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Boundary behaviour of universal Taylor series on multiply connected domains

Stephen J. Gardiner and Myrto Manolaki

ABSTRACT. A holomorphic function on a planar domain Ω is said to possess a universal Taylor series about a point ζ of Ω if subsequences of the partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compact sets in $\mathbb{C}\backslash\Omega$ that have connected complement. In the case where Ω is simply connected such functions are known to be unbounded, and to form a collection that is independent of the choice of ζ. This paper uses tools from potential theory to show that, even for domains Ω of arbitrary connectivity, such functions are unbounded whenever they exist. In the doubly connected case a further analysis of boundary behaviour reveals that the collection of such functions can depend on the choice of ζ. This phenomenon was previously known only for domains that are at least triply connected. Related results are also established for universal Laurent series.

1. Introduction

Let Ω be a proper subdomain of the complex plane $\mathbb{C}$ and let $\zeta \in \Omega$. If f is a holomorphic function on Ω, we denote by $S_N(f,\zeta)(z)$ the partial sum $\sum_{n=0}^{N} c_n(z-\zeta)^n$ of the Taylor series of f about ζ. We then say that f belongs to the collection $\mathcal{U}(\Omega,\zeta)$ of functions with *universal Taylor series* about ζ if, for each compact set $K \subset \mathbb{C}\backslash\Omega$ with connected complement and each continuous function $g : K \to \mathbb{C}$ which is holomorphic on K°, there exists a subsequence $(S_{N_k}(f,\zeta))$ that converges uniformly to g on K.

When Ω is simply connected the existence of such functions was established by Nestoridis [**17**], [**18**], who even showed that $\mathcal{U}(\Omega,\zeta)$ is a dense G_δ subset of the space $\mathcal{H}(\Omega)$ of holomorphic functions on Ω endowed with the topology of local uniform convergence. Further, Müller, Vlachou and Yavrian [**15**] showed that the collection $\mathcal{U}(\Omega,\zeta)$ is independent of the centre of expansion ζ, from which it follows that no function f in $\mathcal{U}(\Omega,\zeta)$ can be extended holomorphically beyond Ω (see also Theorem 8.4 in Melas and Nestoridis [**14**]). Recently, this conclusion was substantially strengthened in [**9**], where it was shown that each such function f is unbounded near every

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point of $\partial\Omega$. (In fact, for every disc D centred at some point of $\partial\Omega$, the set $\mathbb{C}\backslash f(D\cap\Omega)$ was shown to be polar.)

The theory of universal Taylor series is less well developed in the case of multiply connected domains, where the general existence question remains open. There are, however, several partial results. For example, it is known [**15**] that, if the complement Ω^c (with respect to $\mathbb{C}$) of a multiply connected domain Ω is non-thin at infinity, then $\mathcal{U}(\Omega,\zeta)=\emptyset$ for every choice of ζ. On the other hand, Melas [**13**] showed that, if Ω^c is compact and connected, then $\mathcal{U}(\Omega,\zeta)\neq\emptyset$ for all $\zeta\in\Omega$. He also established that the same is true if Ω^c is a discrete set, and that, provided Ω^c is not a singleton, the collection $\mathcal{U}(\Omega,\zeta)$ then depends on ζ. More recently, the first author and Tsirivas [**11**] showed that, if Ω^c is a suitable disjoint union of a non-degenerate continuum and a singleton, then the very existence of functions in $\mathcal{U}(\Omega,\zeta)$ depends on the choice of ζ.

The main purpose of this paper is to study the boundary behaviour of functions in $\mathcal{U}(\Omega,\zeta)$, and the dependence of $\mathcal{U}(\Omega,\zeta)$ on the choice of ζ, for multiply connected domains Ω. We denote by $D(\zeta,R)$ the open disc of centre ζ and radius R.

Our first result concerns boundary behaviour in arbitrary domains.

THEOREM 1. *Let* $f\in\mathcal{U}(\Omega,\zeta)$, *where* $\Omega\subsetneq\mathbb{C}$ *is a domain and* $\zeta\in\Omega$, *and let* $R=\operatorname{dist}(\zeta,\partial\Omega)$ *and* $\xi\in\partial D(\zeta,R)\cap\partial\Omega$. *If* f *is bounded on* $\Omega\cap D(\xi,\rho)$ *for some* $\rho>0$, *then* $\Omega^c\backslash\overline{D(\zeta,R)}$ *must be polar and* f *has a holomorphic continuation to* $D(\xi,\rho)$.

Melas [**13**] showed that, if Ω^c is a discrete subset of $D(0,1)^c$ containing 1, then $\mathcal{U}(\Omega,0)$ contains functions that may be holomorphically extended to $\mathbb{C}\backslash\{1\}$. Thus, in Theorem 1, it is indeed possible for a function in $\mathcal{U}(\Omega,\zeta)$ to be holomorphically extendable at some points of $\partial D(\zeta,R)\cap\partial\Omega$. However, this clearly cannot happen at every point of that set, so we immediately obtain the following conclusion.

COROLLARY 2. *For any domain* $\Omega\subsetneq\mathbb{C}$ *and any point* $\zeta\in\Omega$, *every function* f *in* $\mathcal{U}(\Omega,\zeta)$ *is unbounded.*

As mentioned above, examples are known of domains Ω where the collection $\mathcal{U}(\Omega,\zeta)$, or even the existence of functions in $\mathcal{U}(\Omega,\zeta)$, depends on the choice of ζ. However, all these examples have been at least triply connected. The doubly connected case, which we will now consider, is more delicate. Since $\mathcal{U}(\Omega,\zeta)=\emptyset$ for every $\zeta\in\Omega$ when Ω^c contains both a bounded and an unbounded component, we will restrict our attention to the case where Ω^c is compact and connected. For such a domain Ω and any point ζ in Ω, Melas [**13**] has shown that, as for simply connected domains, $\mathcal{U}(\Omega,\zeta)$ is a dense G_δ subset of $\mathcal{H}(\Omega)$. (See also Vlachou [**21**] for the special case where Ω^c is a singleton.) Further, Bayart [**3**] has shown that $\cap_{\zeta\in\Omega}\mathcal{U}(\Omega,\zeta)$ is a residual subset of $\mathcal{H}(\Omega)$. (See also Costakis [**4**] for the special case where Ω^c is a polygon.) However, it has remained an open problem whether the collection $\mathcal{U}(\Omega,\zeta)$ can depend on ζ. We will answer this question below.

We recall that a function $\gamma:[0,2\pi]\to\mathbb{C}$ is said to be *Dini continuous* if $\int_0^1 t^{-1}\omega_\gamma(t)dt<\infty$, where ω_γ is the modulus of continuity of γ, defined by

$$\omega_\gamma(\delta)=\sup\{|\gamma(t_1)-\gamma(t_2)|:t_1,t_2\in[0,2\pi],|t_1-t_2|\leq\delta\}\qquad(\delta>0).$$

We call Ω an *exterior Jordan domain* if it is the exterior domain of a Jordan curve Γ in $\mathbb{C}$. If Γ has a parametrization $\alpha(t)$ $(0 \leq t \leq 2\pi)$ such that $\alpha'(t)$ is Dini continuous and never zero, and $\alpha'(0) = \alpha'(2\pi)$, then Ω will be called an *exterior Dini domain.*

THEOREM 3. *If Ω is an exterior Dini domain, then, for every $\zeta \in \Omega$, there exists $\zeta_1 \in \Omega$ such that $\mathcal{U}(\Omega, \zeta)\backslash\mathcal{U}(\Omega, \zeta_1) \neq \emptyset$.*

The proof of this result will reveal that, if ξ_1 denotes a point of $\partial\Omega$ at maximum distance from ζ, then we may choose ζ_1 to be any point of the form $\xi_1 + t(\xi_1 - \zeta)$, where $t > 0$. Theorem 3 relies, in part, on the following result, which concerns boundary growth restrictions, on functions in $\mathcal{H}(\Omega)$, that are compatible with membership of the collection $\mathcal{U}(\Omega, \zeta)$.

THEOREM 4. *Let Ω be an exterior Jordan domain and $\zeta \in \Omega$, choose $\xi_1 \in \partial\Omega$ at maximum distance r from ζ and then $\xi_2 \in \partial\Omega\backslash\{\xi_1\}$. Further, let E_1, E_2 be bounded relatively closed subsets of Ω such that $E_1 \cap \overline{D(\zeta, r)} = \emptyset$ and $\overline{E}_k \cap \partial\Omega = \{\xi_k\}$ $(k = 1, 2)$. If $\phi : \Omega \to (1, \infty)$ is a continuous function such that $\phi(z) \to \infty$ as $z \to \xi_k$ $(k = 1, 2)$, then there exists f in $\mathcal{U}(\Omega, \zeta)$ such that $|f| \leq \phi$ on $E_1 \cup E_2$.*

These theorems will be proved below, following some preliminary material. After that we will adapt some of our arguments to obtain new results about universal Laurent series.

2. Preliminaries

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and $G_\omega(\cdot, z)$ denote the Green function of an open set $\omega \subset \widehat{\mathbb{C}}$ with pole at $z \in \omega$, when it exists; that is, when $\mathbb{C}\backslash\omega$ is non-polar. We interpret $G_\omega(\cdot, z)$ as 0 outside the connected component of ω that contains z. We also denote by H^ω_ϕ the (Perron-Wiener-Brelot) solution to the Dirichlet problem on ω with Borel measurable boundary function ϕ. This solution has the representation

$$H^\omega_\phi(z) = \int_{\partial\omega} \phi d\mu^\omega_z \qquad (z \in \omega),$$

where μ^ω_z denotes harmonic measure relative to ω and z. Finally, we denote the characteristic function of a set A by χ_A, and write $\check{v}$ for the upper (respectively, $\widehat{v}$ for the lower) semicontinuous regularization of a function v.

We begin by recalling two results, which can be found in [**8**] and [**9**], respectively.

THEOREM A. *Let $f \in \mathcal{H}(\Omega)$, where $\Omega \subsetneq \mathbb{C}$ is a domain and $0 \in \Omega$, let (S_{N_k}) be a subsequence of $(S_N(f, 0))$ and U be the largest domain containing 0 on which (S_{N_k}) is locally uniformly convergent. Further, suppose that $\Omega\backslash U \neq \emptyset$ and that (S_{N_k}) is uniformly bounded on a compact set K disjoint from U. Then*
(i) U is bounded (and simply connected),
(ii) $\left(\overline{U}\right)^\circ \cap \Omega = U \cap \Omega$, and
(iii) $\check{v}$ is subharmonic on $\Omega \cup (\mathbb{C}\backslash\partial U)$ and continuously vanishes on ∂U,

where

$$v(z)=\begin{cases} -G_U(z,0) & \textit{if } z\in U \\ G_V(z,\infty) & \textit{if } z\in V \\ 0 & \textit{otherwise}\end{cases} \qquad \textit{and} \quad V=\widehat{\mathbb{C}}\backslash(\overline{U}\cup K).$$

Theorem B. *Let $\omega\subsetneq\mathbb{C}$ be simply connected and $\zeta_0\in\omega$, and let ω_0 be a non-empty open subset of ω. Suppose that (v_k) is a decreasing sequence of harmonic functions on ω such that $v_1/G_\omega(\zeta_0,\cdot)$ is bounded above on ω_0 and $\lim_{k\to\infty}v_k<0$ on ω. If $H_{\chi_\omega}^{\omega_0}\not\equiv 1$, then there exists $k'\in\mathbb{N}$ such that the open set $\omega_1=\{z\in\omega_0: v_{k'}(z)<0\}$ is non-empty and $H_{\chi_\omega}^{\omega_1}\not\equiv 1$.*

An important first step in proving Theorem 1 is the following result about harmonic measure for the domain U in Theorem A.

Proposition 5. *Let $\Omega, f, (S_{N_k}), U$ and K satisfy the hypotheses of Theorem A, let $R=\operatorname{dist}(0,\partial\Omega)$ and assume that $(\overline{U}\cup K)\backslash\overline{D(0,R)}$ is non-polar. If $\xi\in\partial D(0,R)\cap\partial\Omega$ and*

$$\mu_0^U(E_r)=0 \quad \textit{for some} \quad r>0, \quad \textit{where} \quad E_r=\partial U\cap\Omega^c\cap\overline{D(\xi,r)},$$

then $\xi\in U$ and so f can be extended holomorphically to a neighbourhood of ξ.

Proof. There is no loss of generality in assuming that $r<R$. Let v and V be as in Theorem A. That result tells us that U is bounded, and that $\check{v}$ is subharmonic on $\Omega\cup(\mathbb{C}\backslash\partial U)$ and vanishes continuously on ∂U. Let

$$l=-\log c(\overline{U}\cup K) \quad \text{and} \quad \mu=\nu+\delta_0-\mu_0^U,$$

where $c(\cdot)$ denotes logarithmic capacity, ν is the equilibrium measure of $\overline{U}\cup K$, and δ_0 the Dirac measure at 0.

We claim that $\check{v}=-\mathbf{U}\mu+l$ on $\mathbb{C}$, where $\mathbf{U}\mu$ denotes the usual logarithmic potential of μ given by

$$\mathbf{U}\mu(z)=-\int\log|z-\zeta|d\mu(\zeta) \qquad (z\in\mathbb{C}).$$

To see this, we first note that $\max\{\check{v},0\}=-\mathbf{U}\nu+l$ on $\mathbb{C}$ (see Lemma 5.8.1 and Corollary 3.2.7 of [**2**]). Also, by Theorem 6.8.1 of [**2**] and the symmetry of the Green function, the function $h=\max\{-\check{v},0\}+\log|\cdot|$, which is assigned its limiting value at 0, can be expressed as

$$h(z)=\begin{cases} -\mathbf{U}\mu_0^U(z) & \text{if } z\in U \\ \log|z| & \text{if } z\in U^c\end{cases}.$$

Moreover, $-\mathbf{U}\mu_0^U(z)=\log|z|$ for all $z\in U^c$, by the regularity of U. Therefore $\max\{-\check{v},0\}=\mathbf{U}\delta_0-\mathbf{U}\mu_0^U$ on $\mathbb{C}$, which yields the desired conclusion that

$$\check{v}=\max\{\check{v},0\}-\max\{-\check{v},0\}=-\mathbf{U}\nu+l-\mathbf{U}\delta_0+\mathbf{U}\mu_0^U=-\mathbf{U}\mu+l.$$

Next, we claim that $\check{v}$ is subharmonic on $D(\xi,r)$. In view of the above representation, it suffices to show that $\mu|_{D(\xi,r)}\geq 0$. Since $\check{v}$ is subharmonic on $\Omega\cup(\mathbb{C}\backslash\partial U)$, we see that $\mu|_{\Omega\cup(\mathbb{C}\backslash\partial U)}\geq 0$. In particular, $\mu|_{D(\xi,r)\backslash E_r}\geq 0$. Since $\mu_0^U(E_r)=0$, by hypothesis, we deduce that $\mu|_{D(\xi,r)}\geq 0$, as required.

Now let $\mathcal{M}(w;\xi,r)$ denote the mean value of a function w over the circle $\partial D(\xi,r)$. Since $D(0,R)\subset U$, we see that

$$-\log\frac{|z|}{R}=G_{D(0,R)}(z,0)\leq G_U(z,0)\quad (z\in D(0,R)),$$

and so $\check{v}\leq\log(|\cdot|/R)$ on $D(0,R)$. Further, using the assumption that $(\overline{U}\cup K)\backslash\overline{D(0,R)}$ is non-polar and the minimum principle, we deduce that

$$\log\frac{|z|}{R}=G_{\widehat{\mathbb{C}}\backslash\overline{D(0,R)}}(z,\infty)>G_V(z,\infty)\quad (z\in\mathbb{C}\backslash\overline{D(0,R)}),$$

so $\check{v}<\log(|\cdot|/R)$ on $\mathbb{C}\backslash\overline{D(0,R)}$. Combining this with the subharmonicity of $\check{v}$ on $D(\xi,r)$, we see that

$$\check{v}(\xi)\leq\mathcal{M}(\check{v};\xi,\frac{r}{2})<\mathcal{M}(\log\frac{|\cdot|}{R};\xi,\frac{r}{2})=\log\frac{|\xi|}{R}=0,$$

since $r/2<R$. Hence $\xi\in U$ and so f has a holomorphic extension to a neighbourhood of ξ. □

Using the proposition we can give the following illustrative example.

Example 6. *Let* $\Omega=([1-i,1+i]\cup P)^c$, *where* P *is a closed countable subset of* $\{1\}\cup\{z:\mathrm{Re}(z)<1$ *and* $|z|>1\}$ *such that the sets* $\overline{P\cap\{z:\mathrm{Im}(z)<0\}}$ *and* $\overline{P\cap\{z:\mathrm{Im}(z)>0\}}$ *each contain* 1. *Then* $\mathcal{U}(\Omega,0)=\emptyset$.

Indeed, if $\mathcal{U}(\Omega,0)\neq\emptyset$, then we could choose $f\in\mathcal{U}(\Omega,0)$ and a subsequence (S_{N_k}) of $(S_N(f,0))$ which converges uniformly to some constant l on the set $K=\Omega^c\cap\overline{D(1,1)}$. Let U and v be as in Theorem A (we can arrange that $K\cap U=\emptyset$ by choosing l appropriately). Using the submean-value property of v we see that $\overline{U}\cup[1-i,1+i]$ has connected complement. This, and the proposition, together imply that $\partial U\cap[1-i,1+i]$ contains more than one point, and so U contains at least one component of the set $D(1,\varepsilon)\cap\{z:\mathrm{Re}(z)<1,|z|>1\}$ for sufficiently small $\varepsilon>0$. This contradicts the fact that $K\cap U=\emptyset$.

3. Proof of Theorem 1

Let Ω,ζ,f,R,ξ and ρ be as in the statement of Theorem 1. The hypotheses ensure that Ω is multiply connected, by [**9**], and all components of Ω^c are bounded, by [**15**]. Without loss of generality we may assume that $\zeta=0$ and $\rho<R$. (In the general case we could then use the removability of polar sets for bounded holomorphic functions to see that f has a holomorphic continuation to $D(\xi,\rho)\backslash\overline{D(0,R)}$, and could apply the original case with ξ replaced by an arbitrary point of $\partial D(0,R)\cap\partial\Omega\cap D(\xi,\rho)$.) Further, by adding a constant if necessary, we may assume that $\limsup_{z\to\xi}|f(z)|>0$. Let $M>1$ be such that $|f|\leq M$ on $\Omega\cap\overline{D(\xi,\rho)}$. Let K be a compact subset of Ω^c containing $\Omega^c\cap\overline{D(\xi,\rho)}$. Further, if $\Omega^c\backslash\overline{D(0,R)}$ is non-polar, then we choose K so that $K\backslash\overline{D(0,R)}$ is also non-polar. If $\Omega^c\cap\overline{D(\xi,\rho)}$ is polar, then f obviously has a holomorphic continuation to $D(\xi,\rho)$; also, from Corollary 1 of [**8**] and the fact that $\mathcal{U}(\Omega,0)\neq\emptyset$, we see that $\Omega^c\backslash\overline{D(0,R)}$ must be polar. We may therefore assume from now on that $\Omega^c\cap\overline{D(\xi,\rho)}$ is non-polar. Since $f\in\mathcal{U}(\Omega,0)$, and all components of Ω^c are bounded, we can use a diagonal sequence argument to choose a subsequence (S_{N_k}) of $(S_N(f,0))$ such that $S_{N_k} \to 0$ locally uniformly on Ω^c and $|S_{N_k}| \leq 1$ on K for all $k \in \mathbb{N}$.

Let U be the largest domain containing 0 on which (S_{N_k}) is locally uniformly convergent. Then U is simply connected, and $\xi \notin U$ since $\limsup_{z\to\xi} |f(z)| > 0$ and $S_{N_k}(\xi) \to 0$. Also, the choice of (S_{N_k}) ensures that $U\backslash\Omega$ is at most countable, for otherwise the identity principle would yield the contradictory conclusion that $f \equiv 0$. Hence, by working with the holomorphic extension of f to $U \cup \Omega$ if necessary, we may assume without loss of generality that $U \subset \Omega$. Further, $U \subsetneq \Omega$, since Ω is multiply connected. Thus U is bounded, by Theorem A. For each $r > 0$ we write $V_r = U \cap D(\xi, r)$ and $E_r = \partial U \cap \Omega^c \cap \overline{D(\xi, r)}$ as before. There are two cases to consider.

Case 1 : $\mu_0^U(E_r) > 0$ for all $r > 0$.

Here we firstly observe that

$$H^{V_r}_{\chi_\Omega} \not\equiv 1 \quad \text{on} \quad V_r \quad (r > 0),$$

because $E_{r/2} \subset \partial U \setminus \overline{U \setminus V_r}$ and $E_{r/2}$ is non-negligible (for harmonic measure) with respect to U, whence E_r is non-negligible with respect to V_r by Theorem 6.6.10 of [**2**]. Next, we can choose a connected component U_ρ of V_ρ satisfying

$$H^{U_\rho}_{\chi_\Omega} < 1 \quad \text{on} \quad U_\rho. \tag{1}$$

For each $k \in \mathbb{N}$ we define the subharmonic function

$$u_k(z) = \frac{1}{N_k} \log |S_{N_k}(z) - f(z)| \qquad (z \in \Omega),$$

and observe that $u_k(z) = \log|z| + g_k(z)$ for some subharmonic function g_k on Ω. By considering the functions $u = \limsup_{k\to\infty} u_k$ and $g = \limsup_{k\to\infty} g_k$, we now see that

$$\check{u}(z) = \log|z| + \check{g}(z) \qquad (z \in \Omega),$$

and $\check{u}$, $\check{g}$ are subharmonic on Ω by Corollary 5.7.2 of [**2**] and Bernstein's Lemma (Theorem 5.5.7 of [**19**]). Further, since $S_{N_k} \to f$ locally uniformly on U, we see that $\check{u} \leq 0$ on U and so, by the definition of the Green function,

$$\check{u}(z) \leq -G_U(z, 0) \qquad (z \in U). \tag{2}$$

Let $\Omega_1 = \mathbb{C}\backslash K$. Since K is non-polar, Bernstein's Lemma shows that

$$\begin{aligned} \frac{1}{N_k} \log |S_{N_k}(z)| &\leq \frac{1}{N_k} \log \|S_{N_k}\|_K + G_{\Omega_1 \cup \{\infty\}}(z, \infty) \\ &\leq G_{\Omega_1 \cup \{\infty\}}(z, \infty) \qquad (z \in \Omega_1, k \in \mathbb{N}), \end{aligned}$$

where $\|\cdot\|_K$ denotes the supremum norm on K. Additionally, by Harnack's inequalities and inversion, there is a constant $c > 0$ such that

$$G_{\Omega_1 \cup \{\infty\}}(z, \infty) \leq c G_{\Omega_1}(z, 0) \qquad (z \in \Omega \cap D(\xi, \rho)).$$

Combining the above with the triangle inequality, we conclude that

$$u_k(z) \leq \max\left\{ c G_{\Omega_1}(z, 0), \frac{\log M}{N_k} \right\} + \frac{\log 2}{N_k} \qquad (z \in \Omega \cap D(\xi, \rho), k \in \mathbb{N}). \tag{3}$$

Our goal now is to find a domain $\omega_1 \subset U_\rho$ on which (S_{N_k}) is uniformly bounded, and such that $\partial\omega_1 \cap \partial U_\rho \subset K$ and $H^{\omega_1}_{\chi_{\partial\omega_1 \cap \partial U_\rho}} > 0$ on ω_1. For this purpose we will make use of the Martin boundary of Ω_1, which we denote by Δ. Let $M(\cdot, w)$ denote the Martin kernel with pole at w with respect to Ω_1. (It plays a role analogous to the Poisson kernel for a disc.) Then every positive harmonic function h on Ω_1 has a representation of the form

$$h(z) = \int_{\Delta} M(z,w) d\mu_h(w) \qquad (z \in \Omega_1),$$

where μ_h is a finite measure on Δ. Moreover Δ contains a Borel subset Δ_1 (the set of minimal points) with the property that each such function h has a unique representation of the above form satisfying $\mu_h(\Delta \backslash \Delta_1) = 0$. We will always use this particular measure to represent h. Also, we will make use of the concept of minimal thinness. Denoting the reduction of a positive superharmonic function v on Ω_1 relative to a subset E by

$$R_v^E = \inf \{s : \ s \text{ is positive and superharmonic on } \Omega_1 \text{ and } s \geq v \text{ on } E\},$$

a set $E \subset \Omega_1$ is then said to be *minimally thin* at a point $w \in \Delta_1$ if $\widehat{R}^E_{M(\cdot,w)}$ differs from $M(\cdot, w)$. Further, a function $\phi : \Omega_1 \to [-\infty, +\infty]$ is said to have a *minimally fine limit* (denoted by mf lim) l at w with respect to Ω_1 if there is a set E, minimally thin at w, such that $\lim_{z \to w, z \in \Omega_1 \backslash E} \phi(z) = l$. (For a detailed account of these notions we refer to Chapters 8 and 9 of [**2**].)

Let μ_1 be the measure in the Martin representation of the constant function 1. Thus

$$1 = \int_{\Delta} M(z,w) d\mu_1(w) = \int_{\Delta_1} M(z,w) d\mu_1(w) \qquad (z \in \Omega_1).$$

By Corollary 9.1.4 of [**2**] we see that

$$\widehat{R}_1^{\Omega_1 \backslash U_\rho}(z) = \int_{\Delta_1} \widehat{R}^{\Omega_1 \backslash U_\rho}_{M(\cdot,w)}(z) d\mu_1(w) \qquad (z \in \Omega_1).$$

From (1) and Theorem 6.9.1 of [**2**] we know that $\widehat{R}_1^{\Omega_1 \backslash U_\rho} \not\equiv 1$, since $\partial U_\rho \cap \Omega = \partial U_\rho \cap \Omega_1$. Hence $\mu_1(A) > 0$, where

$$A = \{w \in \Delta_1 : \Omega_1 \backslash U_\rho \text{ is minimally thin at } w \text{ with respect to } \Omega_1\}.$$

Let $\zeta_0 \in U_\rho$. Then, by Theorem 9.6.2 of [**2**],

$$\text{mf} \lim_{z \to w} \frac{G_{U_\rho}(z, \zeta_0)}{G_{\Omega_1}(z, 0)} > 0 \qquad (w \in A).$$

Therefore $A = \cup_{m=1}^{\infty} A_m$, where

$$A_m = \left\{ w \in A : \text{mf} \lim_{z \to w} \frac{G_{U_\rho}(z, \zeta_0)}{G_{\Omega_1}(z, 0)} \geq \frac{1}{m} \right\},$$

and there exists $m' \in \mathbb{N}$ with $\mu_1(A_{m'}) > 0$. We define the set

$$\omega = \left\{ z \in U_\rho : G_{U_\rho}(z, \zeta_0) > \frac{1}{m'+1} G_{\Omega_1}(z, 0) \right\}.$$

Then ω is a simply connected subdomain of U_ρ containing ζ_0 (by the minimum principle and the fact that U_ρ is a simply connected domain which does not contain 0) and satisfying

$$\partial \omega \cap \partial U_\rho \subset K. \tag{4}$$

Further, $\Omega_1\backslash\omega$ is minimally thin at each point of $A_{m'}$ and, since m' was chosen so that $\mu_1(A_{m'})>0$, we see that

$$\begin{aligned} 1-H^{\omega}_{\chi_{\Omega_1}}(z) &= \int_{\Delta_1}\left(M(z,y)-\widehat{R}^{\Omega_1\backslash\omega}_{M(\cdot,y)}(z)\right)d\mu_1(y) \\ &\geq \int_{A_{m'}}\left(M(z,y)-\widehat{R}^{\Omega_1\backslash\omega}_{M(\cdot,y)}(z)\right)d\mu_1(y)>0 \quad (z\in\omega). \end{aligned}$$

Since $\omega\subset U_\rho\subset\Omega_1$ and (4) holds, we observe that $\chi_{U_\rho}=\chi_{\Omega_1}$ on $\partial\omega$. Hence

$$H^{\omega}_{\chi_{U_\rho}}=H^{\omega}_{\chi_{\Omega_1}}<1 \quad \text{on} \;\; \omega.$$

For each $k\in\mathbb{N}$ we consider the sets

$$Y_k=\left\{z\in\overline{\omega}:\operatorname{dist}(z,\partial\Omega)\geq\frac{1}{k}\right\}.$$

Then Y_k is a compact subset of U, in view of (4). Thus, using (2) and choosing a suitable subsequence of (S_{N_k}) if necessary, we can arrange that

$$u_k(z)\leq-\frac{k}{k+1}G_U(z,0) \quad (z\in Y_k). \tag{5}$$

Since $H^{\omega}_{\chi_{\Omega_1}}<1$ on ω, we can use the same argument as before (with ω playing the role of U_ρ) to see that $\mu_1(B)>0$, where

$$B=\{w\in\Delta_1:\Omega_1\backslash\omega \text{ is minimally thin at } w \text{ with respect to } \Omega_1\},$$

and so there exists $m''\in\mathbb{N}$ such that $\mu_1(B_{m''})>0$, where

$$B_m=\left\{w\in B:\mathrm{mf}\lim_{z\to w}\frac{G_\omega(z,\zeta_0)}{G_{\Omega_1}(z,0)}\geq\frac{1}{m}\right\}.$$

Hence the set

$$\omega_0=\left\{z\in\omega:G_\omega(z,\zeta_0)>\frac{1}{m''+1}G_{\Omega_1}(z,0)\right\}$$

is a simply connected subdomain of ω containing ζ_0 such that

$$\partial\omega_0\cap\partial\omega\subset K \tag{6}$$

and

$$H^{\omega_0}_{\chi_\omega}=H^{\omega_0}_{\chi_{\Omega_1}}<1 \quad \text{on} \;\; \omega_0. \tag{7}$$

We define functions on $\overline{\omega}$ by writing

$$s_k(z)=\begin{cases} u_k(z)-\frac{1}{N_k}\log 2M & (z\in\overline{\omega}\cap\Omega) \\ \limsup_{w\to z}u_k(w)-\frac{1}{N_k}\log 2M & (z\in\partial\omega\cap\partial\Omega)\end{cases}.$$

Then s_k is upper bounded on $\overline{\omega}$, by (3), and subharmonic on ω. Thus

$$s_k(z)\leq H^{\omega}_{s_k}(z)\leq H^{\omega}_{\phi_k}(z) \quad (z\in\omega) \tag{8}$$

by (3) and (5), where

$$\phi_k(z)=\begin{cases} cG_{\Omega_1}(z,0) & \text{if } \; z\in\partial\omega\cap(\Omega\backslash Y_k) \\ -\frac{k}{k+1}G_U(z,0) & \text{if } \; z\in\partial\omega\cap Y_k \\ 0 & \text{if } z\in\partial\omega\cap\partial\Omega\end{cases}.$$

Since $\phi_k \leq cG_{\Omega_1}(\cdot, 0)$ on $\partial\omega$, we see that

$$H^{\omega}_{\phi_k} \leq H^{\omega}_{cG_{\Omega_1}(\cdot,0)} = cG_{\Omega_1}(\cdot, 0) \leq c(m'' + 1)G_{\omega}(\cdot, \zeta_0) \quad \text{on} \ \ \omega_0.$$

Also, $(H^{\omega}_{\phi_k})$ is a decreasing sequence of harmonic functions on ω satisfying

$$\lim_{k\to\infty} H^{\omega}_{\phi_k} = H^{\omega}_{\lim \phi_k} < 0.$$

Consequently, we can apply Theorem B to the functions $H^{\omega}_{\phi_k}$ and the domains $\omega_0 \subset \omega$ (see (7)), to conclude that there exists a non-empty domain $\omega_1 \subset \omega_0$ such that

(I) $H^{\omega}_{\phi_k} < 0$ on ω_1, for all sufficiently large k,
(II) $\partial\omega_1 \cap \partial\omega \subset K$ (because (6) holds), and
(III) $H^{\omega_1}_{\chi_{\partial\omega_1\cap\partial\omega}} > 0$ on ω_1.

From (8) and condition (I) we see that, for all sufficiently large $k \in \mathbb{N}$,

$$\frac{1}{N_k} \log |S_{N_k} - f| - \frac{1}{N_k} \log 2M < 0 \quad \text{on} \ \ \omega_1,$$

or, equivalently, $|S_{N_k} - f| < 2M$ on ω_1. Also, $|f| \leq M$ on $\Omega \cap \overline{D(\xi, \rho)}$ and so $\log |S_{N_k}| \leq M_1$ on ω_1 (where $M_1 = \log 3M$). Using the subharmonicity of $\log |S_{N_k}|$, we now conclude that

$$\begin{aligned} \log |S_{N_k}(z)| &\leq H^{\omega_1}_{\log |S_{N_k}|}(z) \\ &\leq H^{\omega_1}_{M_1\chi_{\partial\omega_1\cap\omega}}(z) + H^{\omega_1}_{\log |S_{N_k}|\chi_{\partial\omega_1\cap\partial\omega}}(z) \\ &\leq M_1 + \int_{\partial\omega_1\cap\partial\omega} \log |S_{N_k}| d\mu^{\omega_1}_z \qquad (z \in \omega_1). \end{aligned} \tag{9}$$

Since (II) holds and $|S_{N_k}| \leq 1$ on K, we can apply Fatou's lemma to the non-positive functions $\log |S_{N_k}|$ on $\partial\omega_1 \cap \partial\omega$ to deduce that

$$\limsup_{k\to\infty} \int_{\partial\omega_1\cap\partial\omega} \log |S_{N_k}| d\mu^{\omega_1}_z \leq \int_{\partial\omega_1\cap\partial\omega} \limsup_{k\to\infty} \log |S_{N_k}| d\mu^{\omega_1}_z \qquad (z \in \omega_1).$$

Using (III) we see that $\mu^{\omega_1}_z(\partial\omega_1 \cap \partial\omega) > 0$, and thus the right hand side of the above inequality is identically $-\infty$ because $S_{N_k} \to 0$ on K. Finally, since $\omega_1 \subset U$, we have $S_{N_k} \to f$ on ω_1 and so (9) yields the contradictory conclusion that $\log |f| = -\infty$ on ω_1, or equivalently $f \equiv 0$. Therefore Case 1 cannot occur under the given assumptions.

Case 2 : $\mu^U_0(E_r) = 0$ for some $r > 0$.

In this case $(\overline{U} \cup K)\backslash\overline{D(0, R)}$ must be polar, for otherwise Proposition 5 would yield the contradictory conclusion that $\xi \in U$. Since $D(0, R) \subset U$, we see that $U = D(0, R)$. Also, $\Omega^c\backslash\overline{D(0, R)}$ must now be polar, by our choice of K. In particular, f has a holomorphic extension to $D(\xi, \rho)\backslash\overline{D(0, R)}$. Since harmonic measure for $D(0, R)$ and 0 coincides with normalized arclength measure, the set E_r has zero arclength measure. A subset of a circle which has zero arclength measure also has zero analytic capacity (cf. p.199 of [**7**]), and so is removable for bounded holomorphic functions (see p.198 of [**7**]). Hence f has a holomorphic extension to $D(\xi, \rho)$, as claimed. □

4. Proof of Theorem 4

Here we will adapt and combine arguments from [**10**] and [**13**]. Let Ω, ζ, ξ_1, ξ_2, r, E_1, E_2 and ϕ be as in the statement of the theorem, and let $E = E_1 \cup E_2$. By Carathéodory's theorem we can choose a conformal mapping $\psi : \Omega \cup \{\infty\} \to \mathbb{D}$, where $\mathbb{D} = D(0,1)$, that extends to a homeomorphism from $\overline{\Omega} \cup \{\infty\}$ to $\overline{\mathbb{D}}$. Further, by composition with a suitable automorphism of the disc, we may arrange that $\psi(\xi_1) = 1$ and that the argument of $\psi(\xi_2)$ is a rational multiple of π. What we will use below is the property that $(\psi(\xi_k))^l = 1$ $(k = 1, 2)$ for some $l \in \mathbb{N}$.

Let (L_m) be an exhaustion of Ω by compacts such that $\zeta \in L_1^\circ$ and each set L_m^c consists of a bounded component containing Ω^c and an unbounded component. Given $m, N \in \mathbb{N}$ we can use Cauchy's estimates to find $\eta(N, m) \in (0, 1)$ small enough so that, if $g \in \mathcal{H}(L_1^\circ)$ and $\|g\|_{L_1^\circ} < \eta(N, m)$, then

$$\|S_n(g, \zeta)\|_{\Omega^c} \leq 2^{-m} \quad (n = 0, 1, ..., N).$$

For each $m \in \mathbb{N}$ let U_m denote the union of the bounded components ω of $\Omega \backslash (L_m \cup E)$ satisfying $\overline{\omega} \cap \partial\Omega \subset \{\xi_1, \xi_2\}$. We define

$$M_m = L_m \cup E \cup U_m \cup \Omega^c \quad (m \in \mathbb{N}).$$

By our assumptions on E_1 and E_2 the open set M_m^c has at least one, and at most two, bounded components, each of which intersects $\overline{D(\zeta, r)}^c$.

Let (p_m) be an enumeration of the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We inductively define a sequence (q_m) of rational functions, as follows.

Let $N_0 = 1$ and suppose we are given $m, N_{m-1} \in \mathbb{N}$ and rational functions $q_1, ..., q_{m-1}$ such that $\sum_1^{m-1} q_j$ (which we interpret as 0 if $m = 1$) has at most two poles, and these lie in the set

$$M_m^c \cap \left(D\left(\zeta, r + \frac{1}{m}\right) \backslash \overline{D(\zeta, r)} \right).$$

Since $|\psi| < 1$ on Ω and $\phi(z) \to \infty$ as $z \to \xi_k$ $(k = 1, 2)$, we can choose $n \in \mathbb{N}$ large enough so that

$$|p_m^*| \leq 2^{-m-2}\eta(N_{m-1}, m)\frac{\phi}{\|\phi\|_{L_m}} \quad \text{on } L_m \cup E, \tag{10}$$

where

$$p_m^* = \left\{ \begin{array}{cc} p_m - \sum_1^{m-1} q_j & \text{on } \Omega^c \\ \psi^{nl}\left(p_m - \sum_1^{m-1} q_j\right) & \text{on } \Omega \end{array} \right. . \tag{11}$$

Our particular choice of ψ ensures that p_m^* is continuous at ξ_1 and ξ_2. Let q_m^* denote the sum of the singular parts of $\sum_1^{m-1} q_j$ at its poles. (If there are no poles, then we take q_m^* to be 0.) Since $p_m^* + q_m^*$ is continuous on M_m and holomorphic on M_m°, we know from Mergelyan's theorem that there is a rational function q_m^{**} with at most two poles, lying in the bounded components of M_m^c, such that

$$\|q_m^{**} - (p_m^* + q_m^*)\|_{M_m} < 2^{-m-2}\eta(N_{m-1}, m),$$

and these poles may be chosen to lie in

$$M_{m+1}^c \cap \left(D\left(\zeta, r + \frac{1}{m+1}\right) \backslash \overline{D(\zeta, r)} \right). \tag{12}$$

The rational function $q_m = q_m^{**} - q_m^*$ thus satisfies

$$\|q_m - p_m^*\|_{M_m} < 2^{-m-2}\eta(N_{m-1}, m), \tag{13}$$

and $\sum_1^m q_j$ has the same poles as q_m^{**}. Since $\Omega^c \subset \overline{D(\zeta, r)}$, we can choose $N_m > N_{m-1}$ sufficiently large so that

$$\left\| S_{N_m}\left(\sum_1^m q_j, \zeta\right) - \sum_1^m q_j \right\|_{\Omega^c} < 2^{-m-2}.$$

Hence, by (11) and (13),

$$\begin{aligned} \left\| S_{N_m}\left(\sum_1^m q_j, \zeta\right) - p_m \right\|_{\Omega^c} &< 2^{-m-2} + \left\| \sum_1^m q_j - p_m \right\|_{\Omega^c} \\ &= 2^{-m-2} + \|q_m - p_m^*\|_{\Omega^c} < 2^{-m-1}. \end{aligned} \tag{14}$$

Also, from (10) and (13), we see that

$$\|q_m\|_{L_m} < 2^{-m-1}\eta(N_{m-1}, m) < 2^{-m-1} \tag{15}$$

and

$$|q_m| < 2^{-m-1}\phi \quad \text{on} \quad E, \tag{16}$$

since $\phi > 1$. Our construction ensures that $\sum_1^m q_j$ has at most two poles, and these lie in the set given by (12).

Having constructed the sequence (q_m) as above, we see from (15) that the series $\sum q_m$ converges locally uniformly on Ω to a holomorphic function f, and from (16) that $|f| \le \phi$ on E. Further, since we may arrange that $(\eta(N_{m-1}, m))_{m \ge 1}$ is a decreasing sequence, and since

$$\left| \sum_{m+1}^\infty q_j \right| < \eta(N_m, m+1) \quad \text{on} \quad L_1$$

by (15) again, we see that

$$\begin{aligned} |p_m - S_{N_m}(f, \zeta)| &\le \left| p_m - S_{N_m}\left(\sum_1^m q_j, \zeta\right) \right| + \left| S_{N_m}\left(\sum_{m+1}^\infty q_j, \zeta\right) \right| \\ &\le 2^{-m-1} + 2^{-m-1} < 2^{-m} \qquad \text{on} \quad \Omega^c, \end{aligned}$$

by (14) and the definition of $\eta(N_m, m+1)$.

It follows that $f \in \mathcal{U}(\Omega, \zeta)$, so the proof of Theorem 4 is complete.

5. Proof of Theorem 3

The Poisson kernel for $\mathbb{D}$ is given by

$$P(z, w) = \frac{1 - |z|^2}{|z - w|^2} \qquad (z \in \mathbb{D}, w \in \partial\mathbb{D}).$$

In view of what was said earlier, a set $E \subset \mathbb{D}$ is minimally thin at a point $w \in \partial\mathbb{D}$ with respect to $\mathbb{D}$ if and only if there is a positive superharmonic function u on $\mathbb{D}$ such that

$$\inf_E \frac{u}{P(\cdot, w)} > \inf_{\mathbb{D}} \frac{u}{P(\cdot, w)}.$$

For example, if $D \subset \mathbb{D}$ is a disc that is internally tangent to $\partial\mathbb{D}$ at a point w, then $\mathbb{D}\backslash D$ is minimally thin at w with respect to $\mathbb{D}$. This follows from the facts that D is of the form $\{P(\cdot, w) > c\}$ for some $c > 0$, and that the function $\min\{P(\cdot, w), c\}$ is superharmonic on $\mathbb{D}$.

Let Ω be an exterior Jordan domain, and $\psi : \Omega \cup \{\infty\} \to \mathbb{D}$ be a conformal mapping that extends to $\partial\Omega$ as before. Then $E \subset \Omega$ is minimally thin at $\xi \in \partial\Omega$ with respect to Ω if and only if $\psi(E)$ is minimally thin at $\psi(\xi)$ with respect to $\mathbb{D}$. We now make the stronger requirement that Ω is an exterior Dini domain. If $E \subset \Omega$, we define the enlarged set

$$E_\rho = \bigcup_{z \in E} D\left(z, \rho \operatorname{dist}(z, \partial\Omega)\right) \qquad (0 < \rho < 1).$$

Then, by the Dini condition and Corollary 7.4.6 in [**1**], the set E_ρ is minimally thin at $\xi \in \partial\Omega$ if and only if

$$\int_{E_\rho} |z - \xi|^{-2} \, dA(z) < \infty, \tag{17}$$

and condition (17) is independent of the choice of ρ. In particular, (17) is thus a sufficient condition for E to be minimally thin at ξ with respect to Ω. It follows from the Dini condition that, if D is a disc in Ω such that $\xi \in \partial D \cap \partial\Omega$, then (17) holds with $E = (\Omega\backslash D) \cap D(\xi, 1)$, and therefore $\Omega\backslash D$ is minimally thin at ξ with respect to Ω.

The result below is a variant of Theorem 2 in [**8**], which covered the case where Ω^c is a disc. We will give a substantially new proof.

THEOREM 7. *Let Ω be an exterior Dini domain, let $\zeta_1 \in \Omega$, $t = \operatorname{dist}(\zeta_1, \partial\Omega)$ and $\varepsilon > 0$. Suppose that, for each $r \in (t, t + \varepsilon)$,*

- (i) *the angle interior to $D(\zeta_1, r) \cap \Omega$ at each point of $\partial D(\zeta_1, r) \cap \partial\Omega$ is greater than $\pi/2$;*
- (ii) *the image of $\Omega^c \cap D(\zeta_1, r)$ under inversion in $\partial D(\zeta_1, r)$ is contained in Ω^c.*

If $f \in \mathcal{H}(\Omega)$, then any subsequence $(S_{N_k}(f, \zeta_1))$ which is uniformly bounded on Ω^c must be locally uniformly convergent either on $D(\zeta_1, t + \varepsilon) \cap \Omega$ or on a domain containing $D(\zeta_1, t) \cup (\Omega^c)^\circ$.

PROOF OF THEOREM 7. To see this, let $f \in \mathcal{H}(\Omega)$, suppose that $(S_{N_k}(f, \zeta_1))$ is uniformly bounded on Ω^c, and let U denote the largest domain containing ζ_1 on which $(S_{N_k}(f, \zeta_1))$ is locally uniformly convergent. If $U\backslash\Omega \neq \emptyset$, then $(\Omega^c)^\circ \subset U$ by Montel's theorem, and the proof is complete. If $\Omega \subset U$, then $U = \mathbb{C}$ by the maximum modulus theorem, and again we are finished. We may therefore suppose below that $U \subsetneq \Omega$. Further, we note from the maximum principle and Theorem A that U is simply connected and $\partial U \cap \partial\Omega$ is either a singleton or a (proper) subarc of $\partial\Omega$.

Let $V = (\Omega \cup \{\infty\}) \backslash \overline{U}$ and v be as in the statement of Theorem A. (Thus we are taking $K = \Omega^c$ here.) Then v is subharmonic on $\Omega \cup (\partial U)^c$, that is, on $(\partial U \cap \partial\Omega)^c$. Let

$$R = \sup \{\rho \in (0, t + \varepsilon) : D(\zeta_1, \rho) \subset U \cup \Omega^c\}.$$

Clearly $R \geq t$. We suppose, for the sake of contradiction, that $R < t + \varepsilon$, and can then choose a point η in $\partial D(\zeta_1, R) \cap \partial U \cap \partial V$, in view of part (ii) of Theorem A. Let v_0 denote the composition of v with inversion in $\partial D(\zeta_1, R)$; that is,

$$v_0(z) = v\left(\zeta_1 + R^2/\overline{z - \zeta_1}\right) \qquad (z \in \mathbb{C}\backslash\{\zeta_1\}),$$

and let $s = v + v_0$. We note that $v(\eta) = 0$, so $v_0(\eta) = 0$ and hence $s(\eta) = 0$. Since

$$U \supset D(\zeta_1, R) \cap \Omega \quad \text{and} \quad V \subset (\Omega \cup \{\infty\}) \backslash \overline{D(\zeta_1, R)},$$

and since Green functions are monotone with respect to the underlying domain, and map to Green functions under composition with an inversion, we see that $s < 0$ on $\Omega \backslash \partial D(\zeta_1, R)$ by hypothesis (ii) and the maximum principle. We cannot have $\eta \in \Omega$, for that would imply that $s(\eta) < 0$ by the subharmonic mean value inequality. Hence $\eta \notin \Omega$; that is, $\eta \in \partial U \cap \partial\Omega \cap \partial V$. If $D(\eta, \varepsilon) \cap U$ has a component that does not intersect $D(\zeta_1, R)$, we redefine v to be 0 there. This ensures that v, v_0, s are all subharmonic on $D(\eta, \delta) \cap \Omega$ for some $\delta \in (0, \varepsilon)$, and $s < 0$ there.

By hypothesis (i) there is an open triangular region L, with vertex η, that is contained in both $\Omega \cap D(\zeta_1, R)$ and the image of $\Omega^c \backslash \overline{D(\zeta_1, R)}$ under inversion in $\partial D(\zeta_1, R)$. The set L is certainly not minimally thin at η with respect to Ω (cf. condition (17)). By Theorem 9.6.2 of [**2**] there is a set $E_s \subset L$, minimally thin at η with respect to Ω, and an extended real number l, such that

$$\frac{s(z)}{G_\Omega(\zeta_1, z)} \to l < 0 \qquad (z \to \eta, z \in L \backslash E_s). \tag{18}$$

Since $v \leq G_{\Omega \cup \{\infty\}}(\cdot, \infty)$, the same result, together with Harnack's inequalities, shows that there is a set $E_v \subset \Omega$, minimally thin at η with respect to Ω, such that

$$\frac{v(z)}{G_\Omega(\zeta_1, z)} \to \limsup_{w \to \eta, w \in \Omega} \frac{v(w)}{G_\Omega(\zeta_1, w)} \qquad (z \to \eta, z \in \Omega \backslash E_v). \tag{19}$$

The right hand side of (19) is non-negative because $\eta \in \partial V$. Also, $v_0 = 0$ on L. Since $s = v + v_0$, and the union of two minimally thin sets is minimally thin, we now arrive at a contradiction to (18). This yields the desired conclusion that $R = t + \varepsilon$. □

We need a further tool before we can complete the proof of Theorem 3.

THEOREM 8. *Let Ω be a Jordan domain (or an exterior Jordan domain), let $\zeta_1 \in \Omega$ and $f \in \mathcal{H}(\Omega)$, and h be a positive harmonic function on Ω. Further, let U be the largest domain containing ζ_1 on which a given subsequence $(S_{N_k}(f, \zeta_1))$ is locally uniformly convergent. If $\xi \in \partial\Omega$ satisfies*

(i) *$D(\xi, \delta) \cap \Omega \subset U$ for some $\delta > 0$;*

(ii) $(S_{N_k}(f,\zeta_1))$ *is uniformly bounded on an open arc of* $\partial\Omega$ *containing* ξ, *and*

(iii) *the set* $\{|f|>e^h\}$ *is minimally thin at* ξ *with respect to* Ω,

then $(e^{-h}S_{N_k})$ *is uniformly bounded on a set of the form* $\Omega\backslash E$, *where* E *is minimally thin at* ξ *with respect to* Ω.

COROLLARY 9. *Let* Ω, ζ_1, t *and* ε *be as in Theorem 7. If* $f\in\mathcal{U}(\Omega,\zeta_1)$ *and* h *is a positive harmonic function on* Ω, *then there is at most one point of* $\partial\Omega\cap D(\zeta_1,t+\varepsilon)$ *at which the set* $\{|f|>e^h\}$ *is minimally thin with respect to* Ω.

The proofs of these results are directly analogous to the corresponding results for $f\in\mathcal{U}(\mathbb{D},0)$ (see Theorem 4 and Corollary 5 in [**10**]), except that, in the case of the corollary, Theorem 7 is used as a substitute for the fact that the Taylor series about 0 of a function in $\mathcal{H}(\mathbb{D})$ converges locally uniformly in $\mathbb{D}$. They also rely on the observation that the boundary Harnack principle is valid in Jordan domains, by Carathéodory's theorem.

Now let Ω be an exterior Dini domain and $\zeta\in\Omega$, choose $\xi_1\in\partial\Omega$ at maximum distance r from ζ and let $\zeta_1=\xi_1+t(\xi_1-\zeta)/\left|\xi_1-\zeta\right|$ for some $t>0$. Since $\Omega^c\subset\overline{D(\zeta,r)}$ and $\partial\Omega$ is given locally by the graph of a C^1 function, we can choose $\varepsilon>0$ such that the hypotheses (i) and (ii) of Theorem 7 are satisfied. Next, let D be a disc in Ω such that $\overline{D}\cap\Omega^c=\{\xi_2\}$ for some $\xi_2\in\partial\Omega\cap D(\zeta_1,t+\varepsilon)\backslash\{\xi_1\}$, and let h be a positive harmonic function on Ω that tends to ∞ at both ξ_1 and ξ_2. Further, let $E_1=\overline{D(\zeta_1,t)}\cap\Omega$ and $E_2=\overline{D}\cap\Omega$. By Theorem 4 there exists f in $\mathcal{U}(\Omega,\zeta)$ such that $|f|\leq e^h$ on $E_1\cup E_2$. Since $\partial\Omega$ is Dini smooth, the set $\Omega\backslash(E_1\cup E_2)$ is minimally thin with respect to Ω at both ξ_1 and ξ_2, as was explained at the beginning of this section. It now follows from Corollary 9 that $f\notin\mathcal{U}(\Omega,\zeta_1)$, as required.

6. Universal Laurent series

We will now consider domains $\Omega\subset\mathbb{C}$ of the form $\Omega=\mathbb{C}\backslash\left(\cup_0^k A_j\right)$, where $k\geq 1$, the sets A_j are pairwise disjoint continua in $\widehat{\mathbb{C}}$, and $\infty\in A_0$. (To avoid trivialities we will assume that at least one of these continua is non-degenerate.) Any function $f\in\mathcal{H}(\Omega)$ has a unique decomposition of the form

(20)
$$f=\sum_{j=0}^{k}f_j,\text{ where } f_j\in\mathcal{H}\left(\widehat{\mathbb{C}}\backslash A_j\right)\ (j=0,...,k)\text{ and } f_j(\infty)=0\ (j=1,...,k).$$

We fix $\alpha_j\in A_j$ for each $j\geq 1$. Then f_j has a Laurent expansion outside some closed disc centred at α_j, and the coefficient of $(z-\alpha_j)^{-n}$ in this expansion will be denoted by $b_n(f_j,\alpha_j)$. Denoting the nth degree Taylor coefficient of f_0 about a point $\zeta\in A_0^c$ by $c_n(f_0,\zeta)$, we define

$$M_N(f,\zeta)(z)=\sum_{n=0}^{N}c_n(f_0,\zeta)(z-\zeta)^n+\sum_{j=1}^{k}\sum_{n=1}^{N}\frac{b_n(f_j,\alpha_j)}{(z-\alpha_j)^n}\quad(z\in\mathbb{C}\backslash\{\alpha_1,...,\alpha_k\}).$$

We then say that f has a *universal Laurent series* with respect to $\{\alpha_1,...,\alpha_k\}$ if, for every compact set $K\subset(\Omega\cup\{\alpha_1,...,\alpha_k\})^c$ with K^c connected, and every function $g \in C(K) \cap \mathcal{H}(K^\circ)$, there is a sequence (N_k) in $\mathbb{N}$ such that

$$\sup_{\zeta \in J} \sup_{z \in K} |M_{N_k}(f,\zeta)(z) - g(z)| \to 0 \quad (k \to \infty)$$

for every compact set $J \subset A_0^c$. This notion was introduced by Costakis, Nestoridis and Papadoperakis [**5**], who showed that the collection $\mathcal{U}_L(\Omega; \alpha_1, ..., \alpha_k)$ of functions with this property is a dense G_δ subset of $\mathcal{H}(\Omega)$. Subsequent work on universal Laurent series includes [**6**], [**12**], [**15**], [**16**] and [**20**].

In the light of Section 1 it is natural to ask about boundary behaviour of functions possessing universal Laurent series, and whether the collection $\mathcal{U}_L(\Omega; \alpha_1, ..., \alpha_k)$ depends on the choice of $\alpha_1, ..., \alpha_k$. (We are grateful to Vassili Nestoridis for informing us that the latter question has remained open.) We deal first with boundary behaviour.

Theorem 10. *Let* $f \in \mathcal{U}_L(\Omega; \alpha_1, ..., \alpha_k)$*, where* Ω *is as above. Then, for any disc* D *centred at a point of* $\partial\Omega \backslash \{\alpha_1, ..., \alpha_k\}$*, the set* $\mathbb{C} \backslash f(D \cap \Omega)$ *is polar. In particular,* f *is unbounded.*

We next show that the the collection $\mathcal{U}_L(\Omega; \alpha_1, ..., \alpha_k)$ can depend on the choice of $\alpha_1, ..., \alpha_k$, by analyzing the case of exterior Jordan domains.

Theorem 11. *Let* $f \in \mathcal{U}_L(\Omega; \alpha_1)$*, where* Ω *is an exterior Jordan domain and* $\alpha_1 \in \partial\Omega$*. Then* $\mathcal{U}_L(\Omega; \alpha_1) \backslash \mathcal{U}_L(\Omega; \alpha) \neq \emptyset$ *for every* $\alpha \in \Omega^c \backslash \{\alpha_1\}$*.*

As a first step in the proof of Theorem 10 we observe the following.

Proposition 12. *Let* Ω *be as in the opening paragraph of this section. If* $f \in \mathcal{U}_L(\Omega; \alpha_1, ..., \alpha_k)$*, then, in the notation of (20),*

$$f_0 \in \mathcal{U}(A_0^c, \zeta) \quad (\zeta \in \Omega) \quad \textit{and} \quad f_j \in \mathcal{U}_L(A_j^c; \alpha_j) \quad (j = 1, ..., k).$$

Proof of Proposition 12. Let $f \in \mathcal{U}_L(\Omega; \alpha_1, ..., \alpha_k)$, and fix $j \in \{1, ..., k\}$ and $\zeta \in \Omega$. Next, let $K \subset A_j \backslash \{\alpha_j\}$ be a compact set with connected complement, and let $g \in C(K) \cap \mathcal{H}(K^\circ)$. By Theorem 5 of [**16**] we can choose a sequence (N_k) in $\mathbb{N}$ such that

$$M_{N_k}(f,\zeta) \to g + \sum_{\substack{i=0 \\ i \neq j}}^{k} f_i \quad \text{uniformly on } K \tag{21}$$

and

$$M_{N_k}(f,\zeta) \to f \quad \text{locally uniformly on } \Omega. \tag{22}$$

(The cited result supposes that $A_1, ..., A_k$ are all nondegenerate, but this assumption is not essential.) By applying Cauchy's integral formula to (22) on suitable contours around A_j we see that

$$\sum_{n=1}^{N_k} \frac{b_n(f_j, \alpha_j)}{(z - \alpha_j)^n} \to f_j \quad \text{locally uniformly on } A_j^c,$$

and so

$$M_{N_k}(f - f_j, \zeta) \to \sum_{\substack{i=0 \\ i \neq j}}^{k} f_i \quad \text{locally uniformly on } \Omega. \tag{23}$$

Indeed, by the maximum modulus theorem, the convergence in (23) is locally uniform on $\Omega \cup A_j$. We now see from (21) that

$$M_{N_k}(f_j, \zeta) = \sum_{n=1}^{N_k} \frac{b_n(f_j, \alpha_j)}{(z-\alpha_j)^n} \to g \text{ uniformly on } K.$$

Thus $f_j \in \mathcal{U}_L(A_j^c; \alpha_j)$. An analogous argument shows that $f_0 \in \mathcal{U}(A_0^c, \zeta)$ for all $\zeta \in \Omega$. □

Proof of Theorem 10. Let f be as in the statement of Theorem 10 and let $\zeta \in \Omega$. Then $f_0 \in \mathcal{U}(A_0^c, \zeta)$ by Proposition 12. Now let D be a disc centred at a (finite) point of $A_0 \cap \partial\Omega$, if such a point exists, and small enough to ensure that $\overline{D} \cap A_j = \emptyset$ $(j \geq 1)$. Thus $|f - f_0| \leq C$ on $D \cap \Omega$ for some constant $C \geq 1$, and so

$$\log^+ |f_0| \leq \log^+ (2 \max\{|f_0 - f|, |f|\}) \leq \log(2C) + \log^+ |f| \quad \text{on} \quad D \cap \Omega.$$

Since A_0^c is simply connected, Theorem 1 of [**9**] tells us that $\log^+ |f_0|$ cannot have a harmonic majorant on $D \cap \Omega$. The same must therefore also be true of $\log^+ |f|$. It now follows from Myrberg's theorem that $\mathbb{C} \backslash f(D \cap \Omega)$ is polar.

Next, if $j \in \{1, ..., k\}$, then $f_j \in \mathcal{U}_L(A_j^c; \alpha_j)$ by Proposition 12, and so the function f_j^* given by

$$f_j^*(z) = f_j\left(\alpha_j + \frac{1}{z}\right) \quad (z \in \Omega_j), \text{ where } \Omega_j = \left\{\frac{1}{w - \alpha_j} : w \in \widehat{\mathbb{C}} \backslash A_j\right\},$$

belongs to $\mathcal{U}(\Omega_j, 0)$. It follows, as before, that, for any disc D centred at a point of $(A_j \cap \partial\Omega) \backslash \{\alpha_j\}$, the set $\mathbb{C} \backslash f(D \cap \Omega)$ is polar. □

We will need the following variant of Theorem 4 for the proof of Theorem 11.

Theorem 13. *Let Ω be an exterior Jordan domain, let ξ_1, ξ_2 be distinct points of $\partial\Omega$, and let E be a relatively closed subset of Ω such that $\overline{E} \cap \partial\Omega = \{\xi_1, \xi_2\}$. If $\phi : \Omega \to (1, \infty)$ is a continuous function such that $\phi(z) \to \infty$ as $z \to \xi_2$, then there exists $f \in \mathcal{U}_L(\Omega; \xi_1)$ such that $|f| \leq \phi$ on E.*

Proof. For each $m \in \mathbb{N}$, let Ω_m be an exterior Jordan domain satisfying

$$\Omega \cup \{\xi_1\} \subset \Omega_m \subset \Omega \cup D(\xi_1, m^{-1}) \quad \text{and} \quad \xi_2 \in \partial\Omega_m.$$

We can choose a conformal mapping $\psi_m : \Omega_m \to \mathbb{D}$ that extends to a homeomorphism from $\overline{\Omega}_m$ to $\overline{\mathbb{D}}$ and satisfies $\psi_m(\xi_2) = 1$.

Let (L_m) be an exhaustion of $\Omega \cup \{\infty\}$ by compacts such that $\infty \in L_1^\circ$ and L_m^c is connected for each m. Given $m, N \in \mathbb{N}$ we can use Cauchy's estimates to find $\eta(N, m) \in (0, 1)$ small enough so that, if $g \in \mathcal{H}(L_1^\circ)$ and $\|g\|_{L_1^\circ} < \eta(N, m)$, then

$$\|T_n(g, \xi_1)\|_{\Omega_m^c} \leq 2^{-m} \quad (n = 0, 1, ..., N),$$

where

$$T_n(g, \xi) = \sum_{m=1}^{n} \frac{b_m(g, \xi)}{(z - \xi)^m} \quad (z \neq \xi).$$

For each $m \in \mathbb{N}$ let U_m denote the union of the components U of $\Omega \backslash (L_m \cup E)$ satisfying $\overline{U} \cap \partial\Omega \subset \{\xi_1, \xi_2\}$. We define

$$M_m = L_m \cup \overline{E} \cup U_m \cup \Omega_m^c \qquad (m \in \mathbb{N}).$$

By our assumptions on E the open set M_m^c is connected for each m.

Let (P_m) be an enumeration of the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$, and let $R_m(z) = P_m((z - \xi_1)^{-1})$ for each m. We inductively define a sequence (Q_m) of functions in $\mathcal{H}(\widehat{\mathbb{C}} \backslash \{\xi_1\})$, as follows.

Let $N_0 = 1$, and suppose that we are given $m, N_{m-1} \in \mathbb{N}$ and functions $Q_1, ..., Q_{m-1} \in \mathcal{H}(\widehat{\mathbb{C}} \backslash \{\xi_1\})$. Since $|\psi_m| < 1$ on $\Omega \cup \{\xi_1\}$ and $\phi(z) \to \infty$ as $z \to \xi_2$, we can choose $n \in \mathbb{N}$ large enough so that

$$|R_m^*| \leq 2^{-m-2} \eta(N_{m-1}, m) \frac{\phi}{\|\phi\|_{L_m}} \quad \text{on} \;\; L_m \cup E, \tag{24}$$

where

$$R_m^* = \begin{cases} R_m - \sum_1^{m-1} Q_j & \text{on} \;\; \Omega_m^c \\ \psi_m^n \left\{ R_m(\xi_2) - \sum_1^{m-1} Q_j(\xi_2) \right\} & \text{on} \;\; \Omega \cup \{\xi_1\}. \end{cases} \tag{25}$$

Since R_m^* is continuous on M_m and holomorphic on M_m°, we see from Mergelyan's theorem and an inversion that there is a rational function Q_m^* with exactly one pole, in M_m^c, such that

$$\|Q_m^* - R_m^*\|_{M_m} < 2^{-m-3} \eta(N_{m-1}, m).$$

A pole-pushing argument yields a function Q_m in $\mathcal{H}(\widehat{\mathbb{C}} \backslash \{\xi_1\})$ such that

$$\|Q_m - R_m^*\|_{M_m \backslash \{\xi_1\}} < 2^{-m-2} \eta(N_{m-1}, m). \tag{26}$$

We can choose $N_m > N_{m-1}$ sufficiently large so that

$$\begin{aligned} \left\| T_{N_m} \left(\sum_1^m Q_j, \xi_1 \right) - R_m \right\|_{\Omega_m^c} &< 2^{-m-2} + \left\| \sum_1^m Q_j - R_m \right\|_{\Omega_m^c} \\ &= 2^{-m-2} + \|Q_m - R_m^*\|_{\Omega_m^c} < 2^{-m-1}, \end{aligned} \tag{27}$$

by (25) and (26). Also, by (24) and (26),

$$\|Q_m\|_{L_m} < 2^{-m-1} \eta(N_{m-1}, m) < 2^{-m-1} \tag{28}$$

and, since $\phi > 1$,

$$|Q_m| < 2^{-m-1} \phi \quad \text{on} \;\; E. \tag{29}$$

Having constructed the sequence (Q_m) as above, we see from (28) that the series $\sum Q_m$ converges locally uniformly on $\Omega \cup \{\infty\}$ to a holomorphic function f, and from (29) that $|f| \leq \phi$ on E. Further, since we may assume that $(\eta(N_{m-1}, m))_{m \geq 1}$ is a decreasing sequence, and since

$$\left| \sum_{m+1}^\infty Q_j \right| < \eta(N_m, m+1) \quad \text{on} \;\; L_1$$

by (28) again, we see that

$$\begin{aligned} |R_m - T_{N_m}(f,\xi_1)| &\leq \left| R_m - T_{N_m}\left(\sum_1^m Q_j, \xi_1\right)\right| + \left| T_{N_m}\left(\sum_{m+1}^\infty Q_j, \xi_1\right)\right| \\ &\leq 2^{-m-1} + 2^{-m-1} = 2^{-m} \quad \text{on } \Omega_m^c, \end{aligned}$$

by (27) and the definition of $\eta(N_m, m+1)$.

It now follows from Mergelyan's theorem that $f \in \mathcal{U}_L(\Omega;\xi_1)$, so the theorem is proved. ☐

The next result can be deduced from Theorem 8 in the same way that Corollary 5 in [**10**] was deduced from Theorem 4 of that paper.

COROLLARY 14. *Let Ω be an exterior Jordan domain and let $\xi \in \Omega^c$. If $f \in \mathcal{U}_L(\Omega;\xi)$ and h is a positive harmonic function on Ω, then there is at most one point of $\partial\Omega\backslash\{\xi\}$ at which the set $\{|f| > e^h\}$ is minimally thin with respect to Ω.*

PROOF OF THEOREM 11. Let Ω and α_1 be as in the statement of the theorem, let $\alpha \in \Omega^c\backslash\{\alpha_1\}$ and $\xi \in \partial\Omega\backslash\{\alpha,\alpha_1\}$, and let h be a positive harmonic function on Ω which tends to ∞ at ξ. Further, let E be a relatively closed subset of Ω such that $\overline{E} \cap \partial\Omega = \{\alpha_1, \xi\}$ and $\Omega\backslash E$ is minimally thin at α_1 and ξ with respect to Ω. By Theorem 13 there exists $f \in \mathcal{U}_L(\Omega;\alpha_1)$ such that $|f| \leq e^h$ on E. However, by Corollary 14, $f \notin \mathcal{U}_L(\Omega;\alpha)$, so the theorem is proved. ☐

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