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Control Law Realification for the Feedback Stabilization of a Class of Diagonal Infinite-Dimensional Systems with Delay Boundary Control

Hugo Lhachemi, Robert Shorten, and Christophe Prieur

***Abstract*—Recently, a predictor feedback control strategy has been reported for the feedback stabilization of a class of infinite-dimensional Riesz-spectral boundary control systems exhibiting a finite number of unstable modes by means of a delay boundary control. Nevertheless, for real abstract boundary control systems exhibiting eigenstructures defined over the complex field, the direct application of such a control strategy requires the embedding of the control problem into a complexified state-space which yields a complex-valued control law. This paper discusses the realification of the control law, i.e., the modification of the design procedure for obtaining a real-valued control law for the original real abstract boundary control system. The obtained results are applied to the feedback stabilization of an unstable Euler-Bernoulli beam by means of a delay boundary control.**

***Index Terms*—Distributed parameter systems, Boundary feedback stabilization, Predictor feedback, Euler-Bernoulli beam.**

I. Introduction

FEEDBACK control of finite-dimensional systems in the presence of input delays has been extensively investigated [1], [19]. Its extension to infinite-dimensional systems such as Partial Differential Equations (PDEs) for unbounded control operators has attracted many attention in the recent years [7], [9], [14]–[16] .

In this paper, we are concerned with the feedback stabilization of open-loop unstable infinite-dimensional systems by means of delay boundary control. One of the first contributions in this field dealt with a reaction-diffusion equation where the controller was designed by resorting to the backstepping technique [10]. More recently, the opportunity of designing a predictor feedback control for a linear reaction-diffusion equation presenting a constant input delay was reported in [18]. Inspired by early developments in the undelayed boundary control of PDEs via a truncated model capturing the unstable part of the system dynamics [4], [5], [21], the control law was computed based on a truncated model by applying the Artstein transformation [1], [19] and the classical pole-shifting theorem. The same design procedure has been employed for the delay feedback stabilization of a linearized Kuramoto-Sivashinsky equation in [8]. This idea has been generalized in [11] for the feedback stabilization of boundary control systems [6] for which the associated disturbance-free operator is a Riesz-spectral operator admitting a finite number of unstable eigenvalues.

In the case of the reaction diffusion-equation or the linearized Kuramoto-Sivashinsky equation studied in [18] and [8], respectively, the Riesz-spectral property holds for the associated real Hilbert Space. However, certain systems such as strings and beams inherently present eigenstructures belonging to a complex Hilbert space. In this case, the control law reported in [11] is *a priori* complex-valued, even if the original problem is defined over the real field. To tackle this issue, a naive approach would consist in only keeping the real part of the complex-valued control law for obtaining a real-valued stabilizing feedback of the original real abstract boundary control system. However, by doing so, the resulting control law entangles both real and imaginary parts of the complex abstract boundary control system. Therefore, the obtained real-valued control law to be applied to the original real abstract boundary control system cannot be expressed as a state feedback as it depends on the imaginary part of the complex abstract boundary control system. The objective of this paper is to present a modification of the design procedure to ensure that, even after the incursion into the complex field for studying the eigenstructures of the system, the final control law is real-valued and can be expressed as a state-feedback of the original real boundary control system.

The remaining of the paper is organized as follows. Notations and the concept of complexification of a real Hilbert space are presented in Section II. The problem setting is introduced in Section III while the corresponding feedback control strategy within the complexified Hilbert space is described in Section IV. Then, the realification of the control law is discussed in Section V. The obtained results are illustrated for an unstable Euler-Bernoulli Beam in Section VI. Finally, concluding remarks are formulated in Section VII.

II. Notation and complexification of a real Hilbert space

The field $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. All the finite-dimensional spaces $\mathbb{K}^p$ are endowed with the usual euclidean inner product $\langle x, y \rangle = x^* y$ and the associated 2-norm $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^* x}$, where $x^* = \overline{x}^\top$. For any matrix $M \in \mathbb{K}^{p \times q}$, $\|M\|$ stands for the induced norm of M associated with the above 2-norms. Throughout the paper, we assume that $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space over the field $\mathbb{R}$. The associated norm is denoted by $\| \cdot \|$. The following definition introduces the concept of complexification of the real Hilbert space $\mathcal{H}$, see, e.g., [13], [20].

Definition 2.1 ([20]): The complexification $\mathcal{H}_c$ of $\mathcal{H}$ is the $\mathbb{C}$-vector space $\mathcal{H}^2$ when endowed with the vector addition defined for any $(x_1, y_1), (x_2, y_2) \in \mathcal{H}^2$ by $(x_1, y_1) + (x_2, y_2) =$

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(x_1+x_2, y_1+y_2) and with the scalar multiplication defined for any $(x,y) \in \mathcal{H}^2$ and $\alpha + i\beta \in \mathbb{C}$ with $\alpha, \beta \in \mathbb{R}$ by $(\alpha+i\beta)\cdot(x,y) = (\alpha x - \beta y, \alpha y + \beta x)$.

We introduce for any $(x,y) \in \mathcal{H}_c$ the notations $x + i_c y = (x,y)$ and $x - i_c y = x + i_c(-y) = (x,-y)$. We define the real and imaginary parts of $z = x + i_c y \in \mathcal{H}_c$ as $\operatorname{Re} z = x \in \mathcal{H}$ and $\operatorname{Im} z = y \in \mathcal{H}$. The complex conjugate function $\overline{\cdot} : \mathcal{H}_c \rightarrow \mathcal{H}_c$ is defined for any $z = x + i_c y \in \mathcal{H}_c$ by $\overline{x + i_c y} = x - i_c y$. The complex conjugate function is its own inverse and thus is an involution. Furthermore, we have for all $\lambda \in \mathbb{C}$ and $z \in \mathcal{H}_c$ that $\overline{\lambda \cdot z} = \overline{\lambda} \cdot \overline{z}$. For any $S_1, S_2 \subset \mathcal{H}$, we define $S_1 + i_c S_2 = \{x + i_c y \in \mathcal{H} \,:\, x \in S_1,\ y \in S_2\} \subset \mathcal{H}_c$.

Property 2.2 ([13]): Defining $\langle \cdot, \cdot \rangle : \mathcal{H}_c \times \mathcal{H}_c \rightarrow \mathbb{C}$ by $\langle x_1 + i_c y_1, x_2 + i_c y_2 \rangle_c = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i\left[\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle\right]$, $(\mathcal{H}_c, \langle \cdot, \cdot \rangle)$ is a separable $\mathbb{C}$-Hilbert space.

The associated norm is denoted by $\|\cdot\|_c$ and is such that $\|z\|_c^2 = \|\operatorname{Re} z\|^2 + \|\operatorname{Im} z\|^2$. As for any $x \in \mathcal{H}$, $\|(x,0)\|_c = \|x\|$, we identify $\mathcal{H}$ and the subspace $\mathcal{H} + i_c\{0\}$ of $\mathcal{H}_c$ and we denote, with a slight abuse of notation, $x = x + i_c 0$. For any $x, y \in \mathcal{H}$, $\langle x, y \rangle_c = \langle x, y \rangle$. A straightforward computation shows that, for any $x, y \in \mathcal{H}_c$, $\overline{\langle x, y \rangle_c} = \langle \overline{x}, \overline{y} \rangle_c$.

Definition 2.3: Let $(\mathcal{H}_k, \langle \cdot, \cdot \rangle_k)$, $k \in \{1,2\}$, be two $\mathbb{R}$-Hilbert spaces. For a given $\mathbb{R}$-linear (eventually unbounded) operator $A : D(A) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_2$, its complexification $A_c : D(A_c) \subset \mathcal{H}_{1,c} \rightarrow \mathcal{H}_{2,c}$ is defined for any $z \in D(A_c) \triangleq D(A) + i_{1,c} D(A)$ by $A_c z = A \operatorname{Re} z + i_{2,c} A \operatorname{Im} z$.

Lemma 2.4 ([13]): The complexification A_c of the $\mathbb{R}$-linear operator A is a $\mathbb{C}$-linear operator. Furthermore, if $A \in \mathcal{L}_{\mathbb{R}}(\mathcal{H}_1, \mathcal{H}_2)$, then $A_c \in \mathcal{L}_{\mathbb{C}}(\mathcal{H}_{1,c}, \mathcal{H}_{2,c})$.

Lemma 2.5: Let $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ be given. Let A_c be the complexification of A. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A_c with associated eigenvector $\phi \in \mathcal{H}_c$. Then $\overline{\lambda} \in \mathbb{C}$ is an eigenvalue of A_c with associated eigenvector $\overline{\phi} \in \mathcal{H}_c$.

Proof: $A_c \overline{\phi} = A_c(\operatorname{Re}\phi - i_c \operatorname{Im}\phi) = A \operatorname{Re}\phi - i_c A \operatorname{Im}\phi = \overline{A \operatorname{Re}\phi + i_c A \operatorname{Im}\phi} = \overline{A_c(\operatorname{Re}\phi + i_c \operatorname{Im}\phi)} = \overline{A_c \phi} = \overline{\lambda\phi} = \overline{\lambda} \cdot \overline{\phi}$ □

Lemma 2.6: Let $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ be the generator of a C_0-semigroup S on $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ (see, e.g., [17]) and let A_c be its complexification. Then $S_c : \mathbb{R}_+ \rightarrow \mathcal{L}(\mathcal{H}_c)$ defined for any $t \geq 0$ and $z \in \mathcal{H}_c$ by $S_c(t) z = S(t) \operatorname{Re} z + i_c S(t) \operatorname{Im} z$ is a C_0-semigroup on $(\mathcal{H}_c, \langle \cdot, \cdot \rangle_c)$ with infinitesimal generator A_c.

Proof: Direct consequence of Lemma 2.4 and of the identity $\|z\|_c^2 = \|\operatorname{Re} z\|^2 + \|\operatorname{Im} z\|^2$ for all $z \in \mathcal{H}_c$. □

III. Problem Setting

A. Real abstract boundary control system

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space. We consider the following real[1] abstract boundary control system [6] with delayed boundary control:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + d(t), & t \geq 0 \\ \mathcal{B}X(t) = u_D(t) \triangleq u(t-D), & t \geq 0 \\ X(0) = X_0 & \end{cases} \tag{1}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ a linear (unbounded) operator, $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \rightarrow \mathbb{R}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ a linear boundary operator, $d : \mathbb{R}_+ \rightarrow \mathcal{H}$ a distributed disturbance, and $u : [-D, +\infty) \rightarrow \mathbb{R}^m$, with a known constant delay $D > 0$ and $u|_{[-D,0)} = 0$, the boundary control. We assume that $(\mathcal{A}, \mathcal{B})$ is a real boundary control system, i.e., 1) the disturbance-free operator $\mathcal{A}_0$, defined over the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$, is the generator of a C_0-semigroup S on $\mathcal{H}$; 2) there exists a bounded operator $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{H})$, called a lifting operator, such that $\mathrm{R}(B) \subset D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(\mathbb{R}^m, \mathcal{H})$, and $\mathcal{B}B = I_{\mathbb{R}^m}$.

[1] I.e., $\mathcal{H}$ is a real Hilbert space and the control input is real-valued.

B. Complexification of the abstract boundary control system

Let $(\mathcal{H}_c, \langle \cdot, \cdot \rangle_c)$ be the complexification of the real Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We introduce the complexified version of the abstract boundary control problem (1) as follows:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t}(t) = \mathcal{A}_c Y(t) + p(t), & t \geq 0 \\ \mathcal{B}_c Y(t) = v_D(t) \triangleq v(t-D), & t \geq 0 \\ Y(0) = Y_0 & \end{cases} \tag{2}$$

where $\mathcal{A}_c$ and $\mathcal{B}_c$ denote the complexified versions of operators $\mathcal{A}$ and $\mathcal{B}$, respectively. In this case, $p : \mathbb{R}_+ \rightarrow \mathcal{H}_c$ is a distributed disturbance and $v : [-D, +\infty) \rightarrow \mathbb{C}^m$ with $v|_{[-D,0)} = 0$ is the boundary control. We introduce the disturbance-free operator $[\mathcal{A}_c]_0$ defined over the domain $D([\mathcal{A}_c]_0) \triangleq D(\mathcal{A}_c) \cap \ker(\mathcal{B}_c)$ by $[\mathcal{A}_c]_0 \triangleq \mathcal{A}_c|_{D([\mathcal{A}_c]_0)}$. Then, based on Lemmas 2.4 and 2.6, it is easy to show that $(\mathcal{A}_c, \mathcal{B}_c)$ is a complex[2] boundary control system with $[\mathcal{A}_c]_0 = [\mathcal{A}_0]_c$ and with associated lifting operator B_c, the complexification of B.

Remark 3.1: If Y with control law v is a classical solution of (2) associated with Y_0 and p, then $X = \operatorname{Re} Y$ with control law $u = \operatorname{Re} v$ is a classical solution of (1) associated with $X_0 = \operatorname{Re} Y_0$ and $d = \operatorname{Re} p$. However, if the control law v is obtained by a state feedback $v = f(Y)$, then $u = \operatorname{Re} v = \operatorname{Re} f(Y)$ depends in general on $\operatorname{Im} Y$ and thus is not a pure state-feedback of $X = \operatorname{Re} Y$, i.e., is not of the form $u = g(X)$. For the problem setting described hereafter and the closed-loop dynamics (6a-6e), the objective of this paper is to show that an appropriate modification of the control design procedure can be used for uncoupling the real part of the dynamics from its imaginary part, and thus obtaining a real-valued control law u under the form of a state-feedback of X. This is achieved by an adequate selection of the eigenstructures and a detailed study of the associated truncated model used to design the predictor feedback.

C. Assumptions on the complexified boundary control system

We assume that the complexified boundary control system $(\mathcal{A}_c, \mathcal{B}_c)$ presents the following diagonal structure, which is typical of many applications such as reaction-diffusion equations and structural vibrations.

Assumption 3.2: The disturbance-free operator $[\mathcal{A}_c]_0 = [\mathcal{A}_0]_c$ is a Riesz spectral operator [6], i.e., is a linear and closed operator with simple eigenvalues λ_n and corresponding eigenvectors $\phi_n \in D([\mathcal{A}_0]_c)$, $n \in \mathbb{N}^*$, that satisfy:

[2] I.e., $\mathcal{H}_c$ is a complex Hilbert space and the control input is complex-valued.

1) $\{\phi_n,\ n \in \mathbb{N}^*\}$ is a Riesz basis [3]:
   a) $\overline{\operatorname{span}_{\mathbb{C}}}_{n\in\mathbb{N}^*} \phi_n = \mathcal{H}_c$;
   b) there exist constants $m_R, M_R \in \mathbb{R}_+^*$ such that for all $N \in \mathbb{N}^*$ and all $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$,

$$m_R \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|_c^2 \leq M_R \sum_{n=1}^{N} |\alpha_n|^2. \tag{3}$$

2) The closure of $\{\lambda_n,\ n \in \mathbb{N}^*\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$, $[a,b] \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$.

As (3) yields that $\sqrt{m_R} \leq \|\phi_n\|_c \leq \sqrt{M_r}$, we can assume without loss of generality (by normalizing the vectors and changing $m_R, M_R \in \mathbb{R}_+^*$) that $\|\phi_n\|_c = 1$ for all $n \in \mathbb{N}^*$.

Lemma 3.3: Let $\mathcal{I} = \{n \in \mathbb{N}^* \,:\, \lambda_n \in \mathbb{R}\}$. For any $n \in \mathcal{I}$, we define $\tilde{\phi}_n \in \mathcal{H}\backslash\{0\} \subset \mathcal{H}_c$ by $\tilde{\phi}_n = \phi_n$ if $\phi_n \in \mathcal{H}$, $\tilde{\phi}_n = \operatorname{Im}\phi_n / \|\operatorname{Im}\phi_n\|$ otherwise. Then, $\tilde{\phi}_n$ is an eigenvector associated with λ_n. Furthermore, $\Phi \triangleq \{\phi_n,\ n \in \mathbb{N}^*\backslash\mathcal{I}\} \cup \left\{\tilde{\phi}_n,\ n \in \mathcal{I}\right\}$ forms a Riesz basis composed of unit eigenvectors of $[\mathcal{A}_c]_0$.

Proof: Let $n \in \mathcal{I}$ with $\|\operatorname{Im}\phi_n\| \neq 0$ be given. As $[\mathcal{A}_0]_c \phi_n = \lambda_n \phi_n$ with $\lambda_n \in \mathbb{R}$, we obtain by taking the imaginary part that $[\mathcal{A}_0]_c \operatorname{Im}\phi_n = \lambda_n \operatorname{Im}\phi_n$ with $\operatorname{Im}\phi_n \neq 0$. As the eigenvalues are assumed simple, we obtain that there exists $\gamma \in \mathbb{C}\backslash\{0\}$ such that $\phi_n = \gamma \operatorname{Im}\phi_n$. Thus $1 = \|\phi_n\|_c = |\gamma| \|\operatorname{Im}\phi_n\|$ and we obtain the existence of a $\theta_n \in [0, 2\pi)$ such that $\gamma = e^{i\theta_n}/\|\operatorname{Im}\phi_n\|$. Consequently, we have $\phi_n = e^{i\theta_n}\tilde{\phi}_n$ and $[\mathcal{A}_0]_c \tilde{\phi}_n = \lambda_n \tilde{\phi}_n$ with $\tilde{\phi}_n \in \mathcal{H}\backslash\{0\} \subset \mathcal{H}_c$. In the case $n \in \mathcal{I}$ with $\|\operatorname{Im}\phi_n\| = 0$, we also have $\phi_n = e^{i\theta_n}\tilde{\phi}_n$ when setting $\theta_n = 0$. In both cases, $\|\tilde{\phi}_n\|_c = \|\phi_n\|_c = 1$. Now, as $\overline{\operatorname{span}_{\mathbb{C}}}_{n\in\mathbb{N}^*} \phi_n = \mathcal{H}_c$, we immediately obtain that $\overline{\operatorname{span}_{\mathbb{C}}} \Phi = \mathcal{H}_c$. Finally, let $N \in \mathbb{N}^*$ and $\beta_1, \ldots, \beta_N \in \mathbb{C}$ be arbitrarily given. We denote $\mathcal{I}_N = \{1, \ldots, N\} \cap \mathcal{I}$ and $\mathcal{J}_N = \{1, \ldots, N\}\backslash\mathcal{I}_N$. We define $\alpha_n = e^{-i\theta_n}\beta_n \in \mathbb{C}$ if $n \in \mathcal{I}$ and $\alpha_n = \beta_n \in \mathbb{C}$ otherwise. Noting that $|\beta_n| = |\alpha_n|$ for all $n \geq 1$ and that $\alpha_n \phi_n = e^{-i\theta_n}\beta_n \times e^{i\theta_n}\tilde{\phi}_n = \beta_n \tilde{\phi}_n$ if $n \in \mathcal{I}$ and $\alpha_n \phi_n = \beta_n \phi_n$ otherwise, (3) yields

$$m_R \sum_{n=1}^{N} |\beta_n|^2 \leq \left\| \sum_{n\in\mathcal{I}_N} \beta_n \tilde{\phi}_n + \sum_{n\in\mathcal{J}_N} \beta_n \phi_n \right\|_c^2 \leq M_R \sum_{n=1}^{N} |\beta_n|^2,$$

which completes the proof. □

Based on the latter lemma, we assume without loss of generality that the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$ is selected such that: $\lambda_n \in \mathbb{R} \Rightarrow \phi_n \in \mathcal{H}$. Furthermore, from $\|\phi_n\| = 1$, Lemma 2.5, and the assumption that the eigenvalues are simple, we also assume without loss of generality, by a similar argument as the one employed in the proof of Lemma 3.3, that $\{\phi_n,\ n \in \mathbb{N}^*\}$ is closed under complex conjugation. Thus we have for any $n_1 \neq n_2$ that $\lambda_{n_1} = \overline{\lambda_{n_2}} \Leftrightarrow \phi_{n_1} = \overline{\phi_{n_2}}$.

We make the following assumption that there exists a finite number of unstable modes and that the real part of the stable modes does not accumulate at 0.

Assumption 3.4: There exist $N_0 \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}_+^*$ such that $\operatorname{Re}\lambda_n \leq -\alpha$ for all $n \geq N_0 + 1$.

Based on Lemma 2.5, we assume without loss of generality (by an adequate numbering of the eigenvalues and an appropriate selection of N_0) that $\{\lambda_n \,:\, 1 \leq n \leq N_0\}$ is closed under complex conjugation. As the eigenvalues are simple, we can also assume that there exists $0 \leq n_0 \leq N_0/2$ such that $\lambda_{2k-1} = \overline{\lambda_{2k}} \in \mathbb{C}\backslash\mathbb{R}$ for all $1 \leq k \leq n_0$ and $\lambda_k \in \mathbb{R}$ for all $2n_0 + 1 \leq k \leq N_0$.

From the well-known properties of the Riesz-basis (see, e.g., [3]), we introduce $\{\psi_n,\ n \in \mathbb{N}^*\}$ the biorthogonal sequence associated with the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e., $\langle \phi_k, \psi_l \rangle_c = \delta_{k,l} \in \{0,1\}$ with $\delta_{k,l} = 1 \Leftrightarrow k = l$. Then, we have:

$$\forall x \in \mathcal{H}_c, \qquad x = \sum_{n\geq 1} \langle x, \psi_n \rangle_c \phi_n = \sum_{n\geq 1} \langle x, \phi_n \rangle_c \psi_n. \tag{4}$$

Lemma 3.5: Let $n \in \mathbb{N}^*$ be such that $\lambda_n \in \mathbb{R}$. As $\phi_n \in \mathcal{H}$, then we have $\psi_n \in \mathcal{H}$.

Proof: Consider first the case $\phi_m \in \mathcal{H}$. Then we have that $\delta_{n,m} = \langle \psi_n, \phi_m \rangle_c = \overline{\langle \psi_n, \phi_m \rangle_c} = \left\langle \overline{\psi_n}, \phi_m \right\rangle_c$. Consider now the case $\phi_m \notin \mathcal{H}$. Then we have $m \neq n$ and there exists $m' \neq n, m$ such that $\phi_m = \overline{\phi_{m'}}$. This yields $0 = \langle \psi_n, \phi_{m'} \rangle_c = \overline{\langle \psi_n, \phi_{m'} \rangle_c} = \left\langle \overline{\psi_n}, \phi_m \right\rangle_c$. From the series expansion (4), we deduce that $\overline{\psi_n} = \sum_{m\geq 1} \left\langle \overline{\psi_n}, \phi_m \right\rangle_c \psi_m = \psi_n$. □

Lemma 3.6: Let $n_1 \in \mathbb{N}^*$ be such that $\lambda_{n_1} \in \mathbb{C}\backslash\mathbb{R}$. Let $n_2 \neq n_1$ be the unique integer such that $\lambda_{n_2} = \overline{\lambda_{n_1}}$. Then we have $\phi_{n_1} = \overline{\phi_{n_2}}$ and $\psi_{n_1} = \overline{\psi_{n_2}}$.

Proof: The existence and uniqueness of n_2 follows from the facts that 1) the eigenvalue are simple; 2) both $\{\lambda_n,\ n \in \mathbb{N}^*\}$ and $\{\phi_n,\ n \in \mathbb{N}^*\}$ are closed under complex conjugation. We have that $\delta_{n,n_2} = \langle \phi_n, \psi_{n_2} \rangle_c = \overline{\langle \phi_n, \psi_{n_2} \rangle_c} = \left\langle \overline{\phi_n}, \overline{\psi_{n_2}} \right\rangle_c$. With $n = n_2$ we obtain that $\left\langle \phi_{n_1}, \overline{\psi_{n_2}} \right\rangle_c = 1$. With $n = n_1$ we obtain that $\left\langle \phi_{n_2}, \overline{\psi_{n_2}} \right\rangle_c = 0$. We now consider the case $n \neq n_1, n_2$. If $\phi_n \in \mathcal{H}$ then $\left\langle \phi_n, \overline{\psi_{n_2}} \right\rangle_c = \left\langle \overline{\phi_n}, \overline{\psi_{n_2}} \right\rangle_c = 0$. Otherwise, there exists a unique $n' \in \mathbb{N}^*$ with $n' \notin \{n, n_1, n_2\}$ such that $\phi_{n'} = \overline{\phi_n}$, from which we deduce that $\left\langle \phi_n, \overline{\psi_{n_2}} \right\rangle_c = \left\langle \overline{\phi_{n'}}, \overline{\psi_{n_2}} \right\rangle_c = 0$. Consequently, we obtain from (4) that $\overline{\psi_{n_2}} = \sum_{n\geq 1} \left\langle \overline{\psi_{n_2}}, \phi_n \right\rangle_c \psi_n = \psi_{n_1}$. □

IV. Feedback control strategy within the complexified Hilbert space

We introduce the control strategy reported in [11] for the studied complexified abstract boundary control system.

A. Spectral decomposition

Assuming that $v \in \mathcal{C}^2([-D, +\infty); \mathbb{C}^m)$, $Y_0 \in D(\mathcal{A}_c)$ such that $\mathcal{B}_c Y_0 = v_D(0) = 0$ (i.e., $Y_0 \in D([\mathcal{A}_0]_c)$), and $p \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H}_c)$, we denote by $Y \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A}_c)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H}_c)$ the unique classical solution of (2). We introduce $c_n(t) \triangleq \langle Y(t), \psi_n \rangle_c$ the coefficients of the projection of $Y(t)$ into the Riezs basis $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e., $Y(t) = \sum_{n\geq 1} c_n(t)\phi_n$. We also introduce $p_n(t) \triangleq \langle p(t), \psi_n \rangle_c$. Then $c_n \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{C})$ and, following [12], we have for all $t \geq 0$, $\dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n \langle B_c v_D(t), \psi_n \rangle_c + \langle \mathcal{A}_c B_c v_D(t), \psi_n \rangle_c + p_n(t)$. Let $\mathcal{E} = (e_1, e_2, \ldots, e_m)$ be the canonical basis of the $\mathbb{C}$-vector space $\mathbb{C}^m$. In particular, we have $e_k \in \mathbb{R}^m$ for all $1 \leq k \leq m$.

Introducing $b_{n,k} \triangleq -\lambda_n \langle B_c e_k, \psi_n \rangle_c + \langle \mathcal{A}_c B_c e_k, \psi_n \rangle_c$, we obtain that the following linear ODE holds true for all $t \geq 0$

$$\dot{Y}_{N_0}(t) = A_{N_0} Y_{N_0}(t) + B_{N_0} v(t-D) + P_{N_0}(t), \quad (5)$$

where $A_{N_0} = \operatorname{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{C}^{N_0 \times N_0}$, $B_{N_0} = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq m} \in \mathbb{C}^{N_0 \times m}$, and

$$Y_{N_0}(t) = \begin{bmatrix} \langle Y(t), \psi_1 \rangle_c \\ \vdots \\ \langle Y(t), \psi_{N_0} \rangle_c \end{bmatrix}, \; P_{N_0}(t) = \begin{bmatrix} \langle p(t), \psi_1 \rangle_c \\ \vdots \\ \langle p(t), \psi_{N_0} \rangle_c \end{bmatrix} \in \mathbb{C}^{N_0}.$$

We assume in the sequel that the finite-dimensional truncated subsystem, gathering the unstable modes of the original infinite-dimensional system, is stabilizable.

Assumption 4.1: (A_{N_0}, B_{N_0}) is stabilizable.

B. Dynamics of the closed-loop system and stability result

Let $D, t_0 > 0$ be given. We consider a given transition signal $\varphi \in \mathcal{C}^2([-D, +\infty); \mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi|_{[-D,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$. The closed-loop system dynamics takes for $t \geq 0$ the following form [11]:

$$\frac{\mathrm{d}Y}{\mathrm{d}t}(t) = \mathcal{A}_c Y(t) + p(t), \quad (6a)$$

$$\mathcal{B}_c Y(t) = v_D(t) = v(t-D), \quad (6b)$$

$$v|_{[-D,0]} = 0 \quad (6c)$$

$$v(t) = \varphi(t) K Y_{N_0}(t) + \varphi(t) K \int_{\max(t-D,0)}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} v(s) \, \mathrm{d}s, \quad (6d)$$

$$Y(0) = Y_0 \quad (6e)$$

with gain $K \in \mathbb{C}^{m \times N_0}$ such that $A_{\mathrm{cl}} \triangleq A_{N_0} + e^{-DA_{N_0}} B_{N_0} K$ is Hurwitz. The control input v defined by (6d) takes the form of a predictor feedback for the truncated model (A_{N_0}, B_{N_0}). It is used to control the infinite-dimensional system (6a) by means of the boundary input (6b).

Theorem 4.2 ([11]): Assume that Assumptions 3.2, 3.4, and 4.1 hold. For any $Y_0 \in D([\mathcal{A}_0]_c)$ and $p \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H}_c)$, there exists a unique classical solution $Y \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A}_c)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H}_c)$ of (6a-6e) associated with the initial condition Y_0 and the distributed disturbance p. The associated control law v is the unique solution of (6d), so called the "fixed point implicit equality" in [2], and is in $\mathcal{C}^2([-D, +\infty); \mathbb{C}^m)$. Furthermore, there exist constants $\kappa_0, \overline{C}_1, \overline{C}_2 > 0$, independent of $Y_0 \in D([\mathcal{A}_0]_c)$ and $p \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H}_c)$, such that the following ISS estimate holds for all $t \geq 0$,

$$\|Y(t)\|_c + \|v(t)\| \leq \overline{C}_1 e^{-\kappa_0 t} \|Y_0\|_c + \overline{C}_2 \sup_{\tau \in [0,t]} \|p(\tau)\|. \quad (7)$$

V. Realification of the control law

The objective of this section is to derive a real-valued control law for the original system (1) within the original real Hilbert space $\mathcal{H}$. As discussed in Remark 3.1, the naive approach consisting in taking $u = \operatorname{Re} v$ is not satisfactory. Indeed, as the matrices A_{N_0}, B_{N_0}, and K present complex coefficients, $\operatorname{Re} v$ entangles explicitly both $\operatorname{Re} Y$ and $\operatorname{Im} Y$. Thus, $\operatorname{Re} v$ cannot be expressed as a state feedback for the original system (1) with $X = \operatorname{Re} Y$. Therefore, a modification of the design procedure is required.

A. Stabilization in the complexified state-space $\mathcal{H}_c$ with a real-valued control law v

We introduce the following matrices:

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \quad (8)$$

$$\mathcal{P} = \operatorname{diag}\left[P, P, \ldots, P, 1, 1, \ldots, 1\right] \in \mathbb{C}^{N_0 \times N_0},$$
$$\mathcal{P}^{-1} = \operatorname{diag}\left[P^{-1}, P^{-1}, \ldots, P^{-1}, 1, 1, \ldots, 1\right] \in \mathbb{C}^{N_0 \times N_0},$$

where P (resp. P^{-1}) is repeated n_0 times while 1 is repeated $N_0 - 2n_0$ times. Introducing $\tilde{Y}_{N_0} = \mathcal{P} Y_{N_0}$, $\tilde{P}_{N_0} = \mathcal{P} P_{N_0}$, $\tilde{A}_{N_0} = \mathcal{P} A_{N_0} \mathcal{P}^{-1}$, and $\tilde{B}_{N_0} = \mathcal{P} B_{N_0}$, we obtain that

$$\dot{\tilde{Y}}_{N_0}(t) = \tilde{A}_{N_0} \tilde{Y}_{N_0}(t) + \tilde{B}_{N_0} v(t-D) + \tilde{P}_{N_0}(t). \quad (9)$$

The newly introduced matrices $\tilde{A}_{N_0}$ and $\tilde{B}_{N_0}$ have real coefficients. Indeed, a direct computation shows that

$$\tilde{A}_{N_0} = \operatorname{diag}\left[\mathcal{R}(\lambda_1), \mathcal{R}(\lambda_3), \ldots, \mathcal{R}(\lambda_{2n_0-1}), \lambda_{2n_0+1}, \lambda_{2n_0+2}, \ldots, \lambda_{N_0}\right] \in \mathbb{R}^{N_0 \times N_0},$$

where, for any $\lambda \in \mathbb{C}$,

$$\mathcal{R}(\lambda) = \begin{bmatrix} \operatorname{Re}(\lambda) & -\operatorname{Im}(\lambda) \\ \operatorname{Im}(\lambda) & \operatorname{Re}(\lambda) \end{bmatrix} \in \mathbb{R}^{2 \times 2}.$$

Recalling that $e_k \in \mathbb{R}^m$, we have that $B_c e_k = B e_k \in \mathcal{H}$ and $\mathcal{A}_c B_c e_k = \mathcal{A} B e_k \in \mathcal{H}$. From Lemma 3.5, we have for $2n_0 + 1 \leq n \leq N_0$ that $\psi_n \in \mathcal{H}$. This yields $b_{n,k} = -\lambda_n \langle B_c e_k, \psi_n \rangle_c + \langle \mathcal{A}_c B_c e_k, \psi_n \rangle_c = -\lambda_n \langle B e_k, \psi_n \rangle + \langle \mathcal{A} B e_k, \psi_n \rangle \in \mathbb{R}$. From Lemma 3.6, we have for $1 \leq m \leq n_0$ that $\lambda_{2m-1} = \overline{\lambda_{2m}}$ and $\psi_{2m-1} = \overline{\psi_{2m}}$. This yields $b_{2m,k} = \overline{-\lambda_{2m-1} \left\langle B e_k, \psi_{2m-1} \right\rangle_c + \left\langle \mathcal{A} B e_k, \psi_{2m-1} \right\rangle_c} = \overline{b_{2m-1,k}}$. We deduce that

$$P \begin{bmatrix} b_{2m-1,k} \\ b_{2m,k} \end{bmatrix} = \begin{bmatrix} \operatorname{Re} b_{2m-1,k} \\ \operatorname{Im} b_{2m-1,k} \end{bmatrix} \in \mathbb{R}^2.$$

Consequently, $\tilde{B}_{N_0} \in \mathbb{R}^{N_0 \times m}$.

Introducing the Artstein transformation (see [1], [19]) defined for all $t \geq 0$ by

$$\tilde{Z}(t) = \tilde{Y}_{N_0}(t) + \int_{t-D}^{t} e^{(t-s-D)\tilde{A}_{N_0}} \tilde{B}_{N_0} v(s) \, \mathrm{d}s,$$

a straightforward differentiation shows that

$$\dot{\tilde{Z}}(t) = \tilde{A}_{N_0} \tilde{Z}(t) + e^{-D\tilde{A}_{N_0}} \tilde{B}_{N_0} v(t) + \tilde{P}_{N_0}(t).$$

As (A_{N_0}, B_{N_0}) is assumed stabilizable, so are $(\tilde{A}_{N_0}, \tilde{B}_{N_0})$ and $(\tilde{A}_{N_0}, e^{-D\tilde{A}_{N_0}} \tilde{B}_{N_0})$. As $\tilde{A}_{N_0}$ and $e^{-D\tilde{A}_{N_0}} \tilde{B}_{N_0}$ have real coefficients, we can compute via the pole shifting theorem a feedback gain $\tilde{K} \in \mathbb{R}^{m \times N_0}$ such that $\tilde{A}_{\mathrm{cl}} \triangleq \tilde{A}_{N_0} + e^{-D\tilde{A}_{N_0}} \tilde{B}_{N_0} \tilde{K}$ is Hurwitz. Introducing the control law $v = \varphi \tilde{K} \tilde{Z}$,

$$v(t) = [\varphi \tilde{K} \tilde{Y}_{N_0}](t) + \varphi(t) \tilde{K} \int_{t-D}^{t} e^{(t-s-D)\tilde{A}_{N_0}} \tilde{B}_{N_0} v(s) \, \mathrm{d}s \quad (10)$$

$$= \varphi(t) \tilde{K} \mathcal{P} Y_{N_0}(t) + \varphi(t) \tilde{K} \mathcal{P} \int_{t-D}^{t} e^{(t-s-D)A_{N_0}} B_{N_0} v(s) \, \mathrm{d}s$$

which is exactly the form of the control law employed in (6a-6e) with $K = \tilde{K}\mathcal{P} \in \mathbb{C}^{m\times N_0}$. Noting that

$$\begin{aligned} A_{\rm cl} &= A_{N_0} + e^{-DA_{N_0}} B_{N_0} K \\ &= \mathcal{P}^{-1}\left(\tilde{A}_{N_0} + e^{-D\tilde{A}_{N_0}}\tilde{B}_{N_0}\tilde{K}\right)\mathcal{P} = \mathcal{P}^{-1}\tilde{A}_{\rm cl}\mathcal{P} \end{aligned}$$

is Hurwitz, the conclusions of Theorem 4.2 apply.

Lemma 5.1: In the context of Theorem 4.2, consider the feedback gain $K = \tilde{K}\mathcal{P} \in \mathbb{C}^{m\times N_0}$ where $\tilde{K} \in \mathbb{R}^{m\times N_0}$ is such that $\tilde{A}_{\rm cl} \triangleq \tilde{A}_{N_0} + e^{-D\tilde{A}_{N_0}}\tilde{B}_{N_0}\tilde{K}$ is Hurwitz. Assume that $Y_0 \in D(\mathcal{A}_0)$ and $p \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$. We denote by Y the system trajectory of (6a-6e) and by v the associated boundary input. Then we have $Y \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ and $v \in \mathcal{C}^2([-D,+\infty);\mathbb{R}^m)$.

Proof: Let $Y_0 \in D(\mathcal{A}_0)$ and $p \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ be given. The existence and the uniqueness of the classical solution $Y \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A}_c)) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{H}_c)$ of (6a-6e) and the corresponding control law $v \in \mathcal{C}^2([-D,+\infty);\mathbb{C}^m)$ associated with Y_0 and p is provided by Theorem 4.2. It remains to show that $Y(t) \in \mathcal{H}$ and $v(t) \in \mathbb{R}^m$ for all $t \geq 0$. To do so, we proceed by induction over $n \in \mathbb{N}^*$ to show that $Y(t) \in \mathcal{H}$ for all $t \in [0, nD]$ and $v(t) \in \mathbb{R}^m$ for all $t \in [-D,(n-1)D]$.

Initialization. For $n = 1$, we have $v(t) = 0$ for all $t \in [-D, 0]$. Thus, the closed-loop system (6a-6e) reduces to $\frac{\mathrm{d}Y}{\mathrm{d}t}(t) = [\mathcal{A}_0]_c Y(t) + p(t)$ for $t \in [0, D]$ with the initial condition $Y(0) = Y_0$. As $\operatorname{Im} Y_0 = 0$ and $\operatorname{Im} p = 0$, we obtain by linearity and uniqueness of the imaginary part that $\frac{\mathrm{d}\operatorname{Im} Y}{\mathrm{d}t}(t) = \mathcal{A}_0 \operatorname{Im} Y(t)$ for all $t \in [0, D]$ with the initial condition $\operatorname{Im} Y(0) = 0$. Thus, as $\mathcal{A}_0$ generates a C_0-semigroup, we obtain that $\operatorname{Im} Y(t) = 0$ for all $t \in [0, D]$.

Heredity. Assume now that the claimed property holds true for a given $n \in \mathbb{N}^*$. Thus $\operatorname{Im} Y(t) = 0$ for all $t \in [0, nD]$ and $\operatorname{Im} v(t) = 0$ for all $t \in [-D,(n-1)D]$. Consequently, $\tilde{Y}_{N_0}(t) \in \mathbb{R}^{N_0}$ for all $t \in [0, nD]$ because 1) for $2n_0 + 1 \leq m \leq N_0$ we have $\psi_m \in \mathcal{H}$ and thus $\langle Y(t), \psi_m\rangle_c = \langle Y(t), \psi_m\rangle \in \mathbb{R}$; 2) for $1 \leq m \leq n_0$, we have $\psi_{2m-1} = \overline{\psi_{2m}}$, yielding $\langle Y(t), \psi_{2m-1}\rangle_c = \langle Y(t), \overline{\psi_{2m}}\rangle_c = \overline{\langle Y(t), \psi_{2m}\rangle_c}$, and thus

$$P\begin{bmatrix}\langle Y(t), \psi_{2m-1}\rangle_c \\ \langle Y(t), \psi_{2m}\rangle_c\end{bmatrix} = \begin{bmatrix}\langle Y(t), \operatorname{Re}\psi_{2m-1}\rangle \\ -\langle Y(t), \operatorname{Im}\psi_{2m-1}\rangle\end{bmatrix} \in \mathbb{R}^2. \quad (11)$$

From (10), as $\varphi(t) \in \mathbb{R}$ and $\tilde{Y}_{N_0}(t) \in \mathbb{R}^{N_0}$ for all $t \in [0, nD]$, and as all the involved matrices have real coefficients, the resulting command v is also real-valued over $[-D, nD]$ (see [2]). We deduce that $\operatorname{Im} v_D(t) = 0$ for all $t \in [0,(n+1)D]$. As $\operatorname{Im} p = 0$, we obtain that $\frac{\mathrm{d}\operatorname{Im} Y}{\mathrm{d}t}(t) = \mathcal{A}_0 \operatorname{Im} Y(t)$ for all $t \in [0,(n+1)D]$ with the initial condition $\operatorname{Im} Y(0) = 0$. Thus $\operatorname{Im} Y(t) = 0$ for all $t \in [0,(n+1)D]$. □

B. Feedback stabilization in the original state-space $\mathcal{H}$

We can now present the main result of this paper. Let $D, t_0 > 0$ be given. We consider a given transition signal $\varphi \in \mathcal{C}^2([-D,+\infty);\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi|_{[-D,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$. The closed-loop system dynamics takes the following form:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + d(t), \quad (12a)$$

$$\mathcal{B}X(t) = u_D(t) = u(t-D), \quad (12b)$$

$$u|_{[-D,0]} = 0 \quad (12c)$$

$$\begin{aligned} u(t) &= \varphi(t)\tilde{K}\tilde{Y}_{N_0}(t) \\ &\quad + \varphi(t)\tilde{K}\int_{\max(t-D,0)}^{t} e^{(t-s-D)\tilde{A}_{N_0}}\tilde{B}_{N_0}u(s)\,\mathrm{d}s, \end{aligned} \quad (12d)$$

$$X(0) = X_0 \quad (12e)$$

for any $t \geq 0$. The feedback gain $\tilde{K} \in \mathbb{R}^{m\times N_0}$ is such that $\tilde{A}_{\rm cl} \triangleq \tilde{A}_{N_0} + e^{-D\tilde{A}_{N_0}}\tilde{B}_{N_0}\tilde{K}$ is Hurwitz.

Theorem 5.2: Assume that Assumptions 3.2, 3.4, and 4.1 hold. For any $X_0 \in D(\mathcal{A}_0)$ and $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$, there exists a unique classical solution X of (12a-12e) associated with the initial condition X_0 and the distributed disturbance d. The associated control law u is the unique solution of (6d), is real-valued, and is in $\mathcal{C}^2([-D,+\infty);\mathbb{R}^m)$. Furthermore, there exist constants $\kappa_0, \overline{C}_1, \overline{C}_2 > 0$, independent of $X_0 \in D(\mathcal{A}_0)$ and $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$, such that the following ISS estimate holds for all $t \geq 0$,

$$\|X(t)\| + \|u(t)\| \leq \overline{C}_1 e^{-\kappa_0 t}\|X_0\| + \overline{C}_2 \sup_{\tau\in[0,t]}\|d(\tau)\|. \quad (13)$$

Proof: Introducing the feedback gain $K = \tilde{K}\tilde{P} \in \mathbb{C}^{m\times N_0}$, the matrix $A_{\rm cl} = A_{N_0} + e^{-DA_{N_0}}B_{N_0}K$ is Hurwitz. Thus, let $\overline{C}_1, \overline{C}_2 \in \mathbb{R}_+$ be the constants provided by Theorem 4.2. Let $X_0 \in D(\mathcal{A}_0)$ and $d \in \mathcal{C}^1(\mathbb{R}_+;\mathcal{H})$ be arbitrarily given. Applying Theorem 4.2, we introduce the unique classical solution $Y \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A}_c)) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{H}_c)$ of (6a-6e) and the corresponding control law $v \in \mathcal{C}^2([-D,+\infty);\mathbb{C}^m)$ associated with the initial condition $Y_0 = X_0 \in \mathcal{H}$ and the distributed disturbance $p(t) = d(t) \in \mathcal{H}$. Applying Lemma 5.1, we obtain that $\operatorname{Im} Y(t) = 0$ and $\operatorname{Im} v(t) = 0$ for all $t \geq 0$. Defining $X(t) = Y(t) \in \mathcal{H}$ and $u(t) = v(t) \in \mathbb{R}^m$, and noting that the control law satisfies (10), we deduce that X is the classical solution of (12a-12e), with control law u, which is associated with X_0 and d. Finally, (13) follows from (7). □

VI. Illustrative example

We consider the Euler-Bernoulli Beam with point torque boundary conditions described by

$$\begin{aligned} &y_{tt} + y_{xxxx} - 2\alpha y_{txx} - \beta y_t = d_b, && \text{in } \mathbb{R}_+^* \times (0,1) \\ &y(t,0) = y(t,1) = y_{xx}(t,0) = 0, && t \in \mathbb{R}_+^* \\ &y_{xx}(t,1) = u(t-D), && t \in \mathbb{R}_+^* \\ &(y(0,x), y_t(0,x)) = (y_0(x), y_{t0}(x)), && x \in (0,1) \end{aligned}$$

where $\alpha \in (0,1)$, $\beta \in \mathbb{R}_+^*$, u is the control law, d_b is a distributed disturbance, and y_0, y_{t0} are the initial conditions. Introducing the real Hilbert space $\mathcal{H} = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times L^2(0,1)$ with the inner product defined for all $(y_1,y_2),(\hat{y}_1,\hat{y}_2) \in \mathcal{H}$ by $\langle (y_1,y_2),(\hat{y}_1,\hat{y}_2)\rangle_{\mathcal{H}} = \int_0^1 y_1''(x)\overline{\hat{y}_1''(x)} + y_2(x)\overline{\hat{y}_2(x)}\,\mathrm{d}x$, the distributed parameter system can be written as the abstract boundary control system (1) with $\mathcal{A}(y_1,y_2) = (y_2, -y_1'''' + 2\alpha y_2'' + \beta y_2)$ defined over the domain $D(\mathcal{A}) = \left\{y_1 \in H^4(0,1) \cap H_0^1(0,1) : y_1''(0) = 0\right\} \times \left(H^2(0,1) \cap H_0^1(0,1)\right)$, the boundary operator $\mathcal{B}(y_1,y_2) = y_1''(1)$ defined over the domain $D(\mathcal{B}) = D(\mathcal{A})$, the state vector $X(t) = (y(t,\cdot), y_t(t,\cdot)) \in \mathcal{H}$, the initial condition

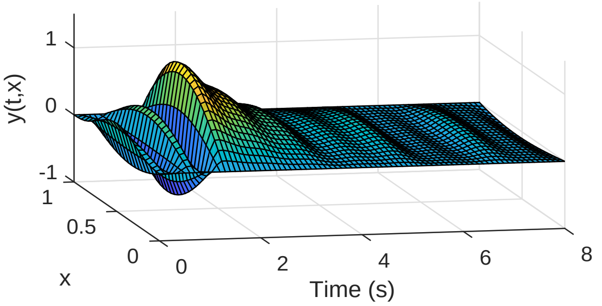

Fig. 1. Time domain evolution of the flexible displacement of the beam

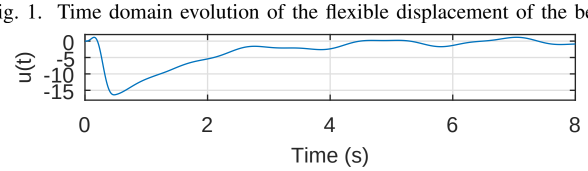

Fig. 2. Command effort of the closed-loop system

$X_0 = (y_0, y_{t0}) \in \mathcal{H}$, the boundary control $u(t) \in \mathbb{R}$, and the distributed perturbation $d(t) = (0, d_b(t)) \in \mathcal{H}$. One can show that $(\mathcal{A}, \mathcal{B})$ is an abstract boundary control system (see, e.g, [12] for a similar setting) and that the linear operator B defined such that $(Bu)(x) = \big(ux(x^2-1)/6, 0\big)$ for all $u \in \mathbb{R}$ and all $x \in (0,1)$ is a lifting operator associated with $(\mathcal{A}, \mathcal{B})$.

Assuming that $\frac{1}{\pi}\sqrt{\frac{\beta}{2(\alpha+1)}} < 1$, $[\mathcal{A}_c]_0$ is a Riesz-Spectral operator with eigenvalues given for $n \geq 1$ and $\epsilon \in \{-1, 1\}$ by $\lambda_{n,\epsilon} = -\alpha n^2\pi^2 + \frac{\beta}{2} + i\epsilon\sqrt{n^4\pi^4 - \left(\alpha n^2\pi^2 - \frac{\beta}{2}\right)^2} \in \mathbb{C}\backslash\mathbb{R}$ with associated unit eigenvectors $\phi_{n,\epsilon} = \frac{1}{n^2\pi^2}(\sin(n\pi\cdot), \lambda_{n,\epsilon}\sin(n\pi\cdot))$ and biorthogonal vectors $\psi_{n,\epsilon} = \frac{2n^2\pi^2}{n^4\pi^4 - \lambda_{n,-\epsilon}^2}(\sin(n\pi\cdot), -\lambda_{n,-\epsilon}\sin(n\pi\cdot))$. It is now easy to show that Assumptions 3.2, 3.4, and 4.1 are satisfied. Therefore, we can apply the result of Theorem 5.2.

Setting $\alpha = 1/2$, $\beta = 12$, and $D = 1\,\mathrm{s}$, the first mode exhibits unstable eigenvalues located (approximately) at $1.0652 \pm 9.8120i$ while all the other modes are stable. The control design is performed for $N_0 = 2$ to place the two first eigenvalues of the closed-loop system at -2 and -2.5. The transition time t_0 is set to $t_0 = 0.5\,\mathrm{s}$ while the switching function $\varphi|_{[0,t_0]}$ is selected as the restriction over $[0, t_0]$ of the unique quintic polynomial function f satisfying $f(0) = f'(0) = f''(0) = f'(t_0) = f''(t_0) = 0$ and $f(t_0) = 1$. The numerical scheme consists in the discretization of (12a-12e) by using the first 20 modes of the beam dynamics. The evolution of the closed-loop system is depicted in Figs. 1-2 for the initial conditions $y_0(x) = 20x^3(1-x)^3$ and $y_{t0}(x) = 10x(3/4-x)(1-x)$, and the distributed disturbance $d_b(t,x) = \sin(2t)\sin(5t)x$. The numerical results are compliant with the theoretical predictions.

VII. Conclusion

This paper discussed the realification of a predictor feedback control law for the stabilization of a class of diagonal abstract boundary control systems. Specifically, assuming that the diagonal structure does not hold for the original real Hilbert space but holds for its complexified version, it has been shown that an adequate selection of the eigenstructures can be used for obtaining a real-valued control law taking the form of a state feedback of the original abstract boundary control system. Future developments toward practical implementations could include either the impact of the discretization of the control law or the design of an observer

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