# $(m, p)$-isometric and $(m, \infty)$-isometric operator tuples on normed spaces

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We generalize the notion of $m$-isometric operator tuples on Hilbert spaces in a natural way to operator tuples on normed spaces. This is done by defining a tuple analogue of $(m, p)$-isometric operators, so-called $(m, p)$-isometric operator tuples. We then extend this definition further by introducing $(m, \infty)$-isometric operator tuples and study properties of and relations between these objects.

*Keywords*: normed space, Banach space, operator tuple, $m$-isometry, $(m, p)$-isometry, $(m, \infty)$-isometry, generalised isometry.

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## 1. Introduction

Let $H$ be a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{N}$ denote the natural numbers including 0. If $m \in \mathbb{N}$, then a bounded linear operator $T \in B(H)$ is called an *$m$-isometry* if, and only if,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0. \tag{1.1}$$

(It is obvious that the case $m = 0$ is trivial.)

Originating in works of Richter [20] (the Dirichlet shift being the standard example of a 2-isometry) and Agler [2] in the 1980s, operators of this kind have been studied extensively by Agler and Stankus in three papers [3, 4, 5] and since then attracted the interest of many other authors (see for example [8], [9] or [13]).

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In recent years, two generalisations of the definition of $m$-isometries have been given. Gleason and Richter in [14] extend the notion of $m$-isometric operators to the case of commuting $d$-tuples of bounded linear operators on a Hilbert space. The defining equation for an *$m$-isometry* (or *$m$-isometric tuple*) $T = (T_1, ..., T_d) \in B(H)^d$ reads:

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha *} T^{\alpha} = 0.$$

Here, $m$ is again a non-negative integer, $\alpha$ is a multi-index, $|\alpha|$ the sum of its entries and $\frac{|\alpha|!}{\alpha!} = \frac{|\alpha|!}{\alpha_1! \cdots \alpha_d!} = \binom{|\alpha|}{\alpha}$) a multinomial coefficient.

On the other hand, the notion of $m$-isometric operators on Hilbert spaces has been generalized to operators on general Banach spaces in papers of Botelho [11], Sid Ahmed [23] and Bayart [6]. In Bayart's definition, given $m \in \mathbb{N}$ and $p \in [1, \infty)$, an operator $T \in B(X)$ on a Banach space $X$ over $\mathbb{K}$ is called an *$(m,p)$-isometry* if, and only if,

$$\forall x \in X, \quad \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^k x\|^p = 0. \tag{1.2}$$

It is easy to see that, if $X = H$ is a Hilbert space and $p = 2$, this definition coincides with the original definition (1.1) of $m$-isometries.[a] In [16] the relationship and intersection class between $(m,p)$- and $(\mu,q)$-isometries is studied. In [10] an example of an unbounded operator satisfying (1.2) is given. (We will, however, assume boundedness for convenience.)

In this paper, we combine both generalisations and consider so-called $(m,p)$-isometric operator tuples on normed spaces, which will be defined in a natural way.

An extension of the definition of $(m,p)$-isometric operators was given in [16] to include the case $p = \infty$: If $m \in \mathbb{N}$ with $m \geq 1$, then an operator $T \in B(X)$ is called an *$(m,\infty)$-isometry* if, and only if,

$$\forall x \in X, \quad \max_{\substack{k=0,...,m \\ k \text{ even}}} \|T^k x\| = \max_{\substack{k=0,...,m \\ k \text{ odd}}} \|T^k x\|.$$

We will generalize this definition to the commuting tuple case in a natural way and give a conjecture on the intersection class of $(m,p)$-isometric and $(m,\infty)$-isometric tuples in the last part of this paper.

In the following, $X$ will denote a normed (not necessarily complete) vector space over $\mathbb{K}$ (unless stated otherwise, for example in section 6). For $d \in \mathbb{N}$, with $d \geq 1$, let $T = (T_1, ..., T_d) \in B(X)^d$ be a tuple of commuting bounded linear operators on $X$. (Boundedness is actually not essential for the definition and the basic properties of the objects we are about to discuss, but plays a role in the later theory.) Greek

[a] In the case $\mathbb{K} = \mathbb{R}$ this holds, because the operator $\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k$ is self-adjoint.

letters like $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ will denote tuples of natural numbers (multi-indices) or their entries, respectively. The norm or 'length' of $\alpha$ will be defined by $|\alpha| = \sum_{j=1}^{d} \alpha_j$ and we set further $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$.

To denote the tuple which we obtain after removing $T_j$ from $T = (T_1, ..., T_d)$, we will write $T'_j$ (that is, $T'_j = (T_1, ..., T_{j-1}, T_{j+1}, ..., T_d)$ ). We use the notation $\alpha'_j$ analogously. Finally (again if not stated otherwise), we take the exponent $p$ to be a positive real number, $p \in (0, \infty)$.

## 2. Definitions and Preliminaries

For $T \in B(X)^d$ commuting, $x \in X$ and $p \in (0, \infty)$ as above, define the sequences $(Q^{n,p}(T,x))_{n\in\mathbb{N}}$ by $Q^{n,p}(T,x) := \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p$. For all $\ell \in \mathbb{N}$, define further the functions $P_\ell^{(p)}(T, \cdot) : X \to \mathbb{R}$, by

$$P_\ell^{(p)}(T,x) := \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} Q^{k,p}(T,x).$$

**Definition 2.1.** Given $m \in \mathbb{N}$ and $p \in (0, \infty)$, a commuting operator tuple $T \in B(X)^d$ is called an $(m,p)$-*isometry* (or $(m,p)$-*isometric tuple*) if, and only if,

$$\begin{aligned} P_m^{(p)}(T,x) &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} Q^{k,p}(T,x) \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^p = 0, \quad \forall x \in X. \end{aligned}$$

Again, it is clear that the case $m = 0$ is trivial. Further, since the operators $T_1, ..., T_d$ are commuting, every permutation of an $(m,p)$-isometric tuple is also an $(m,p)$-isometric tuple.

If the context is clear, we will simply write $P_\ell(x)$ and $Q^n(x)$ instead of $P_\ell^{(p)}(T,x)$ and $Q^{n,p}(T,x)$. This definition coincides with the definition of $m$-isometric tuples by Gleason and Richter if $X$ is a Hilbert space (and $p = 2$) and has, in that context as an equivalent description, essentially already been presented in Lemma 2.1 in [14].

Consequently, as one would expect, the basic theory of $(m,p)$-isometric tuples can be evolved in a similar fashion as in [14]. However, we will use a different approach, based on an idea described in [16].

Let, as in Notation 3.1 in [16], the symbol $\mathfrak{F}$ denote the set of real functions whose domain is a subset of $\mathbb{R}$ which is invariant under the mapping $S : t \to t + 1$. Further, define $D : \mathfrak{F} \to \mathfrak{F}$ by $Dg := g - (g \circ S)$ for each $g \in \mathfrak{F}$ (that is, $D$ is the backward operator with difference interval 1). Then

$$D^m g = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (g \circ S^k)$$

for all $g \in \mathfrak{F}$ and all $m \in \mathbb{N}$. Note that the set of all real sequences $\mathfrak{A}$ is a subset of $\mathfrak{F}$ and that

$$D^m a = \left( \sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{n+k} \right)_{n \in \mathbb{N}}, \quad \forall\, a = (a_n)_{n\in\mathbb{N}} \in \mathfrak{A} \text{ and } \forall\, m \in \mathbb{N}.$$

Then $T \in B(X)^d$ is an $(m,p)$-isometric tuple, if and only if,

$$\left(D^m (Q^n(x))_{n\in\mathbb{N}}\right)_0 = 0, \quad \forall x \in X.$$

Now Proposition 3.2.(ii) in [16] states the following[b]:

**Proposition 2.1.** *Let $a \in \mathfrak{A}$ and $m \in \mathbb{N}$. We have $D^m a = 0$ if, and only if, there exists a (necessarily unique) polynomial function $f$ with $\deg f \le m-1$ such that $f|_{\mathbb{N}} = a$.* [c]

We would like to apply this fact to the sequences $(Q^n(x))_{n\in\mathbb{N}}$, to conclude that, if $T \in B(X)^d$ is an $(m,p)$-isometric tuple, then, for each $x \in X$, there exists a polynomial $f_x$, which interpolates $(Q^n(x))_{n\in\mathbb{N}}$.

Unfortunately, unlike in the situation of $(m,p)$-isometric operators (see Remark 3.6 in [16]), we can not immediately state that $T$ being an $(m,p)$-isometric tuple requires the whole sequence $D^m(Q^n(x))_{n\in\mathbb{N}}$ to be the zero-sequence. This needs some little extra work.

**Lemma 2.1.**

$$Q^{n+1}(x) = \sum_{j=1}^{d} Q^n(T_j x), \quad \forall x \in X,\ \forall n \in \mathbb{N}.$$

**Proof.**

$$\begin{aligned} Q^{n+1}(x) &= \sum_{|\alpha|=n+1} \frac{(n+1)!}{\alpha!} \|T^\alpha x\|^p = \sum_{|\alpha|=n+1} \frac{n!(\alpha_1 + ... + \alpha_d)}{\alpha_1! \cdots \alpha_d!} \|T_1^{\alpha_1} \cdots T_d^{\alpha_d} x\|^p \\ &= \sum_{j=1}^{d} \sum_{\substack{|\alpha|=n+1 \\ \alpha_j \ge 1}} \frac{n! \cdot \alpha_j}{\alpha_1! \cdots \alpha_d!} \|T_1^{\alpha_1} \cdots T_d^{\alpha_d} x\|^p \\ &= \sum_{j=1}^{d} \sum_{\substack{|\alpha|=n+1 \\ \alpha_j \ge 1}} \frac{n! \|T_1^{\alpha_1} \cdots T_{j-1}^{\alpha_{j-1}} T_j^{\alpha_j - 1} T_{j+1}^{\alpha_{j+1}} \cdots T_d^{\alpha_d} T_j x\|^p}{\alpha_1! \cdots \alpha_{j-1}! (\alpha_j - 1)! \alpha_{j+1}! \cdots \alpha_d!} \\ &= \sum_{j=1}^{d} \sum_{|\beta|=n} \frac{n!}{\beta!} \|T^\beta T_j x\|^p = \sum_{j=1}^{d} Q^n(T_j x), \quad \forall x \in X,\ \forall n \in \mathbb{N}. \end{aligned}$$

[b] This proposition is actually a special case of a more general and well-known fact about functions defined on the natural numbers, which can, for example, be found in Satz 3.1 in [1].
[c] To account for the case $m = 0$, set $\deg 0 = -\infty$.

□

**Corollary 2.1.**

$$\left((D^\ell(Q^n(x))_{n\in\mathbb{N}}\right)_{\nu+1} = \sum_{j=1}^{d} \left(D^\ell(Q^n(T_jx))_{n\in\mathbb{N}}\right)_\nu, \quad \forall x \in X,$$

*for all $\ell \in \mathbb{N}$, for all $\nu \in \mathbb{N}$.*

**Proof.**

$$\begin{aligned}\left(D^\ell(Q^n(x))_{n\in\mathbb{N}}\right)_{\nu+1} &= \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}Q^{\nu+1+k}(x)\\ &\overset{\text{Lemma 2.1}}{=} \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\sum_{j=1}^{d}Q^{\nu+k}(T_jx)\\ &= \sum_{j=1}^{d}\left(D^\ell(Q^n(T_jx))_{n\in\mathbb{N}}\right)_\nu, \quad \forall x \in X,\ \forall \ell \in \mathbb{N},\ \forall \nu \in \mathbb{N}.\end{aligned}$$

□

Therefore, $(D^m(Q^n(x))_{n\in\mathbb{N}})_0 = 0$, for all $x \in X$, implies inductively $(D^m(Q^n(x))_{n\in\mathbb{N}})_\nu = 0$, for all $x \in X$, for all $\nu \in \mathbb{N}$. In other words:

**Proposition 2.2.** *$T \in B(X)^d$ is an $(m, p)$-isometry if, and only if, $D^m(Q^n(x))_{n\in\mathbb{N}} = 0$, for all $x \in X$.*

Before we move on, we state the following lemma, which may be of general interest. It is certainly well-known, but lacking a reference, we include the short proof.

**Lemma 2.2.** *Let $D : \mathfrak{F} \to \mathfrak{F}$ be defined as above and $(a_n)_{n\in\mathbb{N}} =: a \in \mathfrak{A}$. Then*

$$\left(D^\ell a\right)_{\nu+1} - \left(D^\ell a\right)_\nu = -\left(D^{\ell+1}a\right)_\nu, \quad \forall \ell, \nu \in \mathbb{N}.$$

**Proof.** By definition

$$D^{\ell+1}a = D(D^\ell a) = D^\ell a - (D^\ell a \circ S), \quad \forall \ell \in \mathbb{N}.$$

Hence,

$$\left(D^{\ell+1}a\right)_\nu = \left(D^\ell a\right)_\nu - \left(D^\ell a\right)_{\nu+1}, \quad \forall \ell, \nu \in \mathbb{N}.$$

□

## 3. Basic Properties of $(m, p)$-isometric tuples

Our preliminary considerations allow us now to derive the basic properties of $(m, p)$-isometric tuples, which are analogous to those given by Gleason and Richter in [14] in the Hilbert space case.

Expressing Lemma 2.2 in terms of $P_\ell(x)$ for $\nu = 0$ reads:

**Proposition 3.1.**

$$P_{\ell+1}(x) = \sum_{j=1}^{d} P_\ell(T_j x) - P_\ell(x), \quad \forall x \in X, \ \forall \ell \in \mathbb{N}.$$

**Proof.** Lemma 2.2 gives for $\nu = 0$ and $a = (Q^n(x))_{n\in\mathbb{N}}$, for all $\ell \in \mathbb{N}$,

$$\left(D^\ell (Q^n(x))_{n\in\mathbb{N}}\right)_1 - \left(D^\ell (Q^n(x))_{n\in\mathbb{N}}\right)_0 = -\left(D^{\ell+1} (Q^n(x))_{n\in\mathbb{N}}\right)_0, \quad \forall x \in X,$$

$$\overset{\text{Corollary 2.1}}{\Leftrightarrow}$$

$$\sum_{j=1}^{d} D^\ell (Q^n(T_j x))_{n\in\mathbb{N}})_0 - \left(D^\ell (Q^n(x))_{n\in\mathbb{N}}\right)_0 = -\left(D^{\ell+1} (Q^n(x))_{n\in\mathbb{N}}\right)_0, \quad \forall x \in X. \tag{3.1}$$

By definition $\left(D^\ell (Q^n(x))_{n\in\mathbb{N}}\right)_0 = (-1)^\ell P_\ell(x)$, for all $\ell \in \mathbb{N}$, for all $x \in X$. Therefore, (3.1) reads

$$(-1)^\ell \sum_{j=1}^{d} P_\ell(T_j x) - (-1)^\ell P_\ell(x) = (-1)^\ell P_{\ell+1}(x), \quad \forall x \in X, \ \forall \ell \in \mathbb{N}.$$

□

Proposition 2.2, as well as Proposition 3.1, imply:

**Corollary 3.1.** *An $(m,p)$-isometry $T \in B(X)^d$ is an $(m+1,p)$-isometry.*

Further, Proposition 2.2 enables us to apply Proposition 2.1 to the sequence $(Q^n(x))_{n\in\mathbb{N}}$, to receive the following fundamental theorem.

**Theorem 3.1.** *$T \in B(X)^d$ is an $(m,p)$-isometry if, and only if, there exists a (necessarily unique) family of polynomials $f_x : \mathbb{R} \to \mathbb{R}$, $x \in X$, of degree $\leq m-1$ with $f_x|_{\mathbb{N}} = (Q^n(x))_{n\in\mathbb{N}}$.*

We remark that this fact has already been stated for $m$-isometric operators on Hilbert spaces by Agler and Stankus in §1, pages 388-389, in [3]. Further, the existence of these polynomials has already been proven by Bayart in Proposition 2.1 in [6] for $(m,p)$-isometric operators on normed spaces, and by Gleason and Richter in Lemma 2.2 and Proposition 2.3 in [14] for $m$-isometric tuples on Hilbert spaces.

Let now for $k, n \in \mathbb{N}$ denote the *(descending) Pochhammer symbol* by $n^{(k)}$. That is,

$$n^{(k)} := \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \ k > n, \\ \binom{n}{k} k!, & \text{if } n > 0, \ k \leq n. \end{cases}$$

**Proposition 3.2.** *Let $m \geq 1$ and $T \in B(X)^d$ be an $(m,p)$-isometry. Then*

(i) $Q^n(x) = \sum_{k=0}^{m-1} n^{(k)}\big(\frac{1}{k!}P_k(x)\big)$, *for all* $x \in X$, *for all* $n \in \mathbb{N}$.
(ii) $\lim_{n\to\infty} \frac{Q^n(x)}{n^{m-1}} = \frac{1}{(m-1)!}P_{m-1}(x) \geq 0$, *for all* $x \in X$.

**Proof.** (ii) follows immediately from (i).
For every $x \in X$, the polynomial $f_x$ interpolates the points $(n, Q^n(x))$. Determining the Newton form of $f_x$ gives (i). □

**Corollary 3.2.** *Let* $m \geq 1$ *and* $(T_1, ..., T_d) \in B(X)^d$ *be an* $(m, p)$*-isometric tuple. Then we have*

(i) $\big(\|T_j^n x\|\big)_{n\in\mathbb{N}} \in \mathcal{O}(n^{m-1})$ *for all* $j \in \{1, ..., d\}$, *for all* $x \in X$.
(ii) $T_j^n(T_j')^\beta x \to 0$ *for* $n \to \infty$, *for all* $\beta \in \mathbb{N}^{d-1}$ *with* $|\beta| \geq m$, *for all* $j \in \{1, ..., d\}$, *for all* $x \in X$.

*If* $X$ *is a Banach space this boundedness and convergence are uniform.*

**Proof.** By the proposition above, for each $x \in X$, the sequence $\left(\frac{Q^n(x)}{n^{m-1}}\right)_{n\in\mathbb{N}} = \left(\frac{1}{n^{m-1}}\left(\sum_{|\alpha|=n}\frac{n!}{\alpha!}\|T^\alpha x\|^p\right)\right)_{n\in\mathbb{N}}$ is a convergent sequence of sums. Since all summands are non-negative, sequences of summands have to be bounded. In particular, the sequences

$$\left(\frac{1}{n^{m-1}}\|T_j^n x\|^p\right)_{n\in\mathbb{N}} \quad\text{and}\quad \left(\frac{1}{n^{m-1}}\left(\frac{n!}{(n-|\beta|)!\beta!}\|T_j^{n-|\beta|}(T_j')^\beta x\|^p\right)\right)_{n\in\mathbb{N}}$$

have to be bounded for all $\beta \in \mathbb{N}^{d-1}$, for all $j \in \{1, ..., d\}$ and all $x \in X$. This immediately gives (i). Noticing that

$$\frac{1}{n^{m-1}}\frac{n!}{(n-|\beta|)!\beta!} = \frac{n^{(|\beta|)}}{n^{m-1}\beta!} \to \infty \quad\text{for}\quad |\beta| \geq m$$

gives (ii).

The last part of the statement follows by the Uniform Boundedness Principle. □

**Proposition 3.3.** *Let* $m \geq 1$ *and* $T = (T_1, ..., T_d) \in B(X)^d$ *be an* $(m, p)$*-isometry. Then* $\ker P_{m-1}$ *is invariant*[d] *for each* $T_j$ *and the tuple*

$$T|_{\ker P_{m-1}} := (T_1|_{\ker P_{m-1}}, ..., T_d|_{\ker P_{m-1}})$$

*is an* $(m-1, p)$*-isometry. Further, if* $M \subset X$ *is invariant for each* $T_j$ *and* $T|_M$ *is an* $(m-1, p)$*-isometry, then* $M \subset \ker P_{m-1}$.

**Proof.** If $T$ is an $(m, p)$-isometry, $P_m \equiv 0$. Then, by Proposition 3.1, $P_{m-1}(x) = \sum_{j=1}^d P_{m-1}(T_j x)$, for all $x \in X$. Let $x_0 \in \ker P_{m-1}$. Since $P_{m-1} \geq 0$ by Proposition 3.2.(ii), $T_j x_0 \in \ker P_{m-1}$ for all $j = 1, ..., d$ follows.

[d]Note that by Proposition 3.2.(ii) and the boundedness of each $T_j$, $\ker P_{m-1}$ is indeed a closed subspace of $X$.

Note further that for every subspace $\mathcal{M}$ which is invariant for all $T_j$, we have that $T^\alpha|_\mathcal{M} = (T|_\mathcal{M})^\alpha$. Thus, if $x \in \ker P^{(p)}_{m-1}(T,\cdot)$, then $P^{(p)}_{m-1}(T|_{\ker P_{m-1}}, x) = P^{(p)}_{m-1}(T,x) = 0$.

Similarly, if $\mathcal{M}$ is an invariant subspace for each $T_j$ such that $T|_\mathcal{M}$ is an $(m-1,p)$-isometry, then, for all $x \in \mathcal{M}$, $0 = P^{(p)}_{m-1}(T|_\mathcal{M}, x) = P^{(p)}_{m-1}(T,x)$. Therefore, $x \in \ker P_{m-1}$. □

An $(m,p)$-isometric operator is by the proof of Theorem 3.3 in [6] an isometry on the quotient space $X/\ker\beta_{m-1}(T,\cdot)$ equipped with the norm $(\beta_{m-1}(T,\cdot))^{1/p}$. Here, for each $x \in X$, $\beta_{m-1}(T,x) = \frac{1}{(m-1)!}\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\|T^jx\|^p$ is the leading coefficient of the polynomial which interpolates the sequence $(\|T^nx\|^p)_{n\in\mathbb{N}}$. Indeed a similar result holds for $(m,p)$-isometric tuples.

We will call a commuting operator tuple $T = (T_1, ..., T_d)$ on a normed space $X$ an $\ell_p$*-spherical isometry* if

$$\sum_{j=1}^{d}\|T_jx\|^p = \|x\|^p, \quad \forall x \in X.$$

In the literature, $\ell_2$-spherical isometries on Hilbert spaces are referred to as just *spherical isometries*. Obviously $\ell_p$-spherical isometries are just $(1,p)$-isometric tuples.

The following has (in equivalent form) already been stated in [21] for $(2,2)$-isometries on Hilbert spaces.

**Proposition 3.4.** *Let $m \geq 1$ and $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m,p)$-isometry. Then $|.|_p := (P^{(p)}_{m-1}(T,\cdot))^{1/p}$ is a semi-norm on $X$ with $|.|_p \leq C\|.\|$ for some constant $C > 0$. Further, $T$ is an $\ell_p$-spherical isometry on the quotient space $X/\ker P^{(p)}_{m-1}(T,\cdot)$.*

**Proof.** By Proposition 3.2.(ii), $|.|_p = (P^{(p)}_{m-1}(T,\cdot))^{1/p}$ is a semi-norm on $X$, hence a norm on $X/(\ker P^{(p)}_{m-1}(T,\cdot))^{1/p}$. That $|.|_p \leq C\|.\|$ for some constant $C > 0$ follows directly from the definition of $P^{(p)}_{m-1}(T,\cdot)$ and the boundedness of $T$. Further, by Proposition 3.1, $\sum_{j=1}^{d}|T_jx|_p^p = |x|_p^p$, for all $x \in X$. □

## 4. Examples of $(m,p)$-isometric tuples

Non-trivial examples of $(m,p)$-isometric operator tuples are in general not easy to find. In the case $m = 1$ this is, however, relatively simple.

**Example 4.1.** Let $X$ be an arbitrary normed space and $I$ the identity operator. The pair $(\frac{1}{2}I, \frac{1}{2}I) \in B(X)^2$ is a $(1,1)$-isometric tuple on $X$.

**Example 4.2.** Let $T_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and $T_2 = \begin{pmatrix} \frac{\sqrt[3]{7}}{2} & 0 \\ 0 & \frac{\sqrt[3]{26}}{3} \end{pmatrix}$. Then the pair $T = (T_1, T_2)$ is a $(1,3)$-isometric tuple on $(\mathbb{K}^2, \|.\|_3)$.

In [21] Richter states (without proof) the following sufficient condition for $(2,2)$-isometric tuples on finite dimensional complex Hilbert spaces:

**Proposition 4.1 (Richter [21]).** *Let $z = (z_1, ..., z_d) \in \mathbb{C}^d$ with $\|z\|_2 = 1$ and consider linear $V_i : \mathbb{C}^m \to \mathbb{C}^n$, $i \in \{1, ..., d\}$, with $\sum_{i=1}^d \overline{z_i} V_i = 0$. Then the operator tuple $S = (S_1, ..., S_d) \in B(\mathbb{C}^{n+m})^d$, with*

$$S_i = \begin{pmatrix} z_i I_n & V_i \\ 0_m & z_i I_m \end{pmatrix},$$

*is a $(2,2)$-isometric tuple.*

This result leads to our next example.

**Example 4.3.** Let $T_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $T_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -1 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then the pair $T = (T_1, T_2)$ is a $(2,2)$-isometry on $(\mathbb{K}^2, \|.\|_2)$.

Further examples for $(m,p)$-isometric tuples can be easily created on the basis of $(m,p)$-isometric operators. This principle was used in Example 3.3, Theorem 4.1 and Theorem 4.2 in [14], however, since it is not stated explicitly there, we include it here.

**Proposition 4.2.** *Let $p \in (0, \infty)$ and $S \in B(X)$ be an $(m,p)$-isometric operator, $d \in \mathbb{N}$ with $d \geq 1$ and $z = (z_1, ..., z_d) \in (\mathbb{K}^d, \|.\|_p)$, such that $\|z\|_p^p := \sum_{j=1}^d |z_j|^p = 1$.* [e] *Then the tuple $T := (z_1 S, ..., z_d S) \in B(X)^d$ is an $(m,p)$-isometric tuple.*

**Proof.** It is clear that the operators $z_j S$ are commuting. Further, by the multinomial theorem, we have $(|z_1|^p + \cdots + |z_d|^p)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} |z^\alpha|^p$. Therefore,

$$\begin{aligned} Q^n(x) &= \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|z^\alpha S^{|\alpha|} x\|^p = \sum_{|\alpha|=n} \frac{n!}{\alpha!} |z^\alpha|^p \|S^n x\|^p \\ &= \|z\|_p^{np} \|S^n x\|^p = \|S^n x\|^p, \quad \forall x \in X, \ \forall n \in \mathbb{N}. \end{aligned}$$

Since $S$ is an $(m,p)$-isometric operator, $D^m\left(Q^n(x)\right)_{n\in\mathbb{N}} = D^m\left(\|S^n x\|^p\right)_{n\in\mathbb{N}} = 0$. □

(Of course, Example 4.1 is also of this kind.)

For examples of $(m,p)$-isometric operators see for instance [20] (the Dirichlet-shift being the standard example), [9] or [23].

We now consider the special case where $T$ is an $(m,p)$-isometric tuple with one of the operators being an isometry.

**Proposition 4.3.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be a tuple of commuting operators and let $T_{j_0}$ be an isometry for some $j_0 \in \{1, ..., d\}$. Then $T$ is an $(m,p)$-isometry*

[e] The fact that $\|.\|_p$ is only a quasi-norm (that is, not convex) if $0 < p < 1$ is not an issue in this case.

*for some* $p \in (0, \infty)$ *if, and only if,* $\left(T'_{j_0}\right)^\beta = 0$ *for all* $\beta \in \mathbb{N}^{d-1}$ *with* $|\beta| = m$. *In this case,* $T$ *is an* $(m, q)$*-isometry for any* $q \in (0, \infty)$.

**Proof.** Without loss of generality, we can assume that $j_0 = 1$. The necessity of $(T'_1)^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = m$ if $T$ is an $(m, p)$-isometric tuple follows from Corollary 3.2.(ii). However, to show equivalence, we proceed by a combinatorial approach.

Note that

$$\sum_{k=\ell}^{m} (-1)^{m-k} \binom{m}{k} \binom{k}{\ell} = \delta_{\ell,m}, \tag{4.1}$$

where $\delta_{\ell,m}$ is the Kronecker-delta. To see this, write $\binom{m}{k}\binom{k}{\ell} = \frac{m!}{(m-\ell)!}\binom{m-\ell}{k-\ell}$, so that the left hand side of (4.1) becomes

$$\frac{m!}{(m-\ell)!} \sum_{k=\ell}^{m} (-1)^{m-\ell-(k-\ell)} \binom{m-\ell}{k-\ell} = \frac{m!}{(m-\ell)!} 0^{m-\ell} = \delta_{\ell,m}.$$

(The convention $0^0 = 1$ applies.)

Now, note further that for all $b_{k,\ell} \in \mathbb{C}$,

$$\sum_{k=0}^{m} \sum_{\ell=0}^{k} b_{k,\ell} = \sum_{\ell=0}^{m} \sum_{k=\ell}^{m} b_{k,\ell}, \quad \forall m \in \mathbb{N}.$$

(This can be easily seen by writing one side out and reordering the summands.) Consequently, by combining this with (4.1), we get for any sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} a_\ell = \sum_{\ell=0}^{m} \sum_{k=\ell}^{m} (-1)^{m-k} \binom{m}{k} \binom{k}{\ell} a_\ell = \sum_{\ell=0}^{m} a_\ell \delta_{\ell,m} = a_m. \tag{4.2}$$

Assume now that $T_1$ is an isometry. Then

$$\begin{aligned} Q^{k,p}(T, x) &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^p = \sum_{\ell=0}^{k} \sum_{\substack{|\alpha'_1|=\ell \\ \alpha_1=k-\ell}} \frac{k!}{\alpha!} \|T^\alpha x\|^p \\ &= \sum_{\ell=0}^{k} \sum_{|\alpha'_1|=\ell} \frac{k!}{(k-\ell)!(\alpha'_1)!} \|(T'_1)^{\alpha'_1} x\|^p, \quad \forall x \in X, \ \forall k \in \mathbb{N}. \end{aligned}$$

(Recall the notations $\alpha'_1 := (\alpha_2, ..., \alpha_d)$ and $T'_1 = (T_2, ..., T_d)$.) Therefore

$$\begin{aligned} P_m^{(p)}(T, x) &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{\ell=0}^{k} \sum_{|\alpha'_1|=\ell} \frac{k! \|(T'_1)^{\alpha'_1} x\|^p}{(k-\ell)!(\alpha'_1)!} \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{|\alpha'_1|=\ell} \frac{\ell! \|(T'_1)^{\alpha'_1} x\|^p}{(\alpha'_1)!}, \quad \forall x \in X. \end{aligned}$$

Then by considering (4.2) for the sequence

$$(a_n)_{n\in\mathbb{N}} = \left( \sum_{|\alpha_1'|=n} \frac{n!}{(\alpha_1')!} \|(T_1')^{\alpha_1'} x\|^p \right)_{n\in\mathbb{N}}$$

it follows that,

$$P_m^{(p)}(T,x) = \sum_{|\alpha_1'|=m} \frac{m!}{(\alpha_1')!} \|(T_1')^{\alpha_1'} x\|^p = 0, \tag{4.3}$$

$$\Leftrightarrow \quad \|(T_1')^{\alpha_1'} x\|^p = 0, \quad \forall \alpha_1' \in \mathbb{N}^{d-1} \text{ with } |\alpha_1'| = m, \tag{4.4}$$

for all $x \in X$, which is the desired equivalence.

The equivalence of (4.3) and (4.4) also shows that, if $T$ is an $(m,p)$-isometry for some $p \in (0,\infty)$, it is an $(m,q)$-isometric tuple for any $q \in (0,\infty)$. □

**Example 4.4.** Let $a \in \mathbb{K}$ and $T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}$. Then the pair $T = (T_1, T_2)$ is a $(2,p)$-isometric tuple for every $p \in (0,\infty)$ on $(\mathbb{K}^3, \|.\|_q)$ for any $q \in [1,\infty]$.[f]

## 5. $(m,\infty)$-isometric tuples

Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m,p)$-isometric tuple. This is equivalent to

$$\left( \sum_{\substack{k=0\\k \text{ even}}}^{m} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^p \right)^{\frac{1}{p}} = \left( \sum_{\substack{k=0\\k \text{ odd}}}^{m} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^p \right)^{\frac{1}{p}}, \quad \forall x \in X.$$

Assuming now that $T$ satisfies this for all $p \in (b,\infty)$ for some $b \geq 0$ and taking the limit for $p$ going to infinity, leads to the following definition.

**Definition 5.1.** Let $m \in \mathbb{N}$ with $m \geq 1$. A tuple $T = (T_1, ..., T_d) \in B(X)^d$ of commuting operators is called an $(m,\infty)$-*isometry* (or $(m,\infty)$-*isometric tuple*) if, and only if,

$$\max_{\substack{|\alpha|=0,...,m\\|\alpha| \text{ even}}} \|T^\alpha x\| = \max_{\substack{|\alpha|=0,...,m\\|\alpha| \text{ odd}}} \|T^\alpha x\|, \quad \forall x \in X.$$

(This definition extends the one appearing in [16] for operators.)

[f] Indeed, if one excludes the Hilbert norm, every isometry on $\mathbb{K}^n$ with respect to the $q$-norm for some $q \neq 2$, will also be an isometry with respect to any other $p$-norm ($p \neq 2$). The reason for this is that the isometric matrices on $(\mathbb{K}^n, \|.\|_q)$ for $q \neq 2$, $q \in [1,\infty]$ are exactly the (generalized) permutation matrices. (A proof in the real case can for example be found in [17].)

By construction, we can immediately give easy examples of these kind of tuples.

**Proposition 5.1.** *Every $(m,p)$-isometric tuple $T = (T_1, ..., T_d) \in B(X)^d$ that includes an isometry is an $(m,\infty)$-isometry.*

**Proof.** If one of the operator $T_1, ..., T_d$ is an isometry, then, by Proposition 4.3, $T$ is an $(m,p)$-isometric tuple for all $p \in (0,\infty)$. That $T$ is then an $(m,\infty)$-isometric tuple follows directly from the construction above which lead to our definition of these objects. □

Analogous to Proposition 4.2, one can construct further examples based on $(m,\infty)$-isometric operators.

**Proposition 5.2.** *Let $S \in B(X)$ be an $(m,\infty)$-isometric operator and $z = (z_1, ..., z_d) \in \mathbb{K}^d$, with $\|z\|_\infty = 1$. Then the tuple $T = (z_1 S, ..., z_d S) \in B(X)^d$ is an $(m,\infty)$-isometry.*

**Proof.** Since $\|z\|_\infty = 1$, we have $\max_{|\alpha|=k} |z^\alpha| = 1$ for all $k \in \mathbb{N}$. Hence,

$$\begin{aligned}
&\max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} \|T^\alpha x\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} |z^\alpha| \|S^{|\alpha|} x\| \\
&= \max\{\max_{|\alpha|=k} |z^\alpha| \|S^k x\| \mid k = 0, ..., m,\ k \text{ even}\} \\
&= \max_{\substack{k=0,\dots,m \\ k \text{ even}}} \|S^k x\| = \max_{\substack{k=0,\dots,m \\ k \text{ odd}}} \|S^k x\| = \max\{\max_{|\alpha|=k} |z^\alpha| \|S^k x\| \mid k = 0, ..., m,\ k \text{ odd}\} \\
&= \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} |z^\alpha| \|S^{|\alpha|} x\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} \|T^\alpha x\|, \quad \forall x \in X.
\end{aligned}$$

□

**Example 5.1.** Let $m \in \mathbb{N}$, $m \geq 1$, $p \in [1,\infty]$ and $T_p \in B(\ell_p)$ be a weighted right-shift operator with a weight sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ such that

$$|\lambda_n| \geq 1, \quad \text{for} \quad n = 1, ..., m-1, \quad \text{and} \quad |\lambda_n| = 1, \quad \text{for} \quad n \geq m.$$

By Example 5.11 in [16], $T_p$ is an $(m,\infty)$-isometric operator. Then, for instance, the tuple $(T_p, \frac{1}{2}T_p, \frac{1}{3}T_p, ..., \frac{1}{d}T_p)$ is an $(m,\infty)$-isometric tuple on $\ell_p$.

Since the definition of $(m,\infty)$-isometric tuples differs from the definition of $(m,\infty)$-isometric operators basically by the replacement of the exponent $k$ by a multi-index $\alpha$, it is not surprising that its basic theory can be developed analogously.

First of all, we have the following (compare page 399 in [16])).

**Proposition 5.3.** *A commuting operator tuple $T \in B(X)^d$ is an $(m,\infty)$-isometry if, and only if, there exists an $m \in \mathbb{N}$, $m \geq 1$, with*

$$\max_{\substack{|\alpha|=\ell,..,\ell+m \\ |\alpha| \ \textit{even}}} \|T^\alpha x\| = \max_{\substack{|\alpha|=\ell,..,\ell+m \\ |\alpha| \ \textit{odd}}} \|T^\alpha x\|, \quad \forall \ell \in \mathbb{N}, \forall x \in X. \tag{5.1}$$

**Proof.** The sufficiency of (5.1) is clear. So assume now that $T \in B(X)^d$ is an $(m,\infty)$-isometry and let $\ell \in \mathbb{N}$. We only prove the case where $\ell$ is even, since the case that $\ell$ is odd is a direct analogue. We have

$$\max_{\substack{|\alpha|=\ell,..,\ell+m\\ |\alpha| \text{ even}}} \|T^\alpha x\| = \max_{\substack{|\beta|=0,..,m\\ |\gamma|=\ell\\ |\beta|+|\gamma| \text{ even}}} \|T^\beta T^\gamma x\| = \max_{|\gamma|=\ell} \max_{\substack{|\beta|=0,..,m\\ |\beta| \text{ even}}} \|T^\beta T^\gamma x\|$$

$$= \max_{|\gamma|=\ell} \max_{\substack{|\beta|=0,..,m\\ |\beta| \text{ odd}}} \|T^\beta T^\gamma x\| = \max_{\substack{|\beta|=0,..,m\\ |\gamma|=\ell\\ |\beta|+|\gamma| \text{ odd}}} \|T^\beta T^\gamma x\| = \max_{\substack{|\alpha|=\ell,..,\ell+m\\ |\alpha| \text{ odd}}} \|T^\alpha x\|.$$

□

Further, using the following lemma we obtain an equivalent description of $(m,\infty)$-isometric tuples (compare Lemma 5.3 in [16]).

**Lemma 5.1.** *Let $\pi(n) = n \bmod 2$ denote the parity of an $n \in \mathbb{N}$. For any family $a = (a_\alpha)_{\alpha\in\mathbb{N}^d} \subset \mathbb{R}$ and $m \in \mathbb{N}$, $m \geq 1$, the following are equivalent.*

*(i) a satisfies* $\max_{\substack{|\alpha|=\ell,\dots,m+\ell\\ |\alpha| \text{ even}}} a_\alpha = \max_{\substack{|\alpha|=\ell,\dots,m+\ell\\ |\alpha| \text{ odd}}} a_\alpha, \;\; \forall \ell \in \mathbb{N}$

*(ii) a attains a maximum and* $\max_{\alpha\in\mathbb{N}^d} a_\alpha = \max_{\substack{|\alpha|=\ell,\dots,m-1+\ell\\ \pi(|\alpha|)=\pi(m-1+\ell)}} a_\alpha, \;\; \forall \ell \in \mathbb{N}.$

**Proof.** (i)⇒(ii): We proceed by induction on $\ell$. Suppose $a = (a_\alpha)_{\alpha\in\mathbb{N}^d} \subset \mathbb{R}$ satisfies (i) and choose $n \in \mathbb{N}$ with $n \geq m$. By (i), $\max_{|\alpha|=n-m,\dots,n} a_\alpha$ is attained for at least two multi-indices, one of even and one of odd norm. Thus, there exists an $r < n$ and a $\beta \in \mathbb{N}^d$ with $|\beta| = r$, such that $a_\beta \geq a_\alpha$ for every $\alpha \in \mathbb{N}^d$ with $|\alpha| = n$. Since this holds for all $n \geq m$, we deduce that the family $(a_\alpha)_{\alpha\in\mathbb{N}^d}$ indeed has a maximum, which is attained at an $\alpha$ with $|\alpha| \leq m-1$. That is, $\max_{\alpha\in\mathbb{N}^d} a_\alpha = \max_{|\alpha|=0,\dots,m-1} a_\alpha$. Since trivially $\max_{|\alpha|=0,\dots,m-1} a_\alpha \leq \max_{|\alpha|=0,\dots,m} a_\alpha$, we actually have equality and by (i) can write

$$\max_{\alpha\in\mathbb{N}^d} a_\alpha = \max_{\substack{|\alpha|=0,\dots,m\\ \pi(|\alpha|)=\pi(m-1)}} a_\alpha = \max_{\substack{|\alpha|=0,\dots,m-1\\ \pi(|\alpha|)=\pi(m-1)}} a_\alpha.$$

Hence, we have (ii) for $\ell_0 = 0$.

Now assume that (ii) holds for some $\ell \in \mathbb{N}$. Then, in particular, $\max_{\alpha\in\mathbb{N}^d} a_\alpha = \max_{|\alpha|=\ell,\dots,m-1+\ell} a_\alpha$ and since $\max_{|\alpha|=\ell,\dots,m-1+\ell} a_\alpha \leq \max_{|\alpha|=\ell,\dots,m+\ell} a_\alpha$, we again have equality. By (i), we can omit the first $\ell$ on the right-hand side, obtaining $\max_{\alpha\in\mathbb{N}^d} a_\alpha = \max_{|\alpha|=\ell+1,\dots,m+\ell} a_\alpha$. But this has to be equal to $\max_{|\alpha|=\ell+1,\dots,m+\ell+1} a_\alpha$ and again by (i), we can write

$$\max_{\alpha\in\mathbb{N}^d} a_\alpha = \max_{\substack{|\alpha|=\ell+1,\dots,m+\ell+1\\ \pi(|\alpha|)=\pi(m+\ell)}} a_\alpha = \max_{\substack{|\alpha|=\ell+1,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m+\ell)}} a_\alpha.$$

This is (ii) for $\ell + 1$.

(ii)⇒(i): If $a$ satisfies (ii), then, for all $\ell \in \mathbb{N}$,

$$\max_{|\alpha|\in\mathbb{N}^d} a_\alpha = \max_{\substack{|\alpha|=\ell,\dots,m-1+\ell\\ \pi(|\alpha|)=\pi(m-1+\ell)}} a_\alpha = \max_{\substack{|\alpha|=\ell,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m-1+\ell)}} a_\alpha$$

and also, by replacing $\ell$ with $\ell+1$,

$$\max_{|\alpha|\in\mathbb{N}^d} a_\alpha = \max_{\substack{|\alpha|=\ell+1,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m+\ell)}} a_\alpha \le \max_{\substack{|\alpha|=\ell,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m+\ell)}} a_\alpha \le \max_{\alpha\in\mathbb{N}^d} a_\alpha.$$

This implies $\max\limits_{\substack{|\alpha|=\ell,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m-1+\ell)}} a_\alpha = \max\limits_{\substack{|\alpha|=\ell,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m+\ell)}} a_\alpha$, which is (i). □

Combining the last two statements gives:

**Corollary 5.1.** *Let $m \in \mathbb{N}$ with $m \ge 1$. A tuple of commuting operators $T = (T_1, ..., T_d) \in B(X)^d$ is an $(m,\infty)$-isometry if, and only if, for each $x \in X$ the family $(\|T^\alpha x\|)_{\alpha\in\mathbb{N}^d}$ attains a maximum and*

$$\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \max_{\substack{|\alpha|=\ell,\dots,m-1+\ell\\ \pi(|\alpha|)=\pi(m-1+\ell)}} \|T^\alpha x\|, \quad \forall \ell \in \mathbb{N}.$$

We easily deduce the following.

**Corollary 5.2.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m,\infty)$-isometry. Then $(\|T^\alpha\|)_{\alpha\in\mathbb{N}^d}$ is bounded. In particular, $T$ is* uniformly power bounded*, that is, there exists a common $C > 0$, such that $\|T_j^n\| \le C$, for all $n \in \mathbb{N}$, for all $j \in \{1, ..., d\}$.*

Note that $C$ is, of course, given by $C = \max_{|\alpha|=0,\dots,m-1} \|T^\alpha\|$. So we do not have to make use of the Uniform Boundedness Principle here and are not even assuming that $X$ is complete.

By simply copying the proof in the single operator case (see Proposition 6.3 in [16]), we show the following:

**Proposition 5.4.** *Let $T \in B(X)^d$ be an $(m,\infty)$-isometric tuple. Then $T$ is an $(m+1,\infty)$-isometric tuple.*

**Proof.** By 5.1, $\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\|$ exists and, for all $\ell \in \mathbb{N}$ and $x \in X$, we have

$$\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \max_{\substack{|\alpha|=\ell+1,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m+\ell)}} \|T^\alpha x\| \le \max_{\substack{|\alpha|=\ell,\dots,m+\ell\\ \pi(|\alpha|)=\pi(m+\ell)}} \|T^\alpha x\| \le \max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\|$$

and the ensuing equality gives the result by invoking 5.1 again. □

The case $m = 1$ deserves some special attention.

We call a commuting operator tuple $T = (T_1, ..., T_d) \in B(X)^d$ an $\ell_\infty$*-spherical isometry* if

$$\max_{j=1,\dots,d} \|T_j x\| = \|x\|, \quad \forall x \in X.$$

Obviously $\ell_\infty$-spherical isometries are just $(1, \infty)$-isometric tuples.

**Proposition 5.5.** *Let $T \in B(X)^d$ be an $\ell_\infty$-spherical isometry (i.e., a $(1, \infty)$-isometric tuple). For each $x \in X$ there exists a $j_x \in \{1, ..., d\}$ such that $\|T_{j_x}^n x\| = \|x\|$ for all $n \in \mathbb{N}$.*[g]

**Proof.** We first show the following claim:

For each $n \in \mathbb{N}$ and each $x \in X$, there exists a $j_{n,x} \in \{1, ..., d\}$ with $\|T_{j_{n,x}}^k x\| = \|x\|$ for all $k \in \mathbb{N}$ with $k \leq n$.

Proof of the claim:

Note first, since we have by definition $\max_{j=1,...,d} \|T_j x\| = \|x\|$, for all $x \in X$, that $\|T_j\| \leq 1$ for all $j \in \{1, ..., d\}$. Clearly then also $\|T^\alpha\| \leq 1$ for all $\alpha \in \mathbb{N}^d$. Further, by Corollary 5.1, $\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \max_{|\alpha|=\ell} \|T^\alpha x\| = \|x\|$, for all $x \in X$ and all $\ell \in \mathbb{N}$ (since $m = 1$).

Therefore, for each $\ell \in \mathbb{N}$ and each $x \in X$, there exists an $\alpha(\ell, x)$ with $|\alpha(\ell, x)| = \ell$ and $\|T^{\alpha(\ell,x)} x\| = \|x\|$. Thus, $\|T^{\alpha(\ell,x)_j} x\| = \|x\|$ for all $j \in \{1, ..., d\}$, as $\|T_j\| \leq 1$ for all $j$ and $\|x\| = \max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\|$. Moreover, setting $n = \ell d$, there exist an index $j_{n,x} \in \{1, ..., d\}$ such that $\alpha_{j_{n,x}} \geq \frac{\ell}{d} = n$. [h] So $\|T_{j_{n,x}}^n x\| = \|T_{j_{n,x}}^k x\| = \|x\|$ for all $k \leq n$, $k \in \mathbb{N}$, again as $\|T_{j_{n,x}}\| \leq 1$. Thus, the claim is proved.

The rest of the proof is essentially the pigeon hole principle: For fixed $x \in X$, we have infinitely many $n \in \mathbb{N}$, but only finitely many $j_{n,x} \in \{1, ..., d\}$.

Fix $x \in X$ and define for each $j \in \{1, ..., d\}$ the set

$$A_j := \{n \in \mathbb{N} \mid \|T_j^k x\| = \|x\|, \text{ for all } k \leq n, \ k \in \mathbb{N}\}.$$

By our claim, for each $n \in \mathbb{N}$ there is an index $j_{n,x}$, that is, every natural number resides in at least one of the $A_j$. Thus, since we have only finitely many sets $A_j$, at least one $A_{j_x}$ is infinite. $\|T_{j_x}\| \leq 1$ then forces $A_{j_x} = \mathbb{N}$ as required. □

Therefore, we have the following remark.

**Remark 5.1.** If $T$ is an $\ell_\infty$-spherical isometry, the space $X$ is the union of the closed subsets $X_j := \{x \in X \mid \|x\| = \|T_j^n x\|, \ \forall n \in \mathbb{N}\}$. [i]

In the uni-variate case, an $(m, \infty)$-isometry $T$ is an isometry under an equivalent norm, given by $\max_{k \in \mathbb{N}} \|T^k x\|$ (see Theorem 5.2 in [16]). Indeed, an analogous result holds in the tuple case.

**Theorem 5.1.** *Let $T \in B(X)^d$ be an $(m, \infty)$-isometric tuple. Then there exists a norm $|.|_\infty$ on $X$ equivalent to $\|.\|$, under which $T$ is an $\ell_\infty$-spherical isometry. $|.|_\infty$ is given by $|x|_\infty = \max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \max_{|\alpha|=0,...,m-1} \|T^\alpha x\|$, for all $x \in X$.*

[g]Note that we make use of the continuity of the operators in the proof.

[h]This certainly holds for any index $j_{\max} \in \{1, ..., d\}$ with $\alpha(\ell, x)_{j_{\max}} := \max_{j=1,...,d} \alpha(\ell, x)_j$, which is, of course, not necessarily uniquely determined.

[i]Note that the $X_j$ are not disjoint (since $0 \in X_j$ for each $j$) and don't necessarily have trivial intersection.

**Proof.** By Corollary 5.1, $\max_{\alpha\in\mathbb{N}^d}\|T^\alpha x\| = \max_{|\alpha|=0,\dots,m-1}\|T^\alpha x\|$, for all $x \in X$. Since $T_j$ is linear for each $j = 1, ..., d$ and the maximum preserves the triangle inequality, $|.|_\infty$ is indeed a norm on $X$. Further, by Corollary 5.1

$$\max_{j=1,\dots,d}\max_{\alpha\in\mathbb{N}^d}\|T^\alpha T_j x\| = \max_{j=1,\dots,d}\max_{|\alpha|=0,\dots,m-1}\|T^\alpha T_j x\|$$
$$= \max_{|\alpha|=1,\dots,m}\|T^\alpha x\| = \max_{\alpha\in\mathbb{N}^d}\|T^\alpha x\|, \quad \forall x \in X,$$

so that $T$ is an $\ell_\infty$-spherical isometry with respect to $|.|_\infty$. Finally, we have

$$\|x\| \le \max_{\alpha\in\mathbb{N}^d}\|T^\alpha x\| = \max_{|\alpha|=0,\dots,m-1}\|T^\alpha x\| \le \max_{|\alpha|=0,\dots,m-1}\|T^\alpha\|\cdot\|x\|, \quad \forall x \in X,$$

and the two norms are equivalent. □

This, of course, implies immediately the next statement.

**Remark 5.2.** If $T$ is an $(m,\infty)$-isometric tuple, the space $X$ is the union of the closed subsets $X_{j,|.|} := \{x \in X \mid |x|_\infty = |T_j^n x|_\infty,\ \forall n \in \mathbb{N}\}$.

## 6. Spectral Properties

Let in this section $X$ be a complex Banach space. As before, let $T = (T_1, ..., T_d) \in B(X)^d$ be a tuple of commuting linear operators on $X$.[j] A first definition of the *joint spectral radius* of such a tuple $T$ was given by Rota and Strang in [22]:

$$\hat{r}(T) := \lim_{k\to\infty}\max_{|\alpha|=k}\|T^\alpha\|^{\frac{1}{k}}$$

(Note that no definition of a joint spectrum is necessary for this expression to make sense.) In [7], Berger and Wang give an alternative definition, which reads as follows.

$$r_*(T) := \lim_{k\to\infty}\max_{|\alpha|=k} r(T^\alpha)^{\frac{1}{k}},$$

where $r(T^\alpha) := r(T_1^{\alpha_1}...T_d^{\alpha_d})$ is the usual spectral radius for operators. However, in the Lemma on page 94 in [25], Soltysiak shows that we indeed have $\hat{r}(T) = r_*(T)$.

We further have the *geometric joint spectral radius*, $r(T)$, defined as

$$r(T) := \max\{\|\lambda\|_2 \mid \lambda \in \sigma(T)\}.$$

Here, $\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d$ and $\sigma(T)$ denotes the *Taylor spectrum* (see [26]).

Other kind of joint spectra include the *Harte spectrum* $\sigma_H(T)$ (see [15]) and the *joint (left) approximate point spectrum*[k]

$$\sigma_\pi(T) := \{(\lambda_1, ..., \lambda_d) \in \mathbb{C}^d \mid \exists\ (x_k)_{k\in\mathbb{N}} \subset X \text{ with } \|x_k\| = 1, \text{ s.th.}$$
$$\lim_{k\to\infty}\sum_{j=1}^d \|(T_j - \lambda_j I)x_k\| = 0\}.$$

[j] Most of the statements that we quote in this section are in general not true, if the operators $T_1, ..., T_d$ do not commute.

[k] Harte refers to this set in [15] as *left approximate point spectrum*.

All three spectra are non-void[l]. For the joint approximate point spectrum this has been shown in Theorem 1.11 in [24].[m]

Further, it was shown in [12] that the convex hulls of all the named spectra above coincide. Thus, the geometric joint spectral radius does not depend on the choice of the joint spectrum. That is, one then can replace in its definition the Taylor spectrum by the Harte spectrum or the joint approximate point spectrum.

Soltysiak generalizes the notion of the geometric joint spectral radius in [25] in the following way: Define for $p \in [1, \infty]$ the *(geometric) joint $\ell_p$-spectral radius* $r_p(T)$ by

$$r_p(T) := \max\{\|\lambda\|_p \mid \lambda \in \sigma_H(T)\}.$$

Again, since we only consider commuting operator tuples, the $\ell_p$-spectral radius does not depend on the chosen spectrum.

Obviously, we have $r_2(T) = r(T)$. Further, Soltysiak shows in Theorem 2 in [25] that $r_\infty(T) = \hat{r}(T)$ $(= r_*(T))$. Thus, the $\ell_p$-spectral radii contain all variations of joint spectral radii named so far. Finally, Müller proves in Theorem 3 in [18] the corresponding equalities for finite $p$:

$$r_p(T) = \lim_{k\to\infty} \left( \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha\|^p \right)^{\frac{1}{pk}} = \lim_{k\to\infty} \left( \sum_{|\alpha|=k} \frac{k!}{\alpha!} r(T^\alpha)^p \right)^{\frac{1}{pk}}, \quad p \in [1, \infty). \tag{6.1}$$

Now, Gleason and Richter prove in Proposition 3.1 and Lemma 3.2 in [14] that the geometric spectral radius $r(T) = r_2(T)$ of an $(m, 2)$-isometric tuple on a complex Hilbert space is equal to 1. They deliver two alternative proofs for this, which can be easily modified to suit the case of $(m, p)$-isometric for $p \geq 1$ and $(m, \infty)$-isometric tuples on complex Banach spaces.

**Proposition 6.1.**

*(i) If $p \in [1, \infty)$, $m \geq 1$ and $T \in B(X)^d$ is an $(m, p)$-isometry, then*

$$r_p(T) = \lim_{k\to\infty} \left( \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha\|^p \right)^{\frac{1}{pk}} = 1.$$

*(ii) If $T \in B(X)^d$ is an $(m, \infty)$-isometry, then*

$$\lim_{k\to\infty} \max_{|\alpha|=k} \|T^\alpha\|^{\frac{1}{k}} = 1.$$

*Consequently, if $p \in [1, \infty]$, $X$ is a complex Banach space and $T \in B(X)^d$ is an $(m, p)$-isometry, the geometric joint $\ell_p$-spectral radius $r_p(T)$ of $T$ is 1.*

[l] Harte gives with Example 1.6 in [15] an example of a non-commuting operator pair with empty Harte spectrum.

[m] The definition of $\sigma_\pi(T)$ given in [24] actually requires the existence of a net instead of a sequence, however, the proof uses a result given in [27], which is stated in terms of sequences.

**Proof.** (i): By (6.1), $r_p(T) = \lim_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha\|^p\Big)^{\frac{1}{pk}}$.

For all $k \in \mathbb{N}$, the number of summands in $\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha\|^p$ is certainly strictly less than $k^d$. Hence,

$$\lim_{k\to\infty}\Bigg(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha\|^p\Bigg)^{\frac{1}{pk}} = \lim_{k\to\infty}\max_{|\alpha|=k}\Big(\frac{k!}{\alpha!}\|T^\alpha\|^p\Big)^{\frac{1}{pk}}$$

$$\text{and } \lim_{k\to\infty}\sup_{\|x\|=1}\Bigg(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha x\|^p\Bigg)^{\frac{1}{pk}} = \lim_{k\to\infty}\sup_{\|x\|=1}\max_{|\alpha|=k}\Big(\frac{k!}{\alpha!}\|T^\alpha x\|^p\Big)^{\frac{1}{pk}}.$$

The following is taken from the proof of Theorem 4 [18]).

$$r_p(T) = \lim_{k\to\infty}\Bigg(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha\|^p\Bigg)^{\frac{1}{pk}} = \lim_{k\to\infty}\max_{|\alpha|=k}\Big(\frac{k!}{\alpha!}\|T^\alpha\|^p\Big)^{\frac{1}{pk}}$$

$$= \lim_{k\to\infty}\max_{|\alpha|=k}\sup_{\|x\|=1}\Big(\frac{k!}{\alpha!}\|T^\alpha x\|^p\Big)^{\frac{1}{pk}} = \lim_{k\to\infty}\sup_{\|x\|=1}\max_{|\alpha|=k}\Big(\frac{k!}{\alpha!}\|T^\alpha x\|^p\Big)^{\frac{1}{pk}}$$

$$= \lim_{k\to\infty}\sup_{\|x\|=1}\Bigg(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha x\|^p\Bigg)^{\frac{1}{pk}} = \lim_{k\to\infty}\sup_{\|x\|=1}\left(Q^{k,p}(T,x)\right)^{\frac{1}{pk}}.$$

Since $\sigma_H(T) \neq \emptyset$ is compact, $r_p(T) < \infty$ exists finitely. Hence, the equation above shows that the function $\left(Q^{k,p}(T,\cdot)\right)^{\frac{1}{pk}}$, restricted to the closed unit ball $\overline{B(0;1)}$ of $X$, converge uniformly to $r_p(T)$. Thus, they converge point-wise.

The remaining parts of the proof are now almost identical to the proof of Proposition 3.1 in [14]. By Proposition 3.2.(ii),

$$\lim_{k\to\infty}\frac{Q^{k,p}(T,x)}{k^{m-1}} = \frac{1}{(m-1)!}P^{(p)}_{m-1}(T,x) \geq 0, \quad \forall x \in X.$$

Assuming that $m$ is the smallest natural number, for which $T$ is $(m,p)$-isometric, prompts that there exist vectors $x \in \overline{B(0;1)}$ for which the inequality on the right is strict. Thus, for all such $x$, $\lim_{k\to\infty}\left(\frac{1}{(m-1)!}P^{(p)}_{m-1}(T,x)\right)^{\frac{1}{pk}} = 1$. Hence, since $\lim_{k\to\infty}\left(k^{m-1}\right)^{\frac{1}{pk}} = 1$, we have for all $x \in \overline{B(0;1)}$ with $P^{(p)}_{m-1}(T,x) \neq 0$,

$$r_p(T) = \lim_{k\to\infty}\left(Q^{k,p}(T,x)\right)^{\frac{1}{pk}} = \lim_{k\to\infty}\left(\frac{(Q^{k,p}(T,x)}{k^{m-1}}\right)^{\frac{1}{pk}}$$

$$= \lim_{k\to\infty}\left(\frac{1}{(m-1)!}P^{(p)}_{m-1}(T,x)\right)^{\frac{1}{pk}} = 1.$$

(ii): By Corollary 5.2 the sequence $(\max_{|\alpha|=n}\|T^\alpha\|)_{\alpha\in\mathbb{N}^d}$ is bounded. The statement follows if we show that this sequence is also bounded below.

By Proposition 5.1, we have $C \cdot \|T^\alpha x\| \geq |T^\alpha x|_\infty$ for all $x \in X$ and $\alpha \in \mathbb{N}^d$ (with $C = \max_{|\alpha|=0,\ldots,m-1}\|T^\alpha\|$). This implies

$$C \cdot \max_{|\alpha|=n}\|T^\alpha x\| \geq \max_{|\alpha|=n}|T^\alpha x|_\infty = |x|_\infty \geq \|x\|, \quad \forall x \in X,\ \forall n \in \mathbb{N}.$$

Here, the equality is due to Corollary 5.1, since $T$ is an $\ell_\infty$-spherical isometry w.r.t. $|.|_\infty$.

In particular, we have $C \cdot \max_{|\alpha|=n} \|T^\alpha x\| \geq \|x\|$ for all $x \in X$, and then

$$\sup_{\|x\|=1} \left( C \cdot \max_{|\alpha|=n} \|T^\alpha x\| \right) = C \cdot \max_{|\alpha|=n} \sup_{\|x\|=1} \|T^\alpha x\| = C \cdot \max_{|\alpha|=n} \|T^\alpha\| \geq 1, \quad \forall n \in \mathbb{N}. \square$$

In Lemma 3.2 in [14] it is shown that, if $T$ is an $(m, 2)$-isometry on a complex Hilbert space, then $\|\lambda\|_2 = 1$, for all $\lambda \in \sigma_\pi(T)$. That is, its joint approximate point spectrum lies in the boundary of the $d$-dimensional unit sphere and, therefore, one gets again that the geometric joint spectral radius of an $(m, 2)$-isometry is equal to 1. To obtain this result, Gleason and Richter show (see page 187 in [14]) that $\lambda \in \sigma_\pi(T)$ if, and only if, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $\|x_n\| = 1$, for all $n \in \mathbb{N}$, such that $(T^\alpha - \lambda^\alpha)x_n \to 0$ $(n \to \infty)$, for all $\alpha \in \mathbb{N}^d$. Using this fact, we obtain a generalisation of Lemma 3.2 in [14]:

**Proposition 6.2.** *Let $p \in [1, \infty]$ and $X$ be a complex Banach space. Then the joint approximate point spectrum $\sigma_\pi(T)$ of an $(m, p)$-isometric tuple $T \in B(X)^d$ is a subset of the $d$-dimensional complex unit sphere with respect to the $p$-norm.*

**Proof.** Let $\lambda \in \sigma_\pi(T)$.

If $p \in [1, \infty)$ and $T \in B(X)^d$ is an $(m, p)$-isometric tuple, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $\|x_n\| = 1$, for all $n \in \mathbb{N}$, such that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x_n\|^p \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} |\lambda^\alpha|^p = (1 - \|\lambda\|_p^p)^m. \end{aligned}$$

$\|\lambda\|_p = 1$, for all $\lambda \in \sigma_\pi(T)$, follows immediately.

If $T \in B(X)^d$ is an $(m, \infty)$-isometric tuple, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $\|x_n\| = 1$, for all $n \in \mathbb{N}$, such that

$$\begin{aligned} &\lim_{n\to\infty} \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} \|T^\alpha x_n\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} |\lambda^\alpha| \\ \text{and} \quad &\lim_{n\to\infty} \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} \|T^\alpha x_n\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} |\lambda^\alpha|. \end{aligned}$$

Since $T$ is an $(m, \infty)$-isometry and by uniqueness of limits, we therefore have

$$\begin{aligned} &\max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} |\lambda^\alpha| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} |\lambda^\alpha| \\ \Leftrightarrow \quad &\max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} \|\lambda\|_\infty^{|\alpha|} = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} \|\lambda\|_\infty^{|\alpha|}. \end{aligned}$$

The fact that we are equating an even power of $\|\lambda\|_\infty$ with an odd power forces $\|\lambda\|_\infty$ to be 0 or 1. However, if $\lambda = 0$, then the maximum on the left hand side is 1, reached at $|\alpha| = 0$, and the right hand side is 0. Therefore, we must have $\|\lambda\|_\infty = 1$. [n] □

## 7. On the intersection class of $(m, p)$- and $(m, \infty)$-isometric tuples

It is known (see Proposition 6.1 in [16]) and easy to see, that an $(m, p)$-isometric operator is simultaneously an $(m, \infty)$-isometric operator if, and only if, it is an isometry.

A natural analogue of this statement would appear to be "an $(m, p)$-isometric tuple is simultaneously an $(m, \infty)$-isometric tuple if, and only if, it is an $\ell_p$-spherical isometry (or an $\ell_\infty$-spherical isometry)". However, such a statement cannot be true.

**Example 7.1.** Let $T_1 \in B(X)$ be an isometric operator and $T = (T_1, ..., T_d) \in B(X)^d$ an $(m, p)$-isometry. By Proposition 4.3, $T$ is an $(m, p)$-isometry for every $p \in (0, \infty)$ and, hence, by definition an $(m, \infty)$-isometry. However, in general $T$ does not need to be an $\ell_p$-spherical or an $\ell_\infty$-spherical isometry, as Example 4.4 shows.

It is currently unknown what the intersection of the set of all $(m, p)$-isometric and all $(m, \infty)$-isometric tuples on a given normed space $X$ actually is. Looking at the joint approximate point spectrum (in the complex Banach space case if $p \geq 1$) gives some information.

**Remark 7.1.** Let $X$ be a complex Banach space and $p \in [1, \infty)$. Let further $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m, p)$-isometric and a $(\mu, \infty)$-isometric tuple. Then every $\lambda \in \sigma_{\text{ap}}(T)$ satisfies $\|\lambda\|_p = \|\lambda\|_\infty = 1$ by Proposition 6.2. Consequently, since $\sigma_{\text{ap}}(T) \subset \sigma_{\text{ap}}(T_1) \times \cdots \times \sigma_{\text{ap}}(T_d)$, one operator $T_{j_0}$ has spectral radius $r(T_{j_0}) \geq 1$ and the remaining operators $T_i$, $i \neq j_0$, are not bounded below and in particular not invertible.

More specific results can so far only be given in special cases.

The case where our tuple is constructed by using an $(m, \infty)$-isometric operator (i.e. by applying Proposition 5.2) is easy and we consider it first.

**Proposition 7.1.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(\mu, \infty)$-isometric tuple of the form of Proposition 5.2. That is, $T_j = z_j S$, where $S \in B(X)$ is an $(\mu, \infty)$-isometric operator and $z := (z_1, ..., z_d) \in \mathbb{K}^d$ with $\|z\|_\infty = 1$. Assume further that $T$ is additionally an $(m, p)$-isometric tuple. Then the operator $S$ is an isometry, $T = (0, ..., 0, z_{j_0} S, 0, ..., 0)$ with $|z_{j_0}| = 1$ for some $j_0 \in \{1, ..., d\}$ and $z_{j_0} S$ is (trivially) also an isometry.*

[n]Note also that, by the proof of Proposition 6.1.(ii), we have $C \cdot \max_{|\alpha|=n} \|T^\alpha x\| \geq \|x\|$ for all $x \in X$ and all $n \in \mathbb{N}$, with $C > 0$. Thus, $\left(\max_{|\alpha|=0,...,m} \|T^\alpha x_n\|\right)_{n\in\mathbb{N}}$ is bounded below, which also shows that $\lambda = 0$ cannot occur.

**Proof.** Since $T$ is an $(m,p)$-isometry, we have for all $x \in X$,

$$Q^n(x) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|z^\alpha S^{|\alpha|} x\|^p = \|z\|_p^{np} \|S^n x\|^p,$$

by the multinomial theorem. Then $D^m\left(Q^n(x)\right)_{n\in\mathbb{N}} = 0$ implies that $\|z\|_p S$ is an $(m,p)$-isometric operator and, consequently, the sequence $\left(\|z\|_p^{np}\|S^n x\|^p\right)_{n\in\mathbb{N}}$ is a polynomial of degree $\leq m-1$ for all $x \in X$.

Since $\|z\|_\infty = 1$, we have $\|z\|_p \geq 1$. If this inequality was strict, the sequence $\left(\|z\|_p^{np}\right)_{n\in\mathbb{N}}$ would grow exponentially. But since $S$ is a $(\mu,\infty)$-isometric operator, we have for all $\ell \in \mathbb{N}$, for all $x \in X$, $\max_{n\in\mathbb{N}} \|S^n x\| = \max_{\ell,\dots,\ell+\mu-1} \|S^n x\|$. This would contradict the polynomial growth of $\left(\|z\|_p^{np}\|Sx\|^p\right)_{n\in\mathbb{N}}$.

Therefore, we have $\|z\|_\infty = \|z\|_p = 1$, which gives $|z_{j_0}| = 1$ for some $j_0 \in \{1,...,d\}$ and $z_i = 0$ for all $i \neq j_0$. In particular, $S$ has to be an $(m,p)$-isometric operator, which forces $S$ by Proposition 6.1 in [16], and therefore $z_{j_0} S$, to be an isometry. □

To prove further results, we first state a series of lemmata.

**Lemma 7.1.** *Let $T = (T_1,...,T_d) \in B(X)^d$ be an $\ell_\infty$-spherical isometry (i.e., a $(1,\infty)$-isometric tuple). For $j \in \{1,...,d\}$ let, as in Remark 5.1,*
$X_j = \{x \in X \mid \|x\| = \|T_j^n x\|,\ \forall n \in \mathbb{N}\}$.

*Then one operator $T_{j_0}$ is an isometry and all other operators $T_i$ with $i \neq j_0$ are nilpotent if, and only if, there exists $\nu \in \mathbb{N}$ such that, $X_j \subset \ker T_i^\nu$, for all $i \neq j$, $i,j \in \{1,...,d\}$.*

**Proof.** "⇒": If one operator $T_{j_0}$ is an isometry and all other operators $T_i$ with $i \neq j_0$ are nilpotent, we have $X_{j_0} = X$ and $X_i = \{0\}$ for all $i \neq j_0$. Setting $\nu := \max\{n \in \mathbb{N} \mid T_i^n = 0,\ i = 1,...,d, i \neq j_0\}$ gives $\ker T_i^\nu = X$ for all $i \neq j_0$. Since $\ker T_{j_0}^\nu = \{0\}$, the statement follows.

"⇐": By Remark 5.1, $X = \bigcup_{j=1,\dots,d} X_j$. Since by assumption each $X_j \subset \ker T_i^\nu$ for all $i \in \{1,...,d\}$ with $i \neq j$, we have

$$X = \bigcup_{j=1,\dots,d} X_j \subset \bigcup_{j=1,\dots,d} \bigcap_{\substack{i=1,\dots,d\\ i\neq j}} \ker T_i^\nu.$$

This forces $\bigcap_{\substack{i=1,\dots,d\\ i\neq j_0}} \ker T_i^\nu = X$ for some $j_0 \in \{1,...,d\}$ (since each intersection is a linear space). Hence, $\ker T_i^\nu = X$ for all $i \neq j_0$, which means $T_i^\nu = 0$ and, thus, $X_i = \{0\}$ for all $i \neq j_0$. Then we must have $X_{j_0} = X$ and $T_{j_0}$ is an isometry. □

**Lemma 7.2.** *Let $T = (T_1,...,T_d) \in B(X)^d$ be an $(m,p)$-isometric tuple and also a $(\mu,\infty)$-isometric tuple. Then for all $\gamma = (\gamma_1,...,\gamma_d) \in \mathbb{N}^d$ with the property $|\gamma_j'| \geq m$ for every $j \in \{1,...,d\}$, we have $T^\gamma = 0$. In particular, $T_i^m T_j^m = 0$ for every $i \neq j$, $i,j \in \{1,...,d\}$.*

**Proof.** Lets first consider the case $\mu = 1$. If $T$ is a $(1,\infty)$-isometric tuple, by Corollary 5.1,

$$\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \|x\| = \max_{|\alpha|=\ell} \|T^\alpha x\|,\ \forall \ell\in\mathbb{N},\ \forall x\in X.$$

So $\big(\max_{|\alpha|=\ell} \|T^\alpha x\|\big)_{\ell\in\mathbb{N}}$ is a constant sequence for all $x \in X$. In particular, $\big(\max_{|\alpha|=\ell} \|T^\alpha T^\gamma x\|\big)_{\ell\in\mathbb{N}}$ is constant for any multi-index $\gamma\in\mathbb{N}^d$, for all $x\in X$.

Since $T$ is an $(m,p)$-isometric tuple, for any $x\in X$, any $\beta\in\mathbb{N}^{d-1}$ with $|\beta|\geq m$ and any $j\in\{1,...,d\}$, by Corollary 3.2.(ii), $T_j^n(T_j')^\beta x \to 0$ for $n\to\infty$. We will show that this implies, given a $\gamma\in\mathbb{N}^d$ with the property $|\gamma_j'|\geq m$ for every $j\in\{1,...,d\}$, that $\|T^\alpha T^\gamma x\|\to 0$ as $|\alpha|\to\infty$, for all $x\in X$.

So take a $\gamma\in\mathbb{N}^d$ with the property $|\gamma_j'|\geq m$ for every $j\in\{1,...,d\}$. Then for any $x\in X$, any $j\in\{1,...,d\}$ and for all $\varepsilon>0$, there exists an $N_\varepsilon(x,j)\in\mathbb{N}$ such that $\|T_j^n(T_j')^{\gamma_j'}x\|\leq\varepsilon$, for all $n\geq N_\varepsilon(x,j)$, by Corollary 3.2.(ii).

But since we have only finitely many $j$, by simply taking the maximum $N_\varepsilon(x)$ of all $N_\varepsilon(x,j)$, we get that for any $x\in X$, for all $\varepsilon>0$, $\|T_j^n(T_j')^{\gamma_j'}x\|\leq\varepsilon$, for all $j\in\{1,...,d\}$, for all $n\geq N_\varepsilon(x)$.

For all $\alpha\in\mathbb{N}^d$ with $|\alpha|=\ell$, we have $\alpha_{j_{\max}} := \max_{j=1,\dots,d}\alpha_j \geq \frac{\ell}{d}$, for all $\ell\in\mathbb{N}$. But then, for any chosen $x\in X$ and for all $\varepsilon>0$, there exists an $M_\varepsilon(x)\in\mathbb{N}$ such that $\|T_{j_{\max}}^{\alpha_{j_{\max}}+\gamma_{j_{\max}}}(T_{j_{\max}}')^{\gamma_{j_{\max}}'}x\|\leq\varepsilon$ for all $\alpha\in\mathbb{N}^d$ with $|\alpha|=\ell$, for all $\ell\geq M_\varepsilon(x)$. [o]

Therefore,

$$\begin{aligned}\|T^\alpha T^\gamma x\| &= \|(T_{j_{\max}}')^{\alpha_{j_{\max}}'} T_{j_{\max}}^{\alpha_{j_{\max}}+\gamma_{j_{\max}}}(T_{j_{\max}}')^{\gamma_{j_{\max}}'}x\| \\ &\leq \|(T_{j_{\max}}')^{\alpha_{j_{\max}}'}\|\cdot\|T_{j_{\max}}^{\alpha_{j_{\max}}+\gamma_{j_{\max}}}(T_{j_{\max}}')^{\gamma_{j_{\max}}'}x\| \\ &\leq \|(T_{j_{\max}}')^{\alpha_{j_{\max}}'}\|\cdot\varepsilon,\quad \forall\alpha\in\mathbb{N}^d \ \text{ with } \ |\alpha|=\ell,\ \forall\ \ell\geq M_\varepsilon(x).\end{aligned}$$

Now, since $T$ is a $(1,\infty)$-isometric tuple, $\|T_j\|\leq 1$, for all $j\in\{1,...,d\}$. Thus, $\|T^\alpha T^\gamma x\|\leq\varepsilon$, for all $\alpha\in\mathbb{N}^d$ with $|\alpha|=\ell$, for all $\ell\geq M_\varepsilon(x)$.

Then

$$\|T^\gamma x\| = \max_{|\alpha|=\ell}\|T^\alpha T^\gamma x\|\to 0,\ \text{as}\ \ell\to\infty.$$

Since $x$ was chosen arbitrarily, $T^\gamma = 0$ follows.

Now consider the case $\mu>1$ and let $T$ be a $(\mu,\infty)$-isometry. By Theorem 5.1, $T$ is a $(1,\infty)$-isometric tuple with respect to the norm $|.|_\infty$ on $X$, where $|.|_\infty$ is equivalent to $\|.\|$. Hence, for any $x\in X$, any $\beta\in\mathbb{N}^{d-1}$ with $|\beta|\geq m$ and any $j\in\{1,...,d\}$, $T_j^n(T_j')^\beta x$ converges to 0 for $n\to\infty$ under $|.|_\infty$. By repeating the argument from above [p], we then get that

$$|T^\gamma x|_\infty = \max_{|\alpha|=\ell}|T^\alpha T^\gamma x|_\infty\to 0,\ \text{as}\ \ell\to\infty.$$

[o]Of course, the index $j_{\max}$ is not uniquely determined and may also be different for every $\alpha$.
[p]Note that we do not have to assume that $T$ is an $(m,p)$-isometry w.r.t. $|.|_\infty$

Again, $T^\gamma = 0$ follows. □

**Corollary 7.1.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m,p)$-isometric tuple for some $m \geq 1$ and also a $(\mu,\infty)$-isometric tuple. If $T^\alpha \neq 0$ with $|\alpha| = n$, then $\alpha$ is a permutation of $(n-|\beta|, \beta_1, ..., \beta_{d-1})$, where $|\beta| \leq m-1$. I.e., $T^\alpha = T_j^{n-|\beta|}(T_j')^\beta$ for some $j \in \{1, ..., d\}$ and some $\beta \in \mathbb{N}^{d-1}$ with $|\beta| \leq m-1$.* [q]

**Proof.** If $d = 1$ there is nothing to show, so assume $d \geq 1$.

Let $|\alpha| = n$ and chose a $j \in \{1, ..., d\}$ with $\alpha_j \neq 0$. Then we can write $\alpha_j = n-|\beta|$ and $T^\alpha = T_j^{n-|\beta|}(T_j')^\beta$ for $\beta = \alpha_j' \in \mathbb{N}^{d-1}$. We have to show that, if $T^\alpha \neq 0$, then $|\beta| \leq m-1$, or we can reorder and write $T^\alpha = T_k^{n-|\tilde\beta|}(T_k')^{\tilde\beta}$ for some $k \in \{1, ..., d\}$ and some $\tilde\beta \in \mathbb{N}^{d-1}$ with $|\tilde\beta| \leq m-1$.

Since $n - |\beta| = \alpha_j \geq 1$, the statement holds trivially if $n \leq m$. So assume $n \geq m+1$.

Certainly, by Lemma 7.2, if $n - |\beta| \geq m$ and $|\beta| \geq m$, $T_j^{n-|\beta|}(T_j')^\beta = 0$. This means, if $T^\alpha \neq 0$, we have we must have $|\beta| \leq m-1$ or $|\beta| \geq n-m+1$.

If we have $|\beta| \leq m-1$ we are done, so assume $|\beta| \geq m$ and $|\beta| \geq n-m+1$.

Now, if the biggest entry of $\beta$, $\beta_{j_{\max}} := \max_{j=1,...,d-1} \beta_j$ (where again, the index $j_{\max}$ is not necessarily unique) satisfies $\beta_{j_{\max}} \geq n-m+1$, then we have $-\beta_{j_{\max}} + n = |\beta'_{j_{\max}}| + n - |\beta| \leq m-1$ and we let $\tilde\beta$ be a multi-index consisting of the entries of $\beta'_{j_{\max}}$ and the entry $n-|\beta|$ w.r.t. some permutation. Then $T^\alpha = T_{j_{\max}}^{\beta_{j_{\max}}}\left(T'_{j_{\max}}\right)^{\tilde\beta} = T_{j_{\max}}^{n-|\tilde\beta|}\left(T'_{j_{\max}}\right)^{\tilde\beta}$ with $|\tilde\beta| \leq m-1$.

If instead $\beta_{j_{\max}} \leq n-m$, then $|\beta_i'| + n - |\beta| \geq m$ for any entry $\beta_i$ of $\beta$. [r] Since, by assumption $|\beta| \geq m$, this means $T^\alpha = T_j^{n-|\beta|}(T_j')^\beta = 0$, by Lemma 7.2. □

We are now able to answer our initial question for the case $m \in \mathbb{N}$ and $\mu = 1$. That is, we can determine the intersection class of $(m,p)$- and $(1,\infty)$-isometric tuples on a given space $X$.

**Theorem 7.1.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m,p)$-isometric tuple and also a $(1,\infty)$-isometric tuple. Then one operator $T_{j_0}$ is an isometry and all other operators satisfy $(T_{j_0}')^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = m$, and are, in particular, nilpotent of order $\leq m$.*

**Proof.** Again, we can assume that $m \geq 1$.

Since $T$ is an $(m,p)$-isometric tuple, by Proposition 3.2.(i),

$$Q^n(x) = \sum_{k=0}^{m-1} n^{(k)} \left(\frac{1}{k!} P_k(x)\right), \quad \forall x \in X, \ \forall n \in \mathbb{N}. \tag{7.1}$$

[q] Of course, we actually have $|\beta| \leq \min\{n, m-1\}$. The main point is, however, that $|\beta| \leq m-1$.
[r] Note that we must have $d-1 \geq 2$ in this case so that the expression $|\beta_i'|$ makes sense.

Where $n^{(k)} = \binom{n}{k}k! = n(n-1)...(n-k+1)$. That is, for all $x \in X$, the sequence $(Q^n(x))_{n\in\mathbb{N}}$ is interpolated by a polynomial of degree of less or equal to $m-1$.

Now, by Corollary 7.1 above, for $n \geq 2m-1$, $n \in \mathbb{N}$, $Q^n(x)$ reduces to

$$Q^n(x) = \sum_{\substack{\beta\in\mathbb{N}^{d-1} \\ |\beta|=0,...,m-1}} \sum_{j=1}^{d} \frac{n!}{(n-|\beta|)!\beta!} \|T_j^{n-|\beta|}(T_j')^\beta x\|^p, \quad \forall x \in X,$$

where $\frac{n!}{(n-|\beta|)!\beta!} = \frac{n^{(|\beta|)}}{\beta!}$. (We set $n \geq 2m-1$, so that we don't get any multi-indices twice in this expression.)

Further, for each $n \in \mathbb{N}$, $k \in \{0,...,m-1\}$, $\beta \in \mathbb{N}^{d-1}$, $j \in \{1,...,d\}$ and all $x \in X$, by Proposition 5.5, there exists an $\ell_j \in \{1,...,d\}$, such that

$$\|T_{\ell_j}^\nu \left(T_j^{n-k}(T_j')^\beta x\right)\| = \|T_j^{n-k}(T_j')^\beta x\|, \quad \forall \; \nu \in \mathbb{N}.$$

By Corollary 7.1, for $n \geq 2m-1$, $n \in \mathbb{N}$, we must have $\ell_j = j$, i.e.

$$\|T_j^{\nu+n-k}(T_j')^\beta x\| = \|T_j^{n-k}(T_j')^\beta x\|, \quad \forall \; n, \nu \in \mathbb{N}, \; n \geq 2m-1.$$

But that means that, for all $k \in \{0,...,m-1\}$, $\beta \in \mathbb{N}^{d-1}$, $j \in \{1,...,d\}$ and all $x \in X$, the sequences $\left(\|T_j^{n-k}(T_j')^\beta x\|^p\right)_{n\in\mathbb{N}}$ becomes constant for $n \geq 2m-1$.

Therefore, for all $x \in X$, for $n \geq 2m-1$, the sequence $(Q^n(x))_{n\in\mathbb{N}}$ is interpolated by the polynomial

$$n \mapsto \sum_{\substack{\beta\in\mathbb{N}^{d-1} \\ |\beta|=0,...,m-1}} \sum_{j=1}^{d} \frac{n^{(|\beta|)}}{\beta!} \|T_j^{2m-1-|\beta|}(T_j')^\beta x\|^p,$$

which is of degree less or equal to $m-1$.

However, this polynomial must be the same as the one in (7.1). In particular, their coefficients have to be equal and, more particularly, equating constants, we must have

$$\sum_{j=1}^{d} \|T_j^{2m-1}x\|^p = \|x\|^p, \quad \forall x \in X.$$

Take now $j_0 \in \{1,...,d\}$ and $x_{j_0} \in X_{j_0}$, for $X_{j_0}$ defined as in Remark 5.1. Then $\sum_{\substack{j=1 \\ j\neq j_0}}^{d} \|T_j^{2m-1}x_{j_0}\|^p = 0$ and, thus, $x_{j_0} \in \ker T_j^{2m-1}$ for all $j \in \{1,...,d\}$ with $j \neq j_0$.

Since $x_{j_0} \in X_{j_0}$ and $j_0 \in \{1,...,d\}$ were chosen arbitrarily, $X_j \in \ker T_i^{2m-1}$ for all $i \neq j$. Then it follows from Lemma 7.1 that one of the operators $T_1,...,T_d$ is an isometry.

Let $T_{j_0}$ be isometric. Then we have $(T_{j_0}')^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = m$ by Proposition 4.3. □

The case $m = 1$ and $\mu \in \mathbb{N}$, $\mu \geq 1$, now follows easily.[s]

**Corollary 7.2.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be a $(1,p)$-isometric tuple and also a $(\mu,\infty)$-isometric tuple. Then one operator $T_{j_0}$ is an isometry and $T = (0, ..., 0, T_{j_0}, 0, ..., 0)$.*

**Proof.** Since $T$ is a $(1,p)$-isometric tuple, Proposition 3.2.(i) gives, $\sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p = \|x\|^p$, for all $n \in \mathbb{N}$ and all $x \in X$. Consequently, $\|x\| \geq \|T^\alpha x\|$, for any multi-index $\alpha \in \mathbb{N}^d$, for all $x \in X$. I.e.

$$\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \|x\|, \ \forall x \in X.$$

Then, since $T$ is a $(\mu,\infty)$-isometric tuple, $T$ is already a $(1,\infty)$-isometric tuple, by Theorem 5.1. The result now follows from the preceding statement and Proposition 4.3. □

We can now prove the case where our tuple is constructed, using an $(m,p)$-isometric operator (i.e., by applying Proposition 4.2).

**Proposition 7.2.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an $(m,p)$-isometric tuple of the form of Proposition 4.2. That is, $T_j = z_j S$, where $S \in B(X)$ is an $(m,p)$-isometric operator and $z := (z_1, ..., z_d) \in \mathbb{K}^d$ with $\|z\|_p = 1$. Assume further that $T$ is additionally a $(\mu,\infty)$-isometric tuple. Then $T = (0, ..., 0, z_{j_0} S, 0, ..., 0)$ with $|z_{j_0}| = 1$ and $S$ being an isometry. In particular, $T_{j_0} = z_{j_0} S$ is an isometry.*

**Proof.** Since $T$ is an $(\mu,\infty)$-isometry, for all $x \in X$, the family $(\|T^\alpha x\|)_{\alpha \in \mathbb{N}^d} = \left(|z^\alpha| \|S^{|\alpha|} x\|\right)_{\alpha \in \mathbb{N}^d}$ attains its maximum. Since $\|z\|_p = 1$ by assumption, we have $\max_{|\alpha|=n} |z^\alpha| = 1$ for all $n \in \mathbb{N}$. This forces $(\|S^n x\|)_{n \in \mathbb{N}}$ to be bounded for every $x \in X$. Then, since $S$ is an $(m,p)$-isometric operator, $S$ is an isometry by Proposition 2.1 in [16]. Hence, by Proposition 4.2, $T$ is a $(1,p)$-isometric tuple. Then Corollary 7.2 forces one operator $z_{j_0} S$ to be an isometry.

Now Proposition 4.3 gives $(z_i S)^m = 0$ for all $i \in \{1, ..., d\}$ with $i \neq j_0$. Since $S$ is an isometry, this is only possible if $z_i = 0$ for all $i \in \{1, ..., d\}$ with $i \neq j_0$. Thus, $T = (0, ..., 0, z_{j_0} S, 0, ..., 0)$ with $|z_{j_0}| = 1$. □

More general results appear difficult to obtain at the moment. We present in the remaining parts some partial results in the case $m = \mu = 2$.

The next lemma simply states that one cannot increase a maximum.

**Lemma 7.3.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be a $(\mu,\infty)$-isometric tuple. For each $x \in X$ and each $\tilde{\alpha}(x) \in \mathbb{N}^d$ with $\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \|T^{\tilde{\alpha}(x)} x\|$, we have $\|T^{\tilde{\alpha}(x)} x\| = |T^{\tilde{\alpha}(x)} x|_\infty$.*

[s]It is actually easy to find an elementary proof for Corollay 7.2. However, it is more elegant to deduce the statement from Proposition 7.1.

**Proof.** Fix $x \in X$ and let $\tilde{\alpha}(x) \in \mathbb{N}^d$, such that $\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \|T^{\tilde{\alpha}(x)}x\|$. We have

$$\|T^{\tilde{\alpha}(x)}x\| \le |T^{\tilde{\alpha}(x)}x|_\infty = \max_{\alpha\in\mathbb{N}^d} \|T^\alpha T^{\tilde{\alpha}(x)}x\| \le \max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \|T^{\tilde{\alpha}(x)}x\|. \qquad \square$$

**Proposition 7.3.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be a $(2, p)$-isometric tuple and a $(2, \infty)$-isometric tuple.*

*(i) The sequences $\big(\|T_j^n x\|\big)_{\substack{n\in\mathbb{N}\\ n\ge 2}}$ are constant for all $x \in X$, for all $j \in \{1, ..., d\}$.*

*(ii) For all $n \ge 2$, $n \in \mathbb{N}$,*

$$\Big\|\sum_{j=1}^d T_j^n x\Big\|^p = \sum_{j=1}^d \|T_j^n x\|^p = \|x\|^p, \quad \forall x \in X.$$

*In particular, the tuple $T^2 := (T_1^2, ..., T_d^2)$ is an $\ell_p$-spherical isometry and the operator $\sum_{j=1}^d T_j^2$ is an isometry.*

*(iii) We have $T_{j_0}^2 = 0$ for some $j_0 \in \{1, ..., d\}$.*

**Proof.** We proof (i) and (ii) together.

(i) + (ii): Since $T$ is a $(2, p)$-isometric tuple, by Proposition 3.2.(i),

$$Q^n(x) = nP_1(x) + \|x\|^p, \quad \forall x \in X, \ \forall n \in \mathbb{N}. \tag{7.2}$$

That is, for all $x \in X$, the sequence $(Q_n(x))_{n\in\mathbb{N}}$ is interpolated by a polynomial of degree less or equal to 1.

Now, by Corollary 7.1, for $n \ge 2$, $n \in \mathbb{N}$, $Q^n(x)$ reduces to

$$Q^n(x) = n\left(\sum_{i=1}^d \sum_{\substack{j=1\\ j\ne i}}^d \|T_i^{n-1}T_j x\|^p\right) + \sum_{j=1}^d \|T_j^n x\|^p, \quad \forall x \in X. \tag{7.3}$$

Since $T$ is a $(2, \infty)$-isometry, for all $x \in X$, for all $i, j \in \{1, ..., d\}$, $\max_{\alpha\in\mathbb{N}^d} \|T^\alpha T_i^2 T_j x\| = \max_{|\alpha|=1} \|T^\alpha T_i^2 T_j x\|$, by Corollary 5.1. If $i \ne j$, by Corollary 7.1, we deduce that $\max_{|\alpha|=1} \|T^\alpha T_i^2 T_j x\| = \|T_i^3 T_j x\|$, for all $x \in X$.

However, then Lemma 7.3 gives that $\|T_i^3 T_j x\| = |T_i^3 T_j x|_\infty$, for all $x \in X$. But then, for all $x \in X$,

$$\|T_i^3 T_j x\| = |T_i^3 T_j x|_\infty = \max_{|\alpha|=1} \|T^\alpha T_i^3 T_j x\| = \|T_i^4 T_j x\| = |T_i^4 T_j x|_\infty,$$

for all $i \ne j$.

By repeating this process ad infinitum, we have for all $i \ne j$,

$$\|T_i^3 T_j x\| = \|T_i^4 T_j x\| = |T_i^4 T_j x|_\infty = \|T_i^5 T_j x\| = |T_i^5 T_j x|_\infty = ... = \|T_i^n T_j x\|,$$

for all $n \ge 3$, $n \in \mathbb{N}$, for all $x \in X$.

Therefore, the sequences $\big(\|T_i^{n-1} T_j x\|\big)_{\substack{n\in\mathbb{N}\\ n\ge 4}}$ are constant, for all $i \ne j$, for all $x \in X$.

By equating (7.2) and (7.3), we get for all $n \geq 2, n \in \mathbb{N}$

$$\sum_{j=1}^{d} \|T_j^n x\|^p = n \left( P_1(x) - \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \|T_i^{n-1} T_j x\|^p \right) + \|x\|^p, \quad \forall x \in X.$$

The left hand side is non-negative and bounded, for all $x \in X$, by Corollary 5.1, thus, so has to be the right hand side. Since $\sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \|T_i^{n-1} T_j x\|^p = \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \|T_i^3 T_j x\|^p$ is constant for $n \geq 4$, this forces

$$P_1(x) = \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \|T_i^3 T_j x\|^p, \quad \forall x \in X. \tag{7.4}$$

Therefore,

$$\sum_{j=1}^{d} \|T_j^n x\|^p = \|x\|^p, \quad \forall n \geq 4, n \in \mathbb{N}, \forall x \in X. \tag{7.5}$$

Since $T_i^2 T_j^2 = 0$ for all $i \neq j$ by Lemma 7.2, replacing $x$ by $T_{j_0}^{\nu} x$ for $\nu \in \mathbb{N}$ with $\nu \geq 2$ in this last equation gives $\|T_{j_0}^{\nu} x\| = \|T_{j_0}^{n+\nu} x\|$ for all $n \geq 4$, $n \in \mathbb{N}$, for all $x \in X$.

Hence, the sequences $\big(\|T_j^k x\|\big)_{\substack{n \in \mathbb{N} \\ n \geq 2}}$ are constant for all $j \in \{1, ..., d\}$, for all $x \in X$. This is (i).

Combining now (i) and (7.5), gives that we actually have

$$\sum_{j=1}^{d} \|T_j^n x\|^p = \|x\|^p, \quad \forall n \geq 2, n \in \mathbb{N}, \forall x \in X,$$

which is one of the two equations we had to show for (ii). Now replace $x$ by $\sum_{i=1}^{d} T_i^{\nu} x$ for $\nu \in \mathbb{N}$ with $\nu \geq 2$ in this last equation. Then, again, since $T_i^2 T_j^2 = 0$ for all $i \neq j$, we get that for all $n, \nu \geq 2$, $n, \nu \in \mathbb{N}$,

$$\Big\| \sum_{j=1}^{d} T_j^{\nu} x \Big\|^p = \sum_{j=1}^{d} \|T_j^{n+\nu} x\|^p = \sum_{j=1}^{d} \|T_j^n x\|^p, \quad \forall x \in X.$$

This is the second equation we had to show.

(iii): The equation in (ii) implies that $\|x\| \geq \max_{j=1,\dots,d} \|T_j^3 x\|$, for all $x \in X$.

(iii.a) If $\|x\| = \|T_j^3 x\|$, for some $j \in \{1, ..., d\}$ then obviously $\|T_i^3 x\| = 0$ and $x \in N(T_i^3)$ for all $i \neq j$, again by (ii).

(iii.b) Assume $\|x\| > \max_{j=1,\dots,d} \|T_j^3 x\|$. (Note that this implies $d > 1$ and that we have more than one non-zero operator.)

Since $|x|_\infty \geq \|x\|$ and $|x|_\infty = \max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \max_{|\alpha|=3} \|T^\alpha x\|$, it follows that

$$|x|_\infty = \max_{\substack{i=1,\dots,d\\ j=1,\dots,d\\ j\neq i}} \|T_i^2 T_j x\|.$$

(Since $T_iT_jT_k = 0$ for distinct $i, j, k$ by Lemma 7.2.)
Let $|x|_\infty = \|T_{i_0}^2 T_{j_0} x\|$ for some $i_0 \neq j_0$, $i_0, j_0 \in \{1, ..., d\}$. Then

$$\|T_{j_0}x\|^p = \sum_{\substack{j=1,\dots,d\\ j\neq i_0,j_0}} \|T_j^2 T_{j_0} x\|^p + \|T_{i_0}^2 T_{j_0} x\|^p + \|T_{j_0}^3 x\|^p$$

$$\Leftrightarrow \ \|T_{j_0}x\|^p = \sum_{\substack{j=1,\dots,d\\ j\neq i_0,j_0}} \|T_j^2 T_{j_0} x\|^p + |x|_\infty^p + \|T_{j_0}^3 x\|^p.$$

Since $|x|_\infty \geq \|T^\alpha x\|$ for all $\alpha \in \mathbb{N}^d$ simply by definition, we certainly have $x \in N(T_{j_0}^3)$.

We conclude that in any case, $x \in N(T_j^3)$ for some $j \in \{1, ..., d\}$.

In other words, $X = \bigcup_{j=1,\dots,d} N(T_j^3)$. Hence, $N(T_{j_0}^3) = X$, i.e. $T_{j_0}^3 = 0$ for one $j_0 \in \{1, ..., d\}$. This gives $T_{j_0}^2 = 0$ by (i). □

**Corollary 7.3.** *Let $T = (T_1, ..., T_d) \in B(X)^d$ be a $(2, p)$-isometric tuple and a $(2, \infty)$-isometric tuple. Assume further that one of the following holds:*

*(i) One of the operators $T_1, ..., T_d$ is injective.*
*(ii) One the operators $T_1, ..., T_d$ is surjective.*
*(iii) $T$ does not contain a non-zero nilpotent operator.*
*(iv) We have $d = 2$.*
*(v) We have $p = 1$ and $X$ is stricly convex.*
*(vi) We have $0 < p < 1$.*

*Then one operator $T_{j_0}$ is an isometry and all other operators satisfy $(T'_{j_0})^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = 2$, and are, in particular, nilpotent of order $\leq 2$. In case (iii), $T$ consists actually of one isometry and zeros. In case (ii), $T_{j_0}$ is actually an isometric isomorphism.*

**Proof.** (i): Let $T_{j_0}$ be injective. Then, by Lemma 7.2, all the other operators $T_j$ for $j \neq j_0$ are nilpotent of degree $\leq 2$. In particular, by Proposition 7.3.(ii), $T_{j_0}^2$ is an isometry. Without loss of generality, assume $T_1^2$ is an isometry and $T_j^2 = 0$, for all $j \in \{2, ..., d\}$. Then, by (7.4) and since $\big(\|T_j^n x\|\big)_{\substack{n\in\mathbb{N}\\ n\geq 2}}$ are constant,

$$P_1(x) = \sum_{i=1}^{d} \sum_{\substack{j=1\\ j\neq i}}^{d} \|T_i^2 T_j x\|^p = \sum_{j=2}^{d} \|T_j x\|^p.$$

Now, by definition, $P_1(x) = -\|x\|^p + \sum_{j=1}^d \|T_j x\|^p$, for all $x \in X$. Therefore,

$$-\|x\|^p + \sum_{j=1}^{d} \|T_j x\|^p = \sum_{j=2}^{d} \|T_j x\|^p \quad \Leftrightarrow \quad -\|x\|^p + \|T_1 x\|^p = 0, \quad \forall x \in X,$$

and $T_1$ is an isometry. Then we have $(T_1')^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = 2$ by Proposition 4.3

(ii): Let $T_{j_0}$ be surjective. Then $T_{j_0}^2$ is surjective and, since $\|T_{j_0}^3 x\| = \|T_{j_0}^2 x\|$ for all $x \in X$, by Proposition 7.3.(i), the operator $T_{j_0}$ is actually an isometry. Then $(T_{j_0}')^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = 2$ by Proposition 4.3

(iii): Assume $T = (T_1, ..., T_d)$ does not contain a non-zero nilpotent operator. Then, by Proposition 7.3.(iii), $T$ must contain an operator which is the zero-operator, say $T_d$. But then we can reduce $T$ to $(T_1, ..., T_{d-1})$ and repeat the argument until $T = (T_{j_0})$ for some isometric operator $T_{j_0}$.

(iv): Let $T = (T_1, T_2)$. By Propposition 7.3.(iii), one operator $T_2$, say, is nilpotent of degree $\leq 2$. Then, 7.3.(ii) forces $T_1^2$ to be an isometry. But then $T_1$ is injective and the result follows from (i).

(v): If $p = 1$, by Proposition 7.3.(ii) we have that, for each $x \in X$, the triangle inequality becomes an equality for the vectors $T_j^2 x$. If $X$ is strictly convex, this implies that there exist $\lambda_{i,j,x} \in \mathbb{R}$ with $T_i^2 x = \lambda_{i,j,x} T_j^2 x$ for all $i, j$, for all $x \in X$. But then $T_j^2 T_i^2 = 0$ for all $i \neq j$ implies $\lambda_{i,j,x} = 0$ or $T_i^2 x = T_j^2 x = 0$. Therefore, there exists a $j_0$ with $T_j^2 = 0$ for all $j \neq j_0$ and, by Proposition 7.3.(ii), $T_{j_0}^2$ is an isometry. Thus, $T_{j_0}$ is injective and the statement follows from (i).

(vi):Assume $0 < p < 1$. Then, by Proposition 7.3.(ii),

$$\|x\| = \left( \sum_{j=1}^{d} \|T_j^2 x\|^p \right)^{1/p} \geq \sum_{j=1}^{d} \|T_j^2 x\| \geq \| \sum_{j=1}^{d} T_j^2 x\| = \|x\|, \quad \forall x \in X.$$

That is, $\|x\| = \sum_{j=1}^d \|T_j^2 x\| = \left( \sum_{j=1}^d \|T_j^2 x\|^p \right)^{1/p}$ for all $x \in X$. (In particular, $(T_1^2, ..., T_d^2)$ is a $(1,p)$-isometry and a $(1,1)$-isometry.) Hence, for all $x \in X$, the vectors $(\|T_1^2 x\|, ..., \|T_d^2 x\|)$ lie on the same circle with radius $\|x\|$ in $\mathbb{R}^d$ with respect to both $\|.\|_1$ and $\|.\|_p$. This means that, for all $x \in X$, $(\|T_1^2 x\|, ..., \|T_d^2 x\|)$ is of the form $(\|x\|, 0, ..., 0)$ with respect to some permutation (which, at this stage of the argument, may depend on $x$). Hence, each $x \in X$ is in the kernel of $d-1$ operators $T_1^2, ..., T_d^2$ while the remaining operator, say $T_{j_0}^2$, acts isometrically on $x$. But then $\max_{j=1,...,d} \|T_j^2 x\| = \|x\|$ for all $x \in X$ and $(T_1^2, ..., T_d^2)$ is a $(1,\infty)$-isometry. By Proposition 7.2, $T_{j_0}^2$ is an isometry and therefore $T_{j_0}$ is injective. □

These results lead to the following question:

**Question.** Is every tuple of commuting bounded linear operators $T = (T_1, ..., T_d) \in B(X)^d$ which is simultaneously an $(m,p)$-isometric and an $(\mu,\infty)$-isometric tuple actually of the form of Proposition 4.3?

That is, is one operator $T_{j_0}$ an isometry and all other operators satisfy $(T'_{j_0})^\beta = 0$ for all $\beta \in \mathbb{N}^{d-1}$ with $|\beta| = m$, and are, in particular, nilpotent of order $\leq m$? Moreover, is an $(m,p)$-isometric tuple an $(m,\infty)$-isometric tuple if, and only if, it is an $(m,q)$-isometric tuple for *all* $q \in (0,\infty)$?

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## References

1. M. Aigner, Diskrete Mathematik, Vieweg, Braunschweig/Wiesbaden 1993.
2. J. Agler, A disconjugacy theorem for Toeplitz operators, *Am. J. Math.* 112(1) (1990), 1-14.
3. J. Agler and M. Stankus, $m$-isometric transformations of Hilbert space, I, *Integr. equ. oper. theory* Vol. 21, No. 4 (1995), 383-429.
4. J. Agler and M. Stankus, $m$-isometric transformations of Hilbert space, II, *Integr. equ. oper. theory* Vol. 23, No. 1 (1995), 1-48.
5. J. Agler and M. Stankus, $m$-isometric transformations of Hilbert space, III, *Integr. equ. oper. theory* Vol. 24, No. 4 (1996), 379-421.
6. F. Bayart, $m$-Isometries on Banach Spaces, *Mathematische Nachrichten*, Volume 284, No. 17-18 (2011), 2141-2147.
7. M. A. Berger and Y. Wang, Bounded semigroups of matrices, *Linear Algebra Appl.* 166 (1992), 21-27.
8. T. Bermúdez, I. Marrero and A. Martinón, On the Orbit of an $m$-Isometry, *Integr. equ. oper. theory* Vol. 64, No. 4 (2009), 487-494.
9. T. Bermúdez, A. Martinón and E. Negrín, Weighted Shift Operators Which are m-Isometries, *Integr. equ. oper. theory* Vol. 68, No. 3 (2010), 301-312.
10. T. Bermúdez, A. Martinón and V. Müller, $(m,q)$-isometries on metric spaces, preprint, *http://www.math.cas.cz/fichier/preprints/IM_20130218145519_37.pdf.*
11. F. Botelho, On the existence of $n$-isometries on $\ell_p$ spaces, *Acta Sci. Math. (Szeged)* 76 : 1-2 (2010), 183-192.
12. M. Chō and W. Żelazko, On geometric spectral radius of commuting $n$-tuples of operators, *Hokkaido Math. J.* 21 (2) (1992), 251-258.
13. M. Faghih Ahmadi and K. Hedayatian, Hypercyclicity and Supercyclicity of $m$-Isometric Operators, *Rocky Mountain J. Math.* Volume 42, Number 1 (2012), 15-23.
14. J. Gleason and S. Richter, $m$-Isometric Commuting Tuples of Operators on a Hilbert Space, *Integr. equ. oper. theory* Vol. 56, No. 2 (2006), 181-196 .
15. R.E. Harte, Spectral mapping theorems, *Poc. Roy. Irish Acad. Sect.* A 72 (1972), 89-107.
16. P. Hoffmann, M. Mackey and M. Ó Searcóid, On the second parameter of an $(m,p)$-isometry, *Integr. equ. oper. theory* Vol. 71, No. 3 (2011), 389-405.

17. C.-K. Li and W. So, Isometries of $\ell_p$-norm, *Amer. Math. Monthly* 101 (1994), 452-453.
18. V. Müller, On the joint spectral radius, *Annales Polonici Mathematici* 66 (1997), 173-182.
19. V. Müller and A. Soltysiak, Spectral radius formula for commuting Hilbert space operators, *Studia Math.* 103 (3) (1992), 329-333.
20. S. Richter, Invariant subspaces of the Dirichlet shift, *J. reine angew. Math.* 386 (1988), 205-220.
21. S. Richter, A model for two-isometric operator tuples with finite defect, talk at the AMS Sectional meeting, Cornell University, September 2011, *http: //www.math.utk.edu/∼richter/talk/Cornell_Sept_2011*.
22. G.-C. Rota and W. G. Strang, A note on the joint spectral radius, *Indag. Math.* 22 (1960), 379-381.
23. O.A. Sid Ahmed, $m$-isometric Operators on Banach Spaces, *Asian-Eur. J. Math.* 3, No. 1 (2010), 1-19.
24. Z. Słodkowski and W. Żelazko, On joint spectra of commuting families of operators, *Studia Math.* 50 (1974), 127-148.
25. A. Soltysiak, On the joint spectral radii of commuting Banach algebra elements, *Studia Math.* 105 (1) (1993), 93-99.
26. J.L. Taylor, A Joint Spectrum for Several Commuting Operators, *J. Functional Analysis* 6 (1970), 172-191.
27. W. Żelazko, On a problem concerning joint approximate point spectra, *Studia Math.* 45 (1973), 239-240.