left(\frac{u_2+w_2}{l}\right)$ , and  $\left(\frac{u_p+w_p}{l}\right)$ . The first two symbols can be calculated using the following two elementary lemmas.

**Lemma 3.4.**  $\left(\frac{u_{-1}+w_{-1}}{l}\right) = 1 \iff (-1)^{\frac{l-1}{2}} l = a^2 - 32b^2$  for some  $a, b \in \mathbb{Z}$  with  $a \equiv 1 \pmod{4}$ .

**Lemma 3.5.**  $\left(\frac{u_2+w_2}{l}\right) = 1 \iff l \equiv \pm 1 \pmod{16}$ .

A little care is necessary in computing  $\left(\frac{u_p+w_p}{l}\right)$ . If  $\left(\frac{(-1)^{\frac{p-1}{2}} p}{l}\right) = 1$ , then the symbol  $\left(\frac{\pi}{l}\right)$  is well defined (see discussion preceding Proposition 3.5 in [2]) and can be computed using

**Lemma 3.6.** For  $\mathcal{K} = \mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}} 2p})$  with  $p \equiv \pm 1 \pmod{8}$  and  $h^+(\mathcal{K})$  the narrow class number of  $\mathcal{K}$ , we have

$$\left(\frac{\pi}{l}\right) = 1 \iff l^{\frac{h^+(\mathcal{K})}{4}} = n^2 - 2pm^2 \text{ for some } n, m \in \mathbb{Z} \text{ with } m \not\equiv 0 \pmod{l}.$$

For  $\mathcal{K} = \mathbb{Q}(\sqrt{-2p})$  with  $p \equiv 7 \pmod{8}$ ,  $\left(\frac{\pi}{l}\right) = -1 \iff l^{\frac{h^+(\mathcal{K})}{4}} = 2n^2 + pm^2$  for some  $n, m \in \mathbb{Z}$  with  $m \not\equiv 0 \pmod{l}$ .

For  $\mathcal{K} = \mathbb{Q}(\sqrt{2p})$  with  $p \equiv 1 \pmod{8}$ ,  $\left(\frac{\pi}{l}\right) = -1 \iff l^{\frac{h^+(\mathcal{K})}{4}} = pn^2 - 2m^2$  for some  $n, m \in \mathbb{Z}$  with  $m \not\equiv 0 \pmod{l}$ .

In fact,

**Lemma 3.7.** If  $\left(\frac{(-1)^{\frac{p-1}{2}} p}{l}\right) = 1$ , then

$$\left(\frac{u_p+w_p}{l}\right) = \begin{cases} \left(\frac{\pi}{l}\right) & \text{for } p \equiv 1 \pmod{8} \\ \left(\frac{u_{-1}+w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) & \text{for } p \equiv 7 \pmod{8}. \end{cases}$$

The case where  $\left(\frac{(-1)^{\frac{p-1}{2}}}{l}p\right) = -1$  can be done by finding  $u_p$  and  $w_p$  from the presentation

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_p + w_p\sqrt{2}) = -(-1)^{\frac{p-1}{2}}pl.$$

Combining Remarks 3.1, 3.2, and 3.3, we have

**Theorem 3.8.** For  $d = (-1)^n(2)^m \prod_{i=1}^t p_i$ , with each  $p_i \equiv \pm 1 \pmod{8}$ , we have

$$(-d, u+w)_{p_k} = \left(\frac{u_{-1}+w_{-1}}{p_k}\right)^{n+(-1)^{\frac{p_k+1}{2}}} \left(\frac{u_2+w_2}{p_k}\right)^m \prod_{i \neq k} \left(\frac{u_{p_i}+w_{p_i}}{p_k}\right).$$

**Example 3.9.** Consider the cases  $d = \pm pl, \pm 2pl$  with  $p \equiv 7 \pmod{8}$ ,  $l \equiv 1 \pmod{8}$ , and  $\left(\frac{l}{p}\right) = 1$  (see Proposition 4.6 in [2]). Note that  $\left(\frac{\pi}{l}\right)$  is well defined and so Lemma 3.7 is applicable.

For  $d = pl$ , we have  $n = 0$ ,  $m = 0$  and so

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1}+w_{-1}}{l}\right)^{-1} \left(\frac{u_2+w_2}{l}\right)^0 \left(\frac{u_{-1}+w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{\pi}{l}\right). \end{aligned}$$

For  $d = 2pl$ , we have  $n = 0$ ,  $m = 1$ . Thus

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1}+w_{-1}}{l}\right)^{-1} \left(\frac{u_2+w_2}{l}\right)^1 \left(\frac{u_{-1}+w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{2+\sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

For  $d = -pl$ , we have  $n = 1$ ,  $m = 0$ . This yields

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1}+w_{-1}}{l}\right)^0 \left(\frac{u_2+w_2}{l}\right)^0 \left(\frac{u_{-1}+w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{1+\sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

Finally, for  $d = -2pl$ , we have  $n = 1$ ,  $m = 1$ . So

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1}+w_{-1}}{l}\right)^0 \left(\frac{u_2+w_2}{l}\right)^1 \left(\frac{u_{-1}+w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{2+\sqrt{2}}{l}\right) \left(\frac{1+\sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

**Example 3.10.** Consider the cases  $d = \pm pl$  with  $p \equiv l \equiv 1 \pmod{8}$ , and  $\left(\frac{l}{p}\right) = 1$  (see Proposition 4.4 in [7]). Again  $\left(\frac{\pi}{l}\right)$  is well defined and so Lemma 3.7 is applicable.

For  $d = pl$ , we have  $n = 0$ ,  $m = 0$ , and so

$$\begin{aligned} (-d, u + w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_p + w_p}{l}\right) \\ &= \left(\frac{1 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

For  $d = -pl$ , we have  $n = 1$ ,  $m = 0$ . Thus

$$\begin{aligned} (-d, u + w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_p + w_p}{l}\right) \\ &= \left(\frac{\pi}{l}\right). \end{aligned}$$

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