



<b>Title</b>	Convergence of rational Bernstein operators
<b>Authors(s)</b>	Render, Hermann
<b>Publication date</b>	2014-04-01
<b>Publication information</b>	Render, Hermann. "Convergence of Rational Bernstein Operators." Elsevier, April 1, 2014. <a href="https://doi.org/10.1016/j.amc.2014.01.152">https://doi.org/10.1016/j.amc.2014.01.152</a> .
<b>Publisher</b>	Elsevier
<b>Item record/more information</b>	<a href="http://hdl.handle.net/10197/5478">http://hdl.handle.net/10197/5478</a>
<b>Publisher's statement</b>	This is the author.s version of a work that was accepted for publication in Applied Mathematics and Computation. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in H. Render : Convergence of rational Bernstein operators. Applied Mathematics and Computation, 232 2014-04, pp.1076-1089. DOI:10.1016/j.amc.2014.01.152
<b>Publisher's version (DOI)</b>	10.1016/j.amc.2014.01.152

Downloaded 2026-05-02 01:17:28

The UCD community has made this article openly available. Please share how this access benefits you. Your story matters! (@ucd\_oa)



© Some rights reserved. For more information

# CONVERGENCE OF RATIONAL BERNSTEIN OPERATORS

HERMANN RENDER

ABSTRACT. In this paper we discuss convergence properties and error estimates of rational Bernstein operators introduced by P. Pişul and P. Sablonnière. It is shown that the rational Bernstein operators converge to the identity operator if and only if the maximal difference between two consecutive nodes is converging to zero. Further a Voronovskaja theorem is given based on the explicit computation of higher order moments for the rational Bernstein operator.

## 1. INTRODUCTION

A rational Bézier curve  $r_n$  of degree  $n$  is given by the control points  $P_k \in \mathbb{R}^d$  and positive weights  $w_k$  in the form

$$r_n(x) = \frac{\sum_{k=0}^n P_k \cdot w_k \binom{n}{k} x^k (1-x)^{n-k}}{\sum_{k=0}^n w_k \binom{n}{k} x^k (1-x)^{n-k}}$$

for  $x \in [0, 1]$ . Here the weights  $w_k > 0$  for  $k = 0, \dots, n$  can be chosen arbitrarily, and it should be emphasized that the polynomial  $\sum_{k=0}^n w_k \binom{n}{k} x^k (1-x)^{n-k}$  has degree at most  $n$ , and not necessarily exact degree  $n$ . It was the primary goal of several authors to give bounds on the magnitude of the derivative, see e.g. [13], [12], [28]. The rational Bernstein operator is analogously defined as

$$r_n(f)(x) = \frac{\sum_{k=0}^n f\left(\frac{k}{n}\right) \cdot w_k \binom{n}{k} x^k (1-x)^{n-k}}{\sum_{k=0}^n w_k \binom{n}{k} x^k (1-x)^{n-k}}.$$

for any  $f \in C[0, 1]$ , the set of all continuous real-valued functions on the interval  $[0, 1]$ . As pointed out in [27, p. 40], the main drawback of this definition is that  $r_n$  preserves

---

2000 Mathematics Subject Classification: *Primary:* 41A20, *Secondary:* 41A36.

Key words and phrases: Rational approximants, Bernstein operator, positive operator.

only the constant function but in general not the identity function  $f(x) = x$ . For special weights  $w_k$ , convergence of  $r_n(f)$  to  $f$  for  $f \in C[0, 1]$  can be deduced from results about the classical Bernstein operator  $B_n$  defined on  $C[0, 1]$  by

$$B_n(f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Indeed, if we assume that  $w_k = \varphi\left(\frac{k}{n}\right)$  for  $k = 0, \dots, n$  for a *positive* function  $\varphi \in C[0, 1]$ , then clearly

$$r_n(f)(x) = \frac{B_n(f \cdot \varphi)(x)}{B_n(\varphi)(x)}.$$

As  $r_n(f) - f = (B_n(f \cdot \varphi) - B_n(\varphi) \cdot f) / B_n(\varphi)$  one obtains convergence from the simple inequality

$$|r_n(f) - f| \leq \frac{1}{B_n(\varphi)} (|B_n(f \cdot \varphi) - \varphi \cdot f| + |\varphi - B_n(\varphi)| \cdot |f|).$$

Let us mention that G. Tachev has discussed in [29] estimates of  $|r_n(f) - f|$  in terms of the second modulus of continuity. It seems that no results are available or arbitrary weights  $w_k$ .

Recently P. Pițul and P. Sablonnière introduced in [27] a special type of *rational Bernstein operator* which is a positive operator of the form

$$(1) \quad R_n f(x) := \sum_{k=0}^n f(x_{n,k}) \bar{w}_{n,k} \binom{n}{k} \frac{x^k (1-x)^{n-k}}{Q_{n-1}(x)}$$

which indeed preserves the constant function and the identity function when the polynomial  $Q_{n-1}$  and the weights  $\bar{w}_{n,k}$  for  $k = 0, \dots, n$  are chosen in a proper way. More precisely, it is assumed at first that the polynomial  $Q_{n-1}(x)$  has degree at most  $n-1$ , so it can be written in the form

$$(2) \quad Q_{n-1}(x) = \sum_{k=0}^{n-1} w_{n-1,k} \binom{n-1}{k} x^k (1-x)^{n-1-k}.$$

Further it is assumed that  $Q_{n-1}$  has two additional properties: (i) the Bernstein coefficients  $w_{n-1,k}$  in the representation are strictly positive and (ii) the sequence  $w_{n-1,k}$ ,  $k = 0, \dots, n-1$  satisfies the inequality

$$(W) \quad \frac{w_{n-1,k-1} w_{n-1,k+1}}{w_{n-1,k}^2} < \left(\frac{k+1}{k}\right) \left(\frac{n-k}{n-k-1}\right)$$

for  $k = 0, \dots, n-1$ . Then, according to results [27], there exist positive weights  $\bar{w}_{n,k}$ ,  $k = 0, \dots, n$ , and increasing nodes  $0 = x_{n,0} < x_{n,1} < \dots < x_{n,n} = 1$  such that  $R_n$  reproduces the constant function  $e_0(x) = 1$  and the linear function  $e_1(x) = x$ , i.e. that

$$(3) \quad R_n e_j = e_j \text{ for } j = 0, 1.$$

The weights  $\bar{w}_{n,k}$  and the nodes are uniquely defined through the condition (3) and they are given by the formula

$$x_{n,k} = \frac{k w_{n,k-1}}{k w_{n,k-1} + (n-k) w_{n,k}} \text{ for } k = 1, \dots, n-1$$

$$\bar{w}_{n,k} = \frac{k}{n} w_{n,k-1} + \left(1 - \frac{k}{n}\right) w_{n,k} \text{ for } k = 1, \dots, n-1$$

and the conditions  $x_{n,0} = 0$  and  $x_{n,n} = 1$  and  $w_{n,0} = Q_{n-1}(0)$  and  $w_{n,n} = Q_{n-1}(1)$ . Using that  $x^k(1-x)^{n-1-k} = x^k(1-x)^{n-k} + x^{k+1}(1-x)^{n-1-k}$  it easily follows that

$$R_n f(x) = \frac{\sum_{k=0}^n f(x_{n,k}) \bar{w}_{n,k} \binom{n}{k} x^k (1-x)^{n-k}}{\sum_{k=0}^n \bar{w}_{n,k} \binom{n}{k} x^k (1-x)^{n-k}},$$

so  $R_n$  is indeed a rational Bernstein operator as defined in the beginning of the introduction.

It was shown in [27] that the rational Bernstein operators  $R_n$  have the same shape preserving properties as the classical Bernstein operator  $B_n$ . Moreover it was proved that  $R_n$  converges to the identity operator and that a Voronovskaja-type theorem holds under the *additional assumption* that

(\*) there exists a positive continuous function  $\varphi$  such that

$$w_{n-1,k} = \varphi\left(\frac{k}{n-1}\right) \binom{n-1}{k} \text{ for } k = 0, \dots, n-1$$

for all natural numbers  $n$ .

The main purpose of the article is to study the convergence of the rational Bernstein operators of Pițul and Sablonnière in the general case, i.e. without assumption (\*). Our main result states that the operators  $R_n$  converge to the identity operator if and only if

$$(4) \quad \Delta_n = \sup_{k=0, \dots, n-1} |x_{n,k+1} - x_{n,k}|$$

converges to 0. The main innovation is the computation and estimation of the moments

$$R_n(e_1 - x)^r(x) \text{ and } R_n(e_r)(x) - x^r$$

for the rational Bernstein operator  $R_n$  where  $e_r(x) = x^r$ . For example, we shall prove the inequality

$$|R_n(e_2)(x) - x^2| \leq \sup_{0 \leq k \leq n-1} |x_{n,k+1} - x_{n,k}| \cdot x(1-x)$$

which implies the convergence of  $R_n$  provided that  $\Delta_n \rightarrow 0$ . Using convergence results of O. Shisha and B. Mond for positive operators we obtain explicit error estimates and using results of R.G. Mamedov we obtain a Voronovskaja theorem for rational Bernstein operators in the general setting. We shall illustrate the results by examples which are of different type as those in [27].

The paper is organized as follows: in the second section we shall recall briefly the basic construction of the rational Bernstein operators as given in [27]. We shall show that there is a variety of examples of rational Bernstein operators: starting with nodes  $0 = x_{n,0} < x_{n,1} < \dots < x_{n,n-1} < x_{n,n} = 1$  and a constant  $\gamma_{n-1,0} > 0$  let us define

$$\gamma_{n-1,k} := \gamma_{n-1,0} \prod_{l=1}^k \frac{1 - x_{n,l}}{x_{n,l}}$$

for  $k = 1, \dots, n$  and  $Q_{n-1}(x) := \sum_{k=0}^{n-1} \gamma_{n-1,k} x^k (1-x)^{n-1-k}$ . Then  $Q_{n-1}$  satisfy property (W) and

$$R_n f(x) = \sum_{k=0}^n f(x_{n,k}) (\gamma_{n-1,k} + \gamma_{n-1,k-1}) \frac{x^k (1-x)^{n-k}}{Q_{n-1}(x)}$$

is a rational Bernstein operator  $R_n$  fixing  $e_0$  and  $e_1$ . In Section 3 we compute the expressions  $R_n(e_r)(x) - x^r$  explicitly and obtain the above-mentioned criterion for the convergence of  $R_n$ . The short Section 4 is devoted to error estimates. In Section 5 we prove a Voronovskaja result. In Section 6 we discuss the special case of rational Bernstein operators of [27] and improve some results. Further we present an example of a sequence rational Bernstein operators converging to the identity operator where the polynomials  $Q_n(x)$  converges pointwise to a *discontinuous* function. In Section 7, we shall comment on links between rational Bernstein operators and general results about Bernstein operators fixing two functions in the framework of extended Chebyshev systems.

By  $C^r[0, 1]$  we shall denote the set of all  $r$  times continuously differentiable functions on the unit interval  $[0, 1]$ .

## 2. RATIONAL BERNSTEIN OPERATORS

For convenience of the reader we recall the basic construction of the rational Bernstein operator  $R_n$  as outlined in [27]. Let  $Q_{n-1}$  be a polynomial of degree  $\leq n-1$ . Instead of the representation (2) it is more convenient to work with

$$Q_{n-1}(x) = \sum_{k=0}^{n-1} \gamma_{n-1,k} x^k (1-x)^{n-1-k}$$

so  $\gamma_{n-1,k} = w_{n-1,k} \binom{n-1}{k}$ . Since  $x^k (1-x)^{n-1-k} = x^k (1-x)^{n-k} + x^{k+1} (1-x)^{n-1-k}$  we infer that

$$Q_{n-1}(x) = \sum_{k=0}^n (\gamma_{n-1,k} + \gamma_{n-1,k-1}) x^k (1-x)^{n-k}$$

with the convention that  $\gamma_{n-1,-1} = 0$  and  $\gamma_{n-1,n} = 0$ . In view of (1) the requirement  $R_n 1 = 1$  is then equivalent to

$$Q_{n-1}(x) = \sum_{k=0}^n \bar{w}_{n,k} \binom{n}{k} x^k (1-x)^{n-k}$$

and we conclude that

$$\bar{w}_{n,k} \binom{n}{k} = \gamma_{n-1,k} + \gamma_{n-1,k-1}.$$

Further we want that  $R_n e_1 = e_1$  for the linear function  $e_1(x) = x$  which is equivalent to the identity

$$(5) \quad xQ_{n-1}(x) = \sum_{k=0}^n x_{n,k} \cdot \bar{w}_{n,k} \binom{n}{k} x^k (1-x)^{n-k}.$$

Inserting  $x = 0$  implies that  $x_{n,0} = 0$ . From the identity

$$xQ_{n-1}(x) = \sum_{k=0}^{n-1} \gamma_{n-1,k} \cdot x^{k+1} (1-x)^{n-1-k} = \sum_{k=1}^n \gamma_{n-1,k-1} x^k (1-x)^{n-k}$$

and (5) we infer that for  $k = 1, \dots, n$

$$(6) \quad x_{n,k} = \frac{\gamma_{n-1,k-1}}{\bar{w}_{n,k} \binom{n}{k}} = \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k} + \gamma_{n-1,k-1}} = \frac{\frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}}}{1 + \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}}}.$$

Hence, given the polynomial  $Q_{n-1}(x)$ , there is *at most one* choice for the nodes  $x_{n,k}$  and the weights  $\bar{w}_{n,k}$  such that  $R_n$  fixes  $e_0$  and  $e_1$ . However, in general the numbers  $x_{n,k}$  defined by (6) are not in the interval  $[0, 1]$ , and they are in general not increasing numbers, see Theorem 16 in Section 6. From formula (6) and the fact that  $f(x) = x/(1+x)$  is strictly increasing we derive that  $x_{n,k}$  is strictly increasing if and only if

$$(7) \quad \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} = \frac{w_{n-1,k-1}}{w_{n-1,k}} \frac{k}{n-k} \text{ is strictly increasing.}$$

This is exactly condition (W). The construction of the rational Bernstein operator  $R_n$  has the disadvantage that one has to check the condition (W) for the Bernstein coefficients of the polynomial  $Q_{n-1}$  which in general might be cumbersome. We now change our point of view: instead of starting with the polynomial  $Q_{n-1}$  we just start with an increasing sequence

$$0 = x_{n,0} < x_{n,1} < \dots < x_{n,n-1} < x_{n,n} = 1.$$

We use equation (6) to define  $\frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}}$ . Clearly (6) is equivalent to

$$x_{n,k} \left( 1 + \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} \right) = \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} \text{ and } \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} = \frac{x_{n,k}}{1 - x_{n,k}}$$

which is a recursion formula for  $\gamma_{n-1,k}$  provided we have defined  $\gamma_{n-1,0}$ . Hence define

$$(8) \quad \gamma_{n-1,k} := \gamma_{n-1,0} \prod_{l=1}^k \frac{1 - x_{n,l}}{x_{n,l}}.$$

These remarks lead to the following statement:

**Proposition 1.** *Let  $0 = x_{n,0} < x_{n,1} < \dots < x_{n,n-1} < x_{n,n} = 1$ . Let  $\gamma_{n-1,0} > 0$  and define  $\gamma_{n-1,k}$  by (8) for  $k = 1, \dots, n$  and  $\gamma_{n-1,-1} = 0$ , and define*

$$Q_{n-1}(x) = \sum_{k=0}^{n-1} \gamma_{n-1,k} x^k (1-x)^{n-1-k}.$$

Then  $Q_{n-1}$  satisfies property (W) and the operator

$$R_n f(x) = \sum_{k=0}^n f(x_{n,k}) (\gamma_{n-1,k} + \gamma_{n-1,k-1}) \frac{x^k (1-x)^{n-k}}{Q_{n-1}(x)}$$

is the rational Bernstein operator  $R_n$  fixing  $e_0$  and  $e_1$ .

*Proof.* There is not much to show: by definition of  $\gamma_{n-1,k}$  we see that  $\frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} = \frac{x_{n,k}}{1-x_{n,k}}$ . This clearly implies that (6) holds, so the nodes of the operator  $R_n$  are just the given numbers  $x_{n,k}$ . Since  $x_{n,k}$ 's are increasing we see that  $\frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}}$  are increasing and therefore property (W) holds.  $\square$

For example, if we take  $Q_{n-1}(x) = 1 + x^2$  then forward calculations show that

$$\begin{aligned} \gamma_{n-1,k} &= \binom{n-1}{k} \left( 1 + \frac{k(k-1)}{(n-1)(n-2)} \right) \\ \gamma_{n-1,k-1} + \gamma_{n-1,k} &= \binom{n}{k} \frac{n(n-1) + k(k-1)}{n(n-1)} \\ x_{n,k} &= \frac{k}{n-2} \frac{(n-1)(n-2) + (k-1)(k-2)}{n(n-1) + k(k-1)} \end{aligned}$$

and the rational Bernstein operator  $R_n$  is given by

$$R_n f(x) = \sum_{k=0}^n \binom{n}{k} \frac{n(n-1) + k(k-1)}{n(n-1)} f(x_{n,k}) \frac{x^k (1-x)^{n-k}}{1+x^2}.$$

### 3. CONVERGENCE OF RATIONAL BERNSTEIN OPERATORS

The following result is of central importance:

**Theorem 2.** Let  $R_n$  be the rational Bernstein operator for the polynomial  $Q_{n-1}(x)$  of degree  $\leq n-1$  satisfying (i) and (ii) in the introduction and let  $x_{n,k}$  be defined by (6). Then the following identity

$$(9) \quad R_n(e_s)(x) - x^s = \frac{x(1-x)}{Q_{n-1}(x)} \sum_{l=0}^{s-2} x^l \sum_{k=0}^{n-1} \gamma_{n-1,k} (x_{n,k+1}^{s-1-l} - x_{n,k}^{s-1-l}) x^k (1-x)^{n-1-k}$$

holds for the polynomial  $e_s(x) = x^s$  and for any natural number  $s \geq 2$ .

*Proof.* At first we note that  $k \geq 1$

$$\gamma_{n-1,k} + \gamma_{n-1,k-1} = \gamma_{n-1,k-1} \frac{1-x_{n,k}}{x_{n,k}} + \gamma_{n-1,k-1} = \gamma_{n-1,k-1} \frac{1}{x_{n,k}}.$$

It follows that

$$R_n f = f(0) \gamma_{n-1,0} \frac{(1-x)^n}{Q_{n-1}(x)} + \sum_{k=1}^n f(x_{n,k}) \frac{\gamma_{n-1,k-1} x^k (1-x)^{n-k}}{x_{n,k} Q_{n-1}(x)}.$$

Let  $s \geq 1$  and  $e_s(x) = x^s$ . Since  $x_{n,0} = 0$  and  $e_s(x_{n,0}) = x_{n,0}^s = 0$  we have

$$(10) \quad Q_{n-1}(x) R_n(e_s)(x) = \sum_{k=1}^n \gamma_{n-1,k-1} x_{n,k}^{s-1} \cdot x^k (1-x)^{n-k}.$$

Using an index transformation we arrive at

$$(11) \quad Q_{n-1}(x) R_n(e_s)(x) = x \sum_{k=0}^{n-1} \gamma_{n-1,k} x_{n,k+1}^{s-1} \cdot x^k (1-x)^{n-1-k}.$$

Writing  $x^k (1-x)^{n-1-k} = x^k (1-x)^{n-k} + x^{k+1} (1-x)^{n-1-k}$  we obtain

$$\begin{aligned} Q_{n-1}(x) R_n(e_s)(x) &= x \sum_{k=0}^{n-1} \gamma_{n-1,k} x_{n,k+1}^{s-1} \cdot x^k (1-x)^{n-k} \\ &\quad + x \sum_{k=0}^{n-1} \gamma_{n-1,k} x_{n,k+1}^{s-1} \cdot x^{k+1} (1-x)^{n-1-k}. \end{aligned}$$

The second sum is equal to  $x \sum_{k=1}^n \gamma_{n-1,k-1} x_{n,k}^{s-1} \cdot x^k (1-x)^{n-k}$ . Using the convention  $\gamma_{n-1,n} = \gamma_{n-1,-1} = 0$  and the fact that  $x_{n,0} = 0$  we obtain

$$Q_{n-1}(x) R_n(e_s)(x) = x \sum_{k=0}^n (\gamma_{n-1,k} x_{n,k+1}^{s-1} + \gamma_{n-1,k-1} x_{n,k}^{s-1}) \cdot x^k (1-x)^{n-k}.$$

On the other hand, as  $\gamma_{n-1,k} = \gamma_{n-1,k-1} \frac{1-x_{n,k}}{x_{n,k}}$  we have

$$\begin{aligned} \gamma_{n-1,k} (x_{n,k+1}^{s-1} - x_{n,k}^{s-1}) &= \gamma_{n-1,k} \left( x_{n,k+1}^{s-1} + \frac{x_{n,k}}{1-x_{n,k}} x_{n,k}^{s-2} (x_{n,k} - 1) \right) \\ &= \gamma_{n-1,k} \left( x_{n,k+1}^{s-1} + \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} x_{n,k}^{s-2} (x_{n,k} - 1) \right) \\ &= \gamma_{n-1,k} x_{n,k+1}^{s-1} + \gamma_{n-1,k-1} x_{n,k}^{s-1} - \gamma_{n-1,k-1} x_{n,k}^{s-2}. \end{aligned}$$

It follows that

$$\begin{aligned} Q_{n-1}(x) R_n(e_s)(x) &= x \sum_{k=0}^n \gamma_{n-1,k} (x_{n,k+1}^{s-1} - x_{n,k}^{s-1}) \cdot x^k (1-x)^{n-k} \\ &\quad + x \sum_{k=0}^n \gamma_{n-1,k-1} x_{n,k}^{s-2} \cdot x^k (1-x)^{n-k}. \end{aligned}$$

As  $\gamma_{n-1,n} = 0$ , the indices in the first sum range only up to  $n-1$ . The first summand of the second sum is zero. Using (10) for  $s-1$  instead of  $s$  for the second sum we arrive

$$\begin{aligned} Q_{n-1}(x) R_n(e_s)(x) &= x(1-x) \sum_{k=0}^{n-1} \gamma_{n-1,k} (x_{n,k+1}^{s-1} - x_{n,k}^{s-1}) \cdot x^k (1-x)^{n-1-k} \\ &\quad + x \cdot Q_{n-1}(x) R_n(e_{s-1})(x). \end{aligned}$$

Now use this formula inductively and recall that  $R_n(e_1) = e_1$  leading to the statement in theorem.  $\square$

**Corollary 3.** *The rational Bernstein operators  $R_n$  satisfy the inequality*

$$\left| R_n(e_2)(x) - x^2 \right| \leq \sup_{0 \leq k \leq n-1} |x_{n,k+1} - x_{n,k}| \cdot x(1-x).$$

*Proof.* From Theorem 2 for  $s=2$  we see that

$$(12) \quad R_n(e_2)(x) - x^2 = \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1,k} (x_{n,k+1} - x_{n,k}) x^k (1-x)^{n-1-k}$$

and then the statement is obvious.  $\square$

**Corollary 4.** *The rational Bernstein operators  $R_n$  converges to the identity operator if and only if*

$$(13) \quad \Delta_n := \sup_{0 \leq k \leq n-1} |x_{n,k+1} - x_{n,k}|$$

*converges to 0.*

*Proof.* If  $\Delta_n$  converges to zero it follows that  $R_n e_2$  converges uniformly to  $e_2$  and Korovkin's theorem shows that  $R_n$  converges to the identity operator. Conversely, suppose that  $R_n$  converges to the identity operator and suppose that  $\Delta_n$  does not converge to 0. Then there exists  $\delta > 0$  and a subsequence  $(n_l)_l$  such that  $\Delta_{n_l} \geq 2\delta$ . Hence for each  $l$  there  $k_l \in \{0, \dots, n_l - 1\}$  such that

$$(14) \quad |x_{n_l, k_l+1} - x_{n_l, k_l}| \geq \delta.$$

Since  $x_{n, k} \in [0, 1]$  we can pass to a subsequence of  $x_{n_l, k_l}$  which converges to some point  $x_0$  and we can pass again to a subsequence of the subsequence such that  $x_{n_{l_r}, k_{l_r}}$  converges to  $x_0$  and  $x_{n_{l_r}, k_{l_r}+1}$  converges to  $x_1$ . From (14) it follows that  $|x_1 - x_0| \geq \delta$ , and since  $x_{n_{l_r}, k_{l_r}} \leq x_{n_{l_r}, k_{l_r}+1}$  we infer that  $x_0 \leq x_1$ . Now we take a natural number  $r_0$  such that  $|x_0 - x_{n_{l_r}, k_{l_r}}| < \delta/3$  and  $|x_1 - x_{n_{l_r}, k_{l_r}+1}| < \delta/3$  for all  $r \geq r_0$ . From the monotonicity of  $x_{n, k}$  for  $k = 0, \dots, n_l - 1$  it follows that  $x_{n_{l_r}, k} \notin [x_0 + \delta/3, x_1 - \delta/3]$  for all  $k = 0, \dots, n_{l_r}$  and  $l \geq l_0$ . Now construct a continuous non-zero function  $f$  with support in  $[x_0 + \delta/3, x_1 - \delta/3]$  such that  $f(\xi) \neq 0$  for some  $\xi \in [x_0 + \delta/3, x_1 - \delta/3]$ . Then  $R_{n_{l_r}} f(x) = 0$  for all  $x \in [0, 1]$ . By assumption,  $R_{n_{l_r}} f(\xi)$  converges to  $f(\xi) \neq 0$ . Since  $R_{n_{l_r}} f(\xi) = 0$  we obtain a contradiction completing the proof.  $\square$

**Corollary 5.** *The following inequality holds for all  $x \in [0, 1]$  and for all natural numbers  $s \geq 2$ :*

$$0 \leq x^s < R_n(e_s)(x).$$

*Proof.* The right hand side in (9) is strictly positive for  $x \in [0, 1]$  and  $s \geq 2$ .  $\square$

In the rest of this section we shall prove some inequalities needed in Section 5:

**Proposition 6.** *The following inequality holds*

$$0 \leq R_n(e_3)(x) - x^3 \leq 3 \cdot (R_n(e_2)(x) - x^2).$$

*Proof.* From Theorem 2 for  $s = 3$  we see that

$$R_n(e_3)(x) - x^3 = \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k} A_k x^k (1-x)^{n-1-k}$$

where

$$A_k = x_{n, k+1}^2 - x_{n, k}^2 + x(x_{n, k+1} - x_{n, k}) = (x_{n, k+1} - x_{n, k})(x_{n, k+1} + x_{n, k} + x) \geq 0.$$

Since  $0 \leq x_{n, k+1} + x_{n, k} + x \leq 3$  we obtain

$$0 \leq R_n(e_3)(x) - x^3 \leq 3 \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1, k} (x_{n, k+1} - x_{n, k}) x^k (1-x)^{n-1-k}$$

and the last expression is equal to  $3(R_n(e_2)(x) - x^2)$ . The proof is complete.  $\square$

**Proposition 7.** *Let  $r$  be a natural number. Then the expression*

$$A := \frac{x}{Q_{n-1}(x)} \sum_{k=0}^{n-1} (x - x_{n,k+1})^r \gamma_{n-1,k} x^k (1-x)^{n-1-k}$$

is equal to

$$B := \sum_{l=0}^r \binom{r}{l} x^{r-l} (-1)^l [R_n(e_{l+1})(x) - x^{l+1}].$$

*Proof.* Since  $(x - x_{n,k+1})^r = \sum_{l=0}^r \binom{r}{l} x^{r-l} (-1)^l x_{n,k+1}^l$  it is easy to see that

$$A = \sum_{l=0}^r \binom{r}{l} x^{r-l} (-1)^l \frac{x}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1,k} x_{n,k+1}^l x^k (1-x)^{n-1-k}.$$

Using (11) we see that

$$A = \sum_{l=0}^r \binom{r}{l} x^{r-l} (-1)^l R_n(e_{l+1})$$

and the result follows from the fact that

$$\sum_{l=0}^r \binom{r}{l} x^{r-l} (-1)^l x^{l+1} = x(x + (-x))^r = 0.$$

□

In the case of the classical Bernstein operator  $B_n$  it is well known that

$$B_n e_2(x) - x^2$$

is a polynomial of degree  $\leq 2$ . For rational Bernstein operators  $R_n$  with  $n \geq 2$  the expression  $R_n e_2(x) - x^2$  is a never polynomial if  $\deg Q_{n-1}(x)$  has *exact* degree  $n-1$ . Indeed, suppose that  $R_n e_2(x) - x^2 = p_s(x)$  for some polynomial  $p_s(x)$  of degree  $s$ . Then by (12)

$$x(1-x) \sum_{k=0}^{n-1} \gamma_{n-1,k} (x_{n,k+1} - x_{n,k}) x^k (1-x)^{n-1-k} = p_s(x) Q_{n-1}(x)$$

which shows that  $p_s(x) Q_{n-1}(x)$  has degree  $\leq n+1$ . By assumption,  $\deg Q_{n-1} = n-1$  and we infer that  $s \leq 2$ . Clearly  $x(1-x)$  must be a factor of  $p_s(x)$ . Hence  $p_s(x) = Ax(1-x)$  and it follows that  $\gamma_{n-1,k} (x_{n,k+1} - x_{n,k}) = A\gamma_{n-1,k}$ , so  $x_{n,k+1} - x_{n,k} = A$ . Then  $x_{n,k} = k/n$  and

$$\frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k}} = \frac{x_{n,k}}{1-x_{n,k}} = \frac{k}{n(1-\frac{k}{n})} = \frac{k}{n-k}.$$

It follows that  $Q_{n-1}(x)$  is the constant polynomial, which has a contradiction since  $n \geq 2$ .

## 4. ERROR ESTIMATES FOR RATIONAL BERNSTEIN OPERATORS

Next we derive quantitative convergence results for  $R_n$ . By estimates of O. Shisha and B. Mond (see e.g. Theorem 8.1 in [27]) we conclude that

$$(15) \quad |R_n f(x) - f(x)| \leq \left(1 + \frac{1}{h} \sqrt{R_n(e_1 - x)^2(x)}\right) \omega_1(f, h)$$

for all  $f \in C[0, 1]$  and  $h > 0$  where  $\omega_1(f, h)$  is the first modulus of continuity defined by

$$\omega_1(f, h) = \sup_{|x-y| \leq h} |f(x) - f(y)|.$$

Error estimates in terms of the second modulus of continuity defined by

$$\omega_2(f, h) = \sup_{|\delta| \leq h} \{|f(x + \delta) - 2f(x) + f(x - \delta)| : x \pm h \in [a, b]\}.$$

have been given by Gonska et al. [14],[15] and R. Păltănea [25], [26] providing the inequality

$$|R_n f(x) - f(x)| \leq \left(1 + \frac{1}{2h^2} R_n(e_1 - x)^2(x)\right) \omega_2(f, h)$$

for all  $f \in C[0, 1]$  and  $h > 0$ . Since

$$R_n(e_1 - x)^2(x) = R_n(e_2)(x) - 2xR_n e_1(x) + x^2 = R_n e_2(x) - x^2$$

we obtain from (15) for  $h := \sqrt{\Delta_n}$ , defined in (13), and from Corollary 3 the following result:

**Theorem 8.** *The rational Bernstein operators  $R_n$  satisfies the following inequality:*

$$(16) \quad |R_n f(x) - f(x)| \leq \left(1 + \sqrt{x(1-x)}\right) \omega_1\left(f, \sqrt{\Delta_n}\right)$$

for all  $f \in C[0, 1]$ .

Similarly, we obtain

**Theorem 9.** *The rational Bernstein operators  $R_n$  satisfy the following inequality*

$$(17) \quad |R_n f(x) - f(x)| \leq \left(1 + \frac{1}{2}x(1-x)\right) \omega_2\left(f, \sqrt{\Delta_n}\right)$$

for all  $f \in C[0, 1]$ .

## 5. VORONOVSKAJA'S THEOREM

The classical Voronovskaja theorem states the following result for the Bernstein operator  $B_n$ :

**Theorem 10.** *Let  $f : [0, 1] \rightarrow \mathbb{R}$  be bounded and differentiable in a neighborhood of  $x$  and has second derivative  $f''(x)$ . Then*

$$\lim_{n \rightarrow \infty} n \cdot (B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x).$$

We shall need the following generalization due to R.G. Mamedov [20], see also [17] and [29] for quantitative estimates and higher order of differentiability.

**Theorem 11.** *Let  $f \in C^2[0, 1]$  and  $L_n : C[0, 1] \rightarrow C[0, 1]$  be a sequence of positive operators such that  $L_n e_j = e_j$  for  $j = 0, 1$  and*

$$\lim_{n \rightarrow \infty} \frac{L_n(e_1 - x)^4(x)}{L_n(e_1 - x)^2(x)} = 0$$

for each  $x \in [0, 1]$ . Then

$$\frac{L_n f(x) - f(x)}{L_n(e_1 - x)^2(x)} \rightarrow \frac{1}{2} f''(x)$$

when  $n \rightarrow \infty$ .

The classical proof of the Voronovskaja theorem requires the computation of the moments of order  $r$  of the Bernstein operator  $B_n$ :

$$B_n[(e_1 - x)^r](x) = \sum_{k=0}^r \binom{k}{n} x^k (1-x)^{n-k} =: \frac{1}{n^r} T_{n,r}(x).$$

It is well known that  $T_{n,r}(x)$  is a polynomial of degree  $r$  in the variable  $x$  and one can determine  $T_{n,r}(x)$  recursively by the formula

$$T_{n,r+1}(x) = x(1-x) [T'_{n,r}(x) + nrT_{n,r-1}(x)],$$

see [18]. From this it is not difficult to show that for each  $r \in \mathbb{N}$  there exists a constant  $A_r > 0$  such that

$$(18) \quad B_n[(e_1 - x)^r](x) \leq \sqrt{A_r} \frac{1}{\sqrt{n}},$$

see e.g. [29]. In passing we mention that in the recent article [16] the following inequality was established: for  $r \in \mathbb{N}$  there exists a constant  $K_r > 0$  such that

$$B_n[(e_1 - x)^{r+1}](x) \leq \frac{K_r}{\sqrt{n}} B_n[(e_1 - x)^r](x)$$

which clearly implies (18).

In the case of the rational Bernstein operator the moments  $R_n [(e_1 - x)^r](x)$  are usually not polynomials in the variable  $x$  as we have seen already at the end of Section 4 for  $r = 2$ . Nonetheless, we can compute them explicitly but the formulae are much more complicated. Indeed, if we use the binomial theorem for  $(e_1 - x)^r$  we obtain

$$R_n [(e_1 - x)^r](x) = \sum_{s=0}^r \binom{r}{s} (-x)^{r-s} R_n (e_s)(x)$$

and since  $0 = (x - x)^r = \sum_{s=0}^r \binom{r}{s} (-x)^{r-s} x^s$  we have

$$(19) \quad R_n [(e_1 - x)^r](x) = \sum_{s=2}^r \binom{r}{s} (-x)^{r-s} [R_n (e_s)(x) - x^s]$$

where we used the fact that  $R_n (e_s) = e_s$  for  $s = 0, 1$ . Theorem 2 provides then an explicit formula for the moments. But in view of Theorem 11 we have to estimate

$$\frac{R_n (e_1 - x)^4(x)}{R_n (e_1 - x)^2(x)}$$

and it is therefore not sufficient just to estimate the moments.

**Theorem 12.** *The fourth moment satisfies the following inequality:*

$$R_n (e_1 - x)^4(x) \leq \Delta_n \cdot [R_n (e_1 - x)^2(x)] (6x^2 - 15x + 12 + \Delta_n).$$

*Proof.* Formula (19) shows that  $R_n (e_1 - x)^4(x)$  is equal to

$$R_n (e_4)(x) - x^4 - 4x (R_n (e_3)(x) - x^3) + 6x^2 (R_n (e_2)(x) - x^2).$$

By Theorem 2 we can calculate each summand explicitly and we obtain

$$(20) \quad R_n (e_1 - x)^4(x) = \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1,k} x^k (1-x)^{n-1-k} \cdot H_k$$

with

$$H_k = x_{n,k+1}^3 - x_{n,k}^3 + x(x_{n,k+1}^2 - x_{n,k}^2) + x^2(x_{n,k+1} - x_{n,k}) - 4x(x_{n,k+1}^2 - x_{n,k}^2 + x(x_{n,k+1} - x_{n,k})) + 6x^2(x_{n,k+1} - x_{n,k})$$

which simplifies to

$$H_k = (x_{n,k+1}^3 - x_{n,k}^3) - 3x(x_{n,k+1}^2 - x_{n,k}^2) + 3x^2(x_{n,k+1} - x_{n,k}).$$

We write  $H_k = (x_{n,k+1} - x_{n,k}) A_k$  with

$$A_k = x_{n,k+1}^2 + x_{n,k+1}x_{n,k} + x_{n,k}^2 - 3x(x_{n,k+1} + x_{n,k}) + 3x^2.$$

A straightforward calculation shows that

$$A_k = 3 \left( x - \frac{1}{2}(x_{n,k+1} + x_{n,k}) \right)^2 + \frac{1}{4}(x_{n,k+1} - x_{n,k})^2 \geq 0.$$

Hence  $A_k$  is positive and it is easy to see that

$$(21) \quad R_n(e_1 - x)^4(x) \leq \Delta_n \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1,k} x^k (1-x)^{n-1-k} \cdot A_k.$$

We write now

$$A_k = 3(x - x_{n,k+1})^2 + 3(x - x_{n,k+1})(x_{n,k+1} - x_{n,k}) + (x_{n,k+1} - x_{n,k})^2.$$

Proposition 7 and 6 show that

$$\begin{aligned} & \frac{x}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1,k} (x - x_{n,k+1})^2 x^k (1-x)^{n-1-k} \\ &= R_n(e_3)(x) - x^3 - 2x [R_n(e_1)(x) - x^2] \leq (3 - 2x) [R_n(e_1)(x) - x^2]. \end{aligned}$$

Formula (21) and the last inequality and the simple estimates  $|x - x_{n,k+1}| \leq 1$  and  $x_{n,k+1} - x_{n,k} \leq \Delta_n$  lead to

$$\begin{aligned} R_n(e_1 - x)^4(x) &\leq \Delta_n (1-x) \cdot 3(3-2x) \cdot [R_n(e_1)(x) - x^2] \\ &\quad + \Delta_n (3 + \Delta_n) \frac{x(1-x)}{Q_{n-1}(x)} \sum_{k=0}^{n-1} \gamma_{n-1,k} (x_{n,k+1} - x_{n,k}) x^k (1-x)^{n-1-k}. \end{aligned}$$

It follows that

$$R_n(e_1 - x)^4(x) \leq \Delta_n [R_n(e_1)(x) - x^2] ((1-x)(9-6x) + 3 + \Delta_n)$$

and the statement is now obvious.  $\square$

Using Theorem 11 and Theorem 12 we obtain

**Theorem 13.** *Let  $f \in C^2[0, 1]$  and assume that  $\Delta_n \rightarrow 0$  for the rational Bernstein operators  $R_n : C[0, 1] \rightarrow C[0, 1]$ . Then*

$$\frac{R_n f(x) - f(x)}{R_n(e_1 - x)^2(x)} \rightarrow \frac{1}{2} f''(x).$$

## 6. SPECIAL CLASSES OF RATIONAL BERNSTEIN OPERATORS

In [27] error estimates and convergence results have been given for rational Bernstein operators  $R_n$  under the assumption that there exists a positive function  $\varphi \in C[0, 1]$  such that

$$Q_{n-1}(x) := B_{n-1}\varphi(x) = \sum_{k=0}^{n-1} \varphi\left(\frac{k}{n-1}\right) \binom{n-1}{k} x^k (1-x)^{n-1-k}$$

where  $B_{n-1}$  is the classical Bernstein operator of degree  $n-1$ . Then  $Q_{n-1}$  has clearly positive Bernstein coefficients but in general one has to assume in addition that property (W) is satisfied.

It is shown in [27, p. 42] that property (W) is satisfied provided that  $n$  is sufficiently large and  $\varphi \in C^2[0, 1]$ . In this section we want to show that it suffices to assume only that  $\varphi \in C^1[0, 1]$ . Moreover we shall show by example that the result is not true for Lipschitz continuous positive functions.

We need the following result which is implicitly contained in [27]:

**Proposition 14.** *Let  $\varphi \in C[0, 1]$  be positive and  $Q_{n-1}(x) = B_{n-1}\varphi(x) = \sum_{k=0}^{n-1} \gamma_{n-1,k} x^k$  with  $\gamma_{n-1,k} = \varphi(k/(n-1)) \binom{n-1}{k}$ . If one defines*

$$x_{n,k} := \frac{\gamma_{n-1,k-1}}{\gamma_{n-1,k-1} + \gamma_{n-1,k}} = \frac{k\varphi\left(\frac{k-1}{n-1}\right)}{k\varphi\left(\frac{k-1}{n-1}\right) + (n-k)\varphi\left(\frac{k}{n-1}\right)}$$

then

$$(22) \quad \Delta_n = \sup_{k=0, \dots, n-1} |x_{n,k+1} - x_{n,k}| \leq \frac{1}{2m} \omega_1\left(\varphi, \frac{1}{n-1}\right) + \frac{1}{n},$$

where  $m = \min_{x \in [0,1]} \varphi(x)$ .

*Proof.* Define a function  $\psi_h$  by

$$\psi_h(x) = \frac{x\varphi(x-h)}{x\varphi(x-h) + (1-x+h)\varphi(x)}.$$

We can estimate with  $m := \min_{y \in [0,1]} \varphi(y) > 0$

$$x\varphi(x-h) + (1-x+h)\varphi(x) \geq (1+h)m.$$

Put  $h = 1/(n-1)$  and  $x = k/(n-1)$ , then

$$(23) \quad x_{n,k} = \psi_{\frac{1}{n-1}}\left(\frac{k}{n-1}\right).$$

Clearly

$$\begin{aligned} \psi_h(x) - \frac{x}{1+h} &= x \frac{(1+h)\varphi(x-h) - x\varphi(x-h) - (1-x+h)\varphi(x)}{(1+h)(x\varphi(x-h) + (1-x+h)\varphi(x))} \\ &= \frac{x \cdot (1-x+h) \cdot (\varphi(x-h) - \varphi(x))}{(1+h) \cdot (x\varphi(x-h) + (1-x+h)\varphi(x))}. \end{aligned}$$

Thus we obtain

$$\left| \psi_h(x) - \frac{x}{1+h} \right| \leq \frac{x(1-x+h)}{(1+h)^2 m} \omega_1(\varphi, h) \leq \frac{1}{4m} \omega_1(\varphi, h).$$

where we used that  $4x(1-x+h) \leq (1+h)^2$  for all  $x \in [0, 1]$  and  $h > 0$ . Using (23) it follows that for all  $k = 0, \dots, n$  and all  $n$  the following inequality

$$\left| x_{n,k} - \frac{k}{n} \right| \leq \frac{1}{4m} \omega_1\left(\varphi, \frac{1}{n-1}\right)$$

holds. Since  $x_{n,k+1} - x_{n,k} = x_{n,k+1} - \frac{k+1}{n} + \frac{1}{n} + \frac{k}{n} - x_{n,k}$  we can estimate

$$(24) \quad |x_{n,k+1} - x_{n,k}| \leq \frac{1}{2m} \omega_1 \left( \varphi, \frac{1}{n-1} \right) + \frac{1}{n}.$$

□

**Theorem 15.** *Let  $\varphi \in C[0,1]$  be strictly positive. If  $\varphi \in C^1[0,1]$  then  $Q_{n-1}(x) := B_{n-1}\varphi(x)$  satisfies property (W) for sufficiently large  $n \in \mathbb{N}$ . If  $\varphi$  is Lipschitz continuous then  $B_{n-1}(a + \varphi)$  satisfies property (W) for sufficiently large  $n \in \mathbb{N}$  and sufficiently large  $a > 0$ .*

*Proof.* We use the notations from the proof of Proposition 14. In view of (23) it suffices to show that the function  $x \mapsto \psi_h(x)$  is increasing if  $h > 0$  is sufficiently small, or equivalently, that for  $\delta > 0$  and  $h > 0$  sufficiently small and for all  $x \in [0,1]$  the inequality

$$(25) \quad \psi_h(x) = \frac{x\varphi(x-h)}{c_h(\varphi)(x)} < \psi_h(x+\delta) = \frac{(x+\delta)\varphi(x+\delta-h)}{c_h(\varphi)(x+\delta)}$$

holds where

$$c_h(\varphi)(x) := x\varphi(x-h) + (1-x+h)\varphi(x) \geq m = \min_{x \in [0,1]} \varphi(x).$$

Note that  $c_h(\varphi)(x)$  converges to  $\varphi(x)$  uniformly in  $x$  when  $h$  tends to zero. Inequality (25) means that

$$D(x, h, \delta, \varphi) := x\varphi(x-h)c_h(\varphi)(x+\delta) - x\varphi(x+\delta-h)c_h(\varphi)(x)$$

satisfies the inequality

$$(26) \quad D(x, h, \delta, \varphi) < \delta\varphi(x+\delta-h)c_h(\varphi)(x).$$

By inserting and subtracting  $x\varphi(x-h)c_h(\varphi)(x)$  we conclude that

$$\begin{aligned} \frac{D(x, h, \delta, \varphi)}{\delta} &= x\varphi(x-h) \frac{c_h(\varphi)(x+\delta) - c_h(\varphi)(x)}{\delta} \\ &\quad + xc_h(\varphi)(x) \frac{\varphi(x-h) - \varphi(x+\delta-h)}{\delta}. \end{aligned}$$

If  $\varphi \in C^1[0,1]$  we can find  $\xi_{x,h,\delta} \in [x-h, x-h+\delta]$  and  $\eta_{x,h,\delta} \in [x, x+\delta]$  with

$$\begin{aligned} \varphi(x-h) - \varphi(x+\delta-h) &= \varphi'(\xi_{x,h,\delta}) \cdot \delta \\ c_h(\varphi)(x+\delta) - c_h(\varphi)(x) &= c_h(\varphi)'(\eta_{x,h,\delta}) \cdot \delta. \end{aligned}$$

It follows that

$$\frac{D(x, h, \delta, \varphi)}{\delta} = x\varphi(x-h)c_h(\varphi)'(\eta_{x,h,\delta}) - x \cdot c_h(\varphi)(x)\varphi'(\xi_{x,h,\delta}).$$

Since  $c_h(\varphi)(x)$  converges to  $\varphi(x)$ , and  $c_h(\varphi)'(x)$  converges to  $\varphi'(x)$  for  $h \rightarrow 0$  we see that  $D(x, h, \delta, \varphi)/\delta$  converges to 0 for  $h \rightarrow 0$  and  $\delta \rightarrow 0$ , hence

$$\frac{D(x, h, \delta, \varphi)}{\delta} < \frac{1}{2}m^2 \leq \varphi(x + \delta - h)c_h(\varphi)(x)$$

for  $m = \min_{x \in [0,1]} \varphi(x)$  and  $h$  sufficiently small, so (26) is satisfied. Now assume that  $\varphi$  is positive and Lipschitz continuous. Clearly  $c_h(\varphi)$  is Lipschitz continuous and there exist  $M > 0$  and  $N > 0$  such that

$$\begin{aligned} |\varphi(x - h) - \varphi(x + \delta - h)| &\leq M\delta \\ |c_h(\varphi)(x + \delta) - c_h(\varphi)(x)| &\leq N\delta \end{aligned}$$

where  $M, N$  do not depend on  $h$ . It follows that  $|D(x, h, \delta, \varphi)|/\delta$  is bounded for all  $x \in [0, 1]$  and  $h > 0$  and  $\delta > 0$ . We will show that (26) is satisfied where we replace  $\varphi$  by  $a + \varphi$ . At first we see that

$$\begin{aligned} c_h(a + \varphi)(x) &= x[a + \varphi(x - h)] + (1 - x + h)[a + \varphi(x)] \\ &= a(1 + h) + c_h(\varphi)(x) \geq a(1 + h). \end{aligned}$$

Then

$$\begin{aligned} D(x, h, \delta, a + \varphi) &= x(a + \varphi(x - h)) \cdot (a(1 + h) + c_h(\varphi)(x + \delta)) \\ &\quad - x(a + \varphi(x + \delta - h))(a(1 + h) + c_h(\varphi)(x)) \end{aligned}$$

can be simplified to

$$\begin{aligned} D(x, h, \delta, a + \varphi) &= D(x, h, \delta, \varphi) + ax[c_h(\varphi)(x + \delta) - c_h(\varphi)(x)] \\ &\quad + xa(1 + h)[\varphi(x - h) - \varphi(x + \delta - h)]. \end{aligned}$$

On the other hand

$$(a + \varphi(x + \delta - h))c_h(a + \varphi)(x) \geq a^2(1 + h)$$

and by taking  $a > 0$  sufficiently large we see that (26) is satisfied for  $a + \varphi$ .  $\square$

We shall give an example of a positive Lipschitz continuous function  $\varphi_a \in C[0, 1]$  such that  $Q_{2n}(x) = B_{2n}\varphi_a$  does not satisfy property (W) for any natural number  $n$ : take  $a \in (0, \frac{1}{2})$  in the following theorem:

**Theorem 16.** Define  $\varphi_a(x) = a + |x - \frac{1}{2}|$  for  $a > 0$ . Then

$$Q_{2n,a}(x) := B_{2n}\varphi_a = \sum_{k=0}^{2n} \left( a + \left| \frac{k}{2n} - \frac{1}{2} \right| \right) \binom{2n}{k} x^k (1-x)^{2n-k}$$

has strictly positive Bernstein coefficients, and if  $Q_{2n,a}$  satisfies property (W) for some natural number  $n$  then  $a > \frac{1}{2}$ .

*Proof.* Let  $\gamma_{2n,k} = \left(a + \left|\frac{k}{2n} - \frac{1}{2}\right|\right) \binom{2n}{k}$ . If  $Q_{2n,a}(x)$  has property (W) then  $\frac{\gamma_{2n,k-1}}{\gamma_{2n,k}}$  is increasing in  $k$  according to (7) and therefore necessarily

$$(27) \quad \frac{\gamma_{2n,n-1}}{\gamma_{2n,n}} < \frac{\gamma_{2n,n}}{\gamma_{2n,n+1}}.$$

Note that

$$\frac{\gamma_{2n,n-1}}{\gamma_{2n,n}} = \frac{\left(a + \frac{1}{2} - \frac{n-1}{2n}\right) \binom{2n}{n-1}}{a \binom{2n}{n}} = \frac{an + \frac{1}{2}}{a(n+1)}$$

and

$$\frac{\gamma_{2n,n}}{\gamma_{2n,n+1}} = \frac{a \binom{2n}{n}}{\left(a - \frac{1}{2} + \frac{n+1}{2n}\right) \binom{2n}{n+1}} = \frac{a(n+1)}{na + \frac{1}{2}}.$$

Hence (27) holds iff

$$\left(an + \frac{1}{2}\right)^2 - a^2(n+1)^2 = \left(\frac{1}{2} - a\right) \left(2an + a + \frac{1}{2}\right)$$

is negative, which is equivalent to  $a > \frac{1}{2}$ .  $\square$

We apply the error estimates now to the case that  $Q_{n-1}(x) = B_{n-1}\varphi(x)$ :

**Theorem 17.** *Suppose that  $\varphi \in C[0,1]$  has the property that  $Q_{n-1}(x) = B_{n-1}\varphi(x)$  satisfies property (W). Then*

$$(28) \quad |R_n f(x) - f(x)| \leq \left(1 + \sqrt{x(1-x)}\right) \omega_1 \left(f, \frac{1}{\sqrt{n}} + \sqrt{\frac{\omega_1\left(\varphi, \frac{1}{n-1}\right)}{2m}}\right).$$

*Proof.* Formula (22) and the inequality  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  for positive numbers  $a, b$ , imply that

$$\sqrt{\Delta_n} \leq \frac{1}{\sqrt{n}} + \sqrt{\frac{\omega_1\left(\varphi, \frac{1}{n-1}\right)}{2m}}.$$

Thus (16) gives the desired estimate.  $\square$

Let us mention that in [27] the following result is proved:

**Theorem 18.** *Suppose that  $\varphi \in C[0,1]$  such that  $Q_{n-1}(x) = B_{n-1}\varphi(x)$  satisfies property (W). Then*

$$(29) \quad |R_n f(x) - f(x)| \leq \left(1 + \frac{1}{2} \sqrt{\frac{\max_{x \in [0,1]} \varphi(x)}{\min_{x \in [0,1]} \varphi(x)}}\right) \omega_1 \left(f, \frac{1}{\sqrt{n}} + \frac{\omega_1\left(\varphi, \frac{1}{n-1}\right)}{2m}\right).$$

Clearly  $\sqrt{x(1-x)} \leq 1/2$  for  $x \in [0, 1]$  implies the inequality

$$\sqrt{x(1-x)} \leq \frac{1}{2} \sqrt{\frac{\max_{x \in [0,1]} \varphi(x)}{\min_{x \in [0,1]} \varphi(x)}}$$

but it seems the modulus of continuity in (28) can not be compared to that in (29).

The section will be finished by an example showing that the rational Bernstein operators  $R_n$  may converge to the identity although that the polynomials  $Q_{n-1}(x)$  do not converge to a *continuous* function. In this example there does not exist a continuous function  $\varphi$  with  $Q_{n-1} = B_{n-1}\varphi$  for all  $n \in \mathbb{N}$  since  $B_{n-1}\varphi$  converges to  $\varphi(x)$ .

**Example 19.** *The rational Bernstein operator  $R_n$  associated to the nodes  $x_{n,k} = \sqrt{\frac{k}{n}}$  for  $k = 0, \dots, n$  converges to the identity operator but the associated polynomials  $Q_{n-1}(x)$  defined by*

$$Q_{n-1}(x) = (1-x)^{n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k} \prod_{l=1}^k \frac{\sqrt{\frac{l}{n}}}{1 + \sqrt{\frac{l}{n}}} x^k (1-x)^{n-1-k}$$

do not converge to a continuous function, in particular  $Q_{n-1}$  is not equal to  $B_{n-1}\varphi$  for some continuous function  $\varphi \in C[0, 1]$ .

*Proof.* Clearly  $1/\sqrt{n} \leq |x_{n,1} - x_{n,0}| \leq \Delta_n$  and

$$|x_{n,k+1} - x_{n,k}| = \sqrt{\frac{k+1}{n}} - \sqrt{\frac{k}{n}} = \frac{\frac{k+1}{n} - \frac{k}{n}}{\sqrt{\frac{k+1}{n}} + \sqrt{\frac{k}{n}}} \leq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{k+1}}.$$

By Corollary 4  $R_n$  converges to the identity operator. Next we consider for  $l = 1, \dots, n-1$

$$\frac{1 - x_{n,l}}{x_{n,l}} = \frac{1 - x_{n,l}^2}{x_{n,l}(1 + x_{n,l})} = \frac{1 - \frac{l}{n}}{\sqrt{\frac{l}{n}}(1 + \sqrt{\frac{l}{n}})} = \frac{n-l}{l} \frac{\sqrt{l}}{\sqrt{n} + \sqrt{l}}.$$

Since  $2\sqrt{l} \leq \sqrt{n} + \sqrt{l}$  we can estimate the last factor by  $1/2$ . It follows that

$$\gamma_{n-1,k} = \prod_{l=1}^k \frac{1 - x_{n,l}}{x_{n,l}} \leq \binom{n-1}{k} \frac{1}{2^k}$$

and

$$Q_{n-1}(x) \leq \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{2^k} x^k (1-x)^{n-1-k} = \left(1 - \frac{x}{2}\right)^n.$$

Now the result is obvious. Then  $Q_{n-1}(0) = 1$  and  $Q_{n-1}(x)$  converges to 0 for  $x \in (0, 1]$ . It follows that  $Q_n(x)$  converges to a discontinuous function.  $\square$

## 7. FINAL COMMENTS

We want to comment on rational Bernstein operators  $R_n$  from a different point of view: Given a strictly positive polynomial  $Q_{n-1}(x)$  we consider the space

$$E_n = \left\{ \frac{p(x)}{Q_{n-1}(x)} : p(x) \text{ is a polynomial of degree } \leq n \right\}.$$

Then  $E_n$  is an extended Chebyshev space over any interval  $[a, b]$ , meaning that each non-zero function  $f \in E_n$  has at most  $n$  zeros (including multiplicities) in  $[a, b]$ . We call a system of functions  $P_{n,k}, k = 0, \dots, n$  in an  $n + 1$  dimensional linear space  $E_n$  of  $C^n[0, 1]$  a Bernstein basis, if each  $P_{n,k}$  has exactly  $k$  zeros in 0 and  $n - k$  zeros in 1. Thus the system of functions

$$\frac{x^k (1-x)^{n-k}}{Q_n(x)}, k = 0, \dots, n-1$$

is a Bernstein basis in  $E_n$ . Bernstein bases in extended Chebyshev spaces have been studied by many authors, see [6], [7], [8], [9], [10], [21], [22].

Recently Bernstein operators for an extended Chebyshev space  $E_n$  of dimension  $n + 1$  have been introduced by J. M. Aldaz, O. Kounchev and the author which by definition are operators of the form

$$S_n f(x) = \sum_{k=0}^n f(x_{n,k}) \alpha_{n,k} p_{n,k}(x)$$

where  $p_{n,k}(x), k = 0, \dots, n$ , is a Bernstein basis for  $E_n$ . The nodes  $x_{n,k}$  and the weights  $\alpha_{n,k}$  are chosen such that  $S_n f_0 = f_0$  and  $S_n f_1 = f_1$  where  $f_0$  is a strictly positive function in  $E_n$  and  $f_1 \in E_n$  has the property that  $f_1/f_0$  is strictly increasing. We refer to [1], [2], [3], [4] and [23] for a systematic study (existence of Bernstein operators fixing two functions and shape preserving properties) and to [24] for a discussion of Schoenberg-type operators in the setting of extended Chebyshev space. It seems to be a difficult task to establish convergence results of Bernstein operators in the setting of extended Chebyshev spaces, and the rational Bernstein operators considered here seems to be the simplest non-trivial example beyond the classical case of Bernstein operators.

**Acknowledgement 20.** *I would like to thank an anonymous referee for a very careful reading, and his or her comments improved the paper considerably.*

## REFERENCES

- [1] J. M. Aldaz, O. Kounchev, H. Render, *Bernstein operators for exponential polynomials*, Constr. Approx. 29 (2009), 345–367.
- [2] J. M. Aldaz, O. Kounchev, H. Render, *Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces*, Numer. Math. 114 (2009), 1–25.

- [3] J. M. Aldaz, O. Kounchev, *Optimality of generalized Bernstein operators*, J. Approx. Theory 162 (2010), 1407–1416.
- [4] J. M. Aldaz, O. Kounchev, H. Render, *Bernstein operators for extended Chebyshev systems*, Appl. Math. Comput. 217 (2010), 790–800.
- [5] O. Kounchev, H. Render, *On the Bernstein polynomial of S. Morigi and M. Neamtu*, Result. Math. 53 (2009), 311–322.
- [6] J.-M. Carnicer, J.-M. Peña, *Shape preserving representations and optimality of the Bernstein basis*, Adv. Comput. Math. 1 (1993), 173–196.
- [7] J.-M. Carnicer, J.-M. Peña, *Totally positive bases for shape preserving curve design and optimality of B-splines*, Comput. Aided Geom. Design 11 (1994), 633–654. (1993), 173–196.
- [8] J.-M. Carnicer, J.-M. Peña, *Total positivity and optimal bases*. In: Total Positivity and its Applications (M. Gasca, C.A. Micchelli, eds.). Dordrecht: Kluwer Academic, 133–145.
- [9] J.-M. Carnicer, E. Mainar, J.M. Peña, *Critical Length for Design Purposes and Extended Chebyshev Spaces*, Constr. Approx. 20 (2004), 55–71.
- [10] J.M. Carnicer, E. Mainar, J.M. Peña, *Shape preservation regions for six-dimensional space*, Adv. Comput. Math. 26 (2007), 121–136.
- [11] P. J. Davis, *Interpolation and Approximation*, Dover Publications, New York 1975.
- [12] G. Farin, *Nurbs Curves and Surfaces*, Morgan Kaufman, Los Altos, CA, 2001.
- [13] M.S. Floater, *Derivatives of rational Bézier curves*, Computer Aided Geometric Design 9 (1992), 161–174.
- [14] H. Gonska, *On approximation by linear operators: improved estimates*, Anal. Numér. Théor. Approx. 14 (1985), 7–32.
- [15] H. Gonska, R.K. Kovacheva, *The second order modulus revisited: remarks applications problems*, Conf. Sem. Mat. Univ. Bari 257 (1994), 1–32.
- [16] I. Gavrea, M. Ivan, *An answer to a conjecture on Bernstein operators*, J. Math. Anal. Appl. 390 (2012), 86–92.
- [17] H. Gonska, G. Tachev, *A quantitative variant of Voronoskaja’s theorem*, Result. Math. 53 (2009), 287–294.
- [18] G.G. Lorentz, *Bernstein polynomials*, Chelsea Publishing Company, New York 1986 (2nd edition).
- [19] E. Mainar, J.M. Peña, J. Sánchez-Reyes, *Shape preserving alternatives to the rational Bézier model*, Comput. Aided Geom. Design 18 (2001), 37–60.
- [20] R.G. Mamedov, *On the asymptotic value of the approximation of repeatedly differentiable functions by positive linear operators*, Dokl. Akad. Nauk. 146 (1962), 1013–1016 (in Russian); Translated in Soviet Math. Dokl. 3 (1962), 1435–1439.
- [21] M.-L. Mazure, *Bernstein bases in Müntz spaces*, Numer. Alg. 22 (1999), 285–304.
- [22] M.-L. Mazure, *Chebyshev Spaces and Bernstein bases*, Constr. Approx. 22 (2005), 347–363.
- [23] M.-L. Mazure, *Bernstein-type operators in Chebyshev spaces*, Numer. Alg. 52 (2009), 1016–1027.
- [24] M.-L. Mazure, *Chebyshev-Schoenberg operators*, Constr. Approx. 34 (2011), 181–208.
- [25] R. Păltănea, *Optimal estimates with moduli of continuity*, Result. Math. 32 (1997), 318–331.
- [26] R. Păltănea, *Approximation Theory Using Positive Linear Operators*, Birkhäuser Verlag, Basel 2004.
- [27] P. Pițul, P. Sablonnière, *A family of univariate rational Bernstein operators*, J. Approx. Theory 160 (2009), 39–55.
- [28] I. Selimovic, *New bounds on the magnitude of the derivative of rational Bézier curves and surfaces*, Comput. Aided Geom. Design 22 (2005), 321–326.
- [29] G.T. Tachev, *The complete asymptotic expansion for Bernstein operators*, J. Math. Anal. Appl. 385 (2012), 1179–1183.

- [30] G.T. Tachev, *The rate of approximation by rational Bernstein functions in terms of second order moduli of continuity*, Numer. Funct. Anal. Optim. 33 (2012), 206–215.

H. RENDER: SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND.

*E-mail address:* `hermann.render@ucd.ie`