



<b>Title</b>	Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations
<b>Authors(s)</b>	Dassios, Ioannis K.
<b>Publication date</b>	2017-11-30
<b>Publication information</b>	Dassios, Ioannis K. "Geometric Relation between Two Different Types of Initial Conditions of Singular Systems of Fractional Nabla Difference Equations." Wiley Online Library, November 30, 2017. <a href="https://doi.org/10.1002/mma.3771">https://doi.org/10.1002/mma.3771</a> .
<b>Publisher</b>	Wiley Online Library
<b>Item record/more information</b>	<a href="http://hdl.handle.net/10197/10650">http://hdl.handle.net/10197/10650</a>
<b>Publisher's statement</b>	This is the peer reviewed version of the following article: Dassios, I. K. ( 2017) Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations. Math. Meth. Appl. Sci., 40: 6085– 6095. doi: 10.1002/mma.3771. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Self-Archiving.
<b>Publisher's version (DOI)</b>	10.1002/mma.3771

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# Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations.

I. K. Dassios\*

In this article we study the geometric relation between two different types of initial conditions (IC) of a class of singular linear systems of fractional nabla difference equations whose coefficients are constant matrices. For this kind of systems, we analyze how inconsistent and consistent IC are related to the column vector space of the finite and the infinite eigenvalues of the pencil of the system and analyze the geometric connection between these two different types of IC. Numerical examples are given to justify the results. Copyright © 2015 John Wiley & Sons, Ltd.

**Keywords:** singular, fractional, discrete time system, nabla, initial conditions.

## 1. Introduction

Difference equations of fractional order have recently proven to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism and so forth, see [3], [12], [15], [21], [27], [28]. There has been a significant development in the study of fractional differential/difference equations and inclusions in recent years; For some recent contributions on fractional differential/difference equations, see [5], [6], [7], [8], [13], [14], [16], [18], [19], [20], [22], [23], [24], [25], [26], [27] and the references therein.

If we define  $\mathbb{N}_\alpha$  by  $\mathbb{N}_\alpha = \{\alpha, \alpha + 1, \alpha + 2, \dots\}$ ,  $\alpha$  integer, and  $n$  such that  $0 < n < 1$  or  $1 < n < 2$ , then the nabla fractional operator in the case of Riemann-Liouville fractional difference of  $n$ -th order for any  $Y_k : \mathbb{N}_\alpha \rightarrow \mathbb{R}^m$  is defined by, see [2],

$$\nabla_\alpha^{-n} Y_k = \frac{1}{\Gamma(n)} \sum_{j=\alpha}^k (k-j+1)^{\overline{n-1}} Y_j.$$

We denote  $\mathbb{R}^{m \times 1}$  with  $\mathbb{R}^m$ . Where the raising power function is defined by

$$k^{\bar{\alpha}} = \frac{\Gamma(k+\alpha)}{\Gamma(k)}.$$

We consider the singular fractional discrete time system of the form

$$F \nabla_0^n Y_k = G Y_k, \quad k = 1, 2, \dots, \quad (1)$$

with known IC. Where  $F, G \in \mathbb{R}^{r \times m}$  and  $Y_k \in \mathbb{R}^m$ . The matrices  $F, G$  can be non-square ( $r \neq m$ ) or square ( $r = m$ ) with  $F$  singular ( $\det F = 0$ ).

In this article we will study the geometric relation between two different types of IC of system (1), the consistent and the inconsistent. The paper is organized as follows: section 2 provides the necessary preliminaries used throughout the paper. section 3 contains the main results. We analyze how inconsistent and consistent IC are related to the column vector space of the finite and the infinite eigenvalues of the pencil of the system and provide a geometric connection of these two different types of IC. section 4 contains examples to justify the results of the previous section and we close the paper with section 5 and the conclusions.

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## 2. Preliminaries

Throughout the paper we will use in several parts matrix pencil theory to establish our results. A matrix pencil is a family of matrices  $sF - G$ , parametrized by a complex number  $s$ , see [10], [11].

**Definition 2.1.** Given  $F, G \in \mathbb{R}^{r \times m}$  and an arbitrary  $s \in \mathbb{C}$ , the matrix pencil  $sF - G$  is called:

1. Regular when  $r = m$  and  $\det(sF - G) \neq 0$ ;
2. Singular when  $r \neq m$  or  $r = m$  and  $\det(sF - G) \equiv 0$ .

In this article we consider the system (1) with a *regular pencil*, where the class of  $sF - G$  is characterized by a uniquely defined element, known as the Weierstrass canonical form, see [10], [11], specified by the complete set of invariants of  $sF - G$ . This is the set of elementary divisors of type  $(s - a_j)^{p_j}$ , called *finite elementary divisors*, where  $a_j$  is a finite eigenvalue of algebraic multiplicity  $p_j$  ( $1 \leq j \leq \nu$ ), and the set of elementary divisors of type  $\hat{s}^q = \frac{1}{s^q}$ , called *infinite elementary divisors*, where  $q$  is the algebraic multiplicity of the infinite eigenvalue.  $\sum_{j=1}^{\nu} p_j = p$  and  $p + q = m$ .

From the regularity of  $sF - G$ , there exist non-singular matrices  $P, Q \in \mathbb{R}^{m \times m}$  such that

$$\begin{aligned} PFQ &= \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix}, \\ PGQ &= \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}. \end{aligned} \tag{2}$$

$J_p, H_q$  are appropriate matrices with  $H_q$  a nilpotent matrix with index  $q_*$ ,  $J_p$  a Jordan matrix and  $p + q = m$ . With  $0_{q,p}$  we denote the zero matrix of  $q \times p$ . The matrix  $Q$  can be written as

$$Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \tag{3}$$

$Q_p \in \mathbb{R}^{m \times p}$  is a matrix with columns the  $p$  linear independent (generalized) eigenvectors of the  $p$  finite eigenvalues of  $sF - G$ ;  $Q_q \in \mathbb{R}^{m \times q}$  is a matrix with columns the  $q$  linear independent (generalized) eigenvectors of the  $q$  infinite eigenvalues of  $sF - G$ . Moreover note that while  $Q$  is a matrix with columns the  $m$  linear independent (generalized) eigenvectors of the  $m$  (finite and infinite) eigenvalues of  $sF - G$ , it is easy to observe that

$$\text{colspan}Q = \mathbb{R}^m. \tag{4}$$

Furthermore from (3), (4)

$$\text{colspan}Q_p \oplus \text{colspan}Q_q = \mathbb{R}^m, \tag{5}$$

where

$$\dim(\text{colspan}Q_p) = p, \quad \dim(\text{colspan}Q_q) = q$$

and  $\oplus$  is the direct sum of  $\text{colspan}Q_p$  and  $\text{colspan}Q_q$ .

**Definition 2.2.** (See [1], [4]) Let  $J_p$  be a Jordan matrix as defined in (2). Then with  $F_{n,n}(J_p(k+n)^{\bar{n}})$  we will denote the discrete Mittag-Leffler function with two parameters defined by

$$F_{n,n}(J_p(k+n)^{\bar{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{(k+n)^{\bar{i}n}}{\Gamma((i+1)n)}. \tag{6}$$

The following results have been proved.

**Theorem 2.1.** (See [5], [6], [7], [8]) We consider the system (1) with a regular pencil. Then, its solution exists if and only if all finite eigenvalues of the pencil are distinct and lie within the open disk  $S = \{s \in \mathbb{R} : |s| < 1\}$ ; Then, the solution of system (1) for  $k \geq 0$ , is given by the formula

$$Y_k = Q_p(k+1)^{\bar{n}-1} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)C.$$

Where  $C \in \mathbb{R}^p$  is a constant vector. The matrices  $Q_p, J_p$  are given by (2), (3). The discrete Mittag-Leffler function with two parameters is defined by (6).

**Definition 2.3.** Consider the system (1) with known IC. Then the IC are called consistent if there exists a solution for the system (1) which satisfies the given conditions.

**Proposition 2.1.** (See [5], [6], [7], [8]) The IC of system (1) are consistent if and only if

$$Y_0 \in \text{colspan}Q_p.$$

**Proposition 2.2.** (See [5], [6], [7], [8]) Consider the system (1) with given IC. Then if there exists a solution for the initial value problem, it is unique if and only if the IC are consistent. Then, the unique solution is given by the formula

$$Y_k = Q_p(k + 1)^{\bar{n}-1} F_{n,n}(J_p(k + n)^{\bar{n}})(I_p - J_p)Z_0^p. \tag{7}$$

Where  $Z_0^p$  is the unique solution of the linear system  $Y_0 = Q_p Z_0^p$ . For inconsistent IC ( $Y_0 \notin \text{colspan}Q_p$ ) it is has been proved that the system (1) has infinite solutions.

From the above already established results, we can conclude that if there exists solutions for system (1), then this solution is unique and given by (7) if and only if the IC are consistent. Inconsistent IC lead to infinite solutions. This makes the relation of this two different type of IC important. This relation has also been studied for singular discrete time systems, see [9]. Another known result that we will use in the next section is the orthogonal projection Theorem.

**Theorem 2.2.** (see [17]) Let  $W$  be an inner product space and let  $V$  be a finite dimensional subspace of  $W$ . Then  $\forall w \in W$  there exists unique vectors  $v_1 \in V$  and  $v_2 \in V^\perp$ , where  $V^\perp$  is the orthogonal complement of  $V$ , such that  $w = v_1 + v_2$  and  $v_1$  is the orthogonal projection of  $w$  on  $V$ , i.e.

$$v_1 = \text{proj}_V w.$$

### 3. Geometric relation between a consistent and an inconsistent initial condition

In this section we will study the relation between a consistent and an inconsistent IC of the singular fractional system (1). It has been proved (see the previous section for references) that if for the singular system (1) with known IC there exists a solution, then it is unique and given by (7) if and only if the IC lie inside the domain  $\text{colspan}Q_p$  (consistent IC). However it is possible for a system to have IC that pro exist and are not in the above mentioned domain; i.e. to be inconsistent. Then the system at  $k = 0$ , almost instantaneously is being transferred into another new situation at time  $k = 1$ , described by system (1). This phenomenon is called impulsive behavior of the system at  $k = 0$ . In order to study the relation of these two different type of IC we have to study further the case of the inconsistent IC of the system.

**Lemma 3.1.** Let  $J_p$  be a Jordan matrix as defined in (2) with  $\|J_p\| < 1$ . Then for  $k = 0$ , the discrete Mittag-Leffler function with two parameters  $F_{n,n}(J_p(k + n)^{\bar{n}})$ , defined in (6), takes the form

$$F_{n,n}(J_p(n)^{\bar{n}}) = \frac{1}{\Gamma(n)}(I_p - J_p)^{-1}.$$

**Proof.** By replacing  $k = 0$  into (6) we get

$$F_{n,n}(J_p(n)^{\bar{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{(n)^{\bar{i}n}}{\Gamma((i + 1)n)},$$

or, equivalently,

$$F_{n,n}(J_p(n)^{\bar{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{\Gamma(n + in)}{\Gamma((i + 1)n)\Gamma(n)},$$

or, equivalently,

$$F_{n,n}(J_p(n)^{\bar{n}}) = \frac{1}{\Gamma(n)} \sum_{i=0}^{\infty} J_p^i$$

and since it is assumed  $\|J_p\| < 1$ ,

$$F_{n,n}(J_p(n)^{\bar{n}}) = \frac{1}{\Gamma(n)}(I_p - J_p)^{-1}.$$

The proof is completed.

**Proposition 3.1.** Assume system (1) and let  $Y_0$  be inconsistent IC. Then if there exist solutions for (1)

$$Y_0 \in N_r \text{colspan}Q_p^{-1}. \tag{8}$$

Where  $Q_p$  is defined by (3),  $Q_p^{-1}$  is the left inverse of the matrix  $Q_p$ , i.e.  $Q_p^{-1}Q_p=I_p$  and  $N_r$  is the right kernel of the set  $\text{colspan}Q_p^{-1}$ .

**Proof.** The IC are assumed inconsistent and thus they don't satisfy (1). Hence

$$Y_0 \notin \text{colspan}Q_p.$$

From (4), (5)

$$Y_0 \in \mathbb{R}^m - \text{colspan}Q_p,$$

or, equivalently,

$$Y_0 \in \text{colspan}Q_q. \tag{9}$$

By using the transform

$$Y_k = QZ_k, \tag{10}$$

if

$$Z_k = \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix},$$

where  $Z_k^p \in \mathbb{R}^p$ ,  $Z_k^q \in \mathbb{R}^q$ , by using (3) we get

$$Y_0 = Q_p Z_0^p + Q_q Z_0^q.$$

But from (9) we have  $Z_0^p=0$  and

$$Y_0 = Q_q Z_0^q. \tag{11}$$

By replacing (10) into (1) we get

$$F \nabla_0^n Q Z_k = G Q Z_k,$$

or, equivalently,

$$F Q \nabla_0^n Z_k = G Q Z_k.$$

Whereby multiplying by  $P$  and using (2) we obtain

$$\begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix} \begin{bmatrix} \nabla_0^n Z_k^p \\ \nabla_0^n Z_k^q \end{bmatrix} = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}.$$

From above expressions, we arrive easily at the subsystems

$$\nabla_0^n Z_k^p = J_p Z_k^p \tag{12}$$

and

$$H_q \nabla_0^n Z_k^q = Z_k^q. \tag{13}$$

The subsystem (12) takes values for  $k \geq 1$  and has the solution

$$Z_k^p = (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\overline{n}})(I_p - J_p)Z_1^p, \quad \forall k \geq 1.$$

For a proof of this solution see [2]. Since  $Z_0^p=0$ , by using the Heaviside function  $H_k$ ,

$$H_k = \begin{cases} 1 & , \quad k \geq 0 \\ 0 & , \quad k < 0 \end{cases},$$

we give the solution the following form

$$Z_k^p = H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\overline{n}})(I_p - J_p)Z_1^p, \quad \forall k \geq 0, \tag{14}$$

i.e. a solution  $\forall k \geq 0$ . The subsystem (13) takes values for  $k \geq 1$  and its solution is given by

$$Z_k^q = 0_{q,1}, \quad \forall k \geq 1$$

For the proof see [5], [6], [7], [8]. But from (11)

$$Z_0^q \neq 0_{q,1}$$

and thus by using the Dirac function  $\delta_k$ ,

$$\delta_k = \begin{cases} 1 & , \quad k = 0 \\ 0 & , \quad k \neq 0 \end{cases},$$

we can have the solution of (13) in the following form

$$Z_k^q = \delta_k Z_0^q, \quad \forall k \geq 0. \tag{15}$$

Therefore the solution of system (1)  $\forall k \geq 0$  can be written as

$$Y_k = QZ_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\overline{n}})(I_p - J_p)Z_1^p \\ \delta_k Z_0^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\overline{n}})(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_k I_q \end{bmatrix} \begin{bmatrix} Z_1^p \\ Z_0^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\overline{n}})(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_k I_q \end{bmatrix} \left( \begin{bmatrix} Z_1^p \\ 0_{q,1} \end{bmatrix} + \begin{bmatrix} 0_{p,1} \\ Z_0^q \end{bmatrix} \right).$$

Let  $Y_1$  be a consistent value for the system (1). Then from Proposition 2.1

$$Y_1 \in \text{colspan} Q_p.$$

By using (11) and the above expression combined with (3), (10), i.e.  $Y_1 = Q_p Z_1^p$ , we have

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\overline{n}})(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_k I_q \end{bmatrix} Q^{-1}(Y_1 + Y_0).$$

Since it is assumed that there exists solutions for system (1), from Theorem 2.1,  $\|J_p\| < 1$ . Then for  $k = 1$  and Lemma 3.1 we obtain

$$Y_1 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (1)^{\overline{n-1}} \frac{1}{\Gamma(\overline{n})} (I_p - J_p)^{-1} (I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_1 I_q \end{bmatrix} Q^{-1}(Y_1 + Y_0),$$

or, equivalently,

$$Y_1 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} \frac{\Gamma(1+n-1)}{\Gamma(1)} \frac{1}{\Gamma(\overline{n})} (I_p - J_p)^{-1} (I_p - J_p) & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1}(Y_1 + Y_0),$$

or, equivalently,

$$Y_1 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1}(Y_1 + Y_0).$$

Let

$$Q^{-1} = \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix},$$

where  $Q_p^{-1} \in \mathbb{R}^{p \times m}$ ,  $Q_q^{-1} \in \mathbb{R}^{q \times m}$ . From  $Q^{-1}Q = I_m$  we have that the matrix  $Q_p^{-1}$  is the left inverse of the matrix  $Q_p$ , i.e.  $Q_p^{-1}Q_p = I_p$ . Then

$$Y_1 = \begin{bmatrix} Q_p & 0_{m,q} \end{bmatrix} \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix} (Y_1 + Y_0),$$

or, equivalently,

$$Y_1 = Q_p Q_p^{-1} (Y_1 + Y_0)$$

and by multiplying from the left by  $Q_p^{-1}$  we arrive at

$$Q_p^{-1} Y_0 = 0_{p,1},$$

or, equivalently,

$$Y_0 \in N_r \text{colspan} Q_p^{-1}.$$

The proof is completed.

**Proposition 3.2.** Assume system (1) and let  $Y_0$  be consistent IC. Then if there exist solutions for (1)

$$Y_0 \in N_r \text{colspan} Q_q^{-1}. \tag{16}$$

$Q_q$  is defined by (3),  $Q_q^{-1}$  is the left inverse of the matrix  $Q_q$ , i.e.  $Q_q^{-1}Q_q=I_q$  and  $N_r$  is the right kernel of the set  $colspanQ_q^{-1}$ .

**Proof.** If  $Z_k = \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}$ , where  $Z_k^p \in \mathbb{R}^p$ ,  $Z_k^q \in \mathbb{R}^q$ , by using the transform (10) and we get

$$Y_0 = Q_p Z_0^p + Q_q Z_0^q.$$

But from Proposition 2.1 we have  $Z_0^q = 0_{q,1}$  and

$$Y_0 = Q_p Z_0^p. \quad (17)$$

Let  $Y_{-1}$  be an inconsistent condition for (1). Then from Proposition 3.1 and (8)

$$Y_{-1} \in colspanQ_q$$

and  $Z_{-1}^p = 0_{p,1}$ . Thus

$$Y_{-1} = Q_q Z_{-1}^q. \quad (18)$$

By replacing (10) into (1) we get

$$F \nabla_0^n Q Z_k = G Q Z_k,$$

or, equivalently,

$$F Q \nabla_0^n Z_k = G Q Z_k.$$

Whereby multiplying by  $P$  and using (2) we obtain

$$\begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix} \begin{bmatrix} \nabla_0^n Z_k^p \\ \nabla_0^n Z_k^q \end{bmatrix} = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}.$$

From the above expressions, we arrive easily at the subsystems (12) and (13). The subsystem (12) takes values for  $k \geq 0$  and has the solution

$$Z_k^p = (k+1)^{\overline{n-1}} F_{n,n} (J_p(k+n)^{\overline{n}}) (I_p - J_p) Z_0^p, \quad k \geq 0.$$

Since  $Z_{-1}^p = 0$ , by using the Heaviside function  $H_k$  we can give to the solution the following form

$$Z_k^p = H_k Z_k^p = (k+1)^{\overline{n-1}} F_{n,n} (J_p(k+n)^{\overline{n}}) (I_p - J_p) Z_0^p, \quad \forall k \geq -1 \quad (19)$$

and thus have a solution for every  $k \geq -1$ . The subsystem (13) takes values for  $k \geq 0$  and has the solution

$$Z_k^q = 0_{q,1}, \quad k \geq 0.$$

But as we stated earlier,  $Z_{-1}^q \neq 0_{q,1}$  and thus by using the Dirac function  $\delta_k$  we can give to the solution the following form

$$Z_k^q = \delta_{k-1} Z_{-1}^q, \quad \forall k \geq -1 \quad (20)$$

and thus have a solution for every  $k \geq -1$ . Then by using (10), (19) and (20), the solution of system (1) can be written as

$$Y_k = Q Z_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n} (J_p(k+n)^{\overline{n}}) (I_p - J_p) Z_0^p \\ \delta_{k-1} Z_{-1}^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n} (J_p(k+n)^{\overline{n}}) (I_p - J_p) Z_0^p & 0_{p,q} \\ 0_{q,p} & \delta_{k-1} I_q \end{bmatrix} \begin{bmatrix} Z_0^p \\ Z_{-1}^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n} (J_p(k+n)^{\overline{n}}) (I_p - J_p) Z_0^p & 0_{p,q} \\ 0_{q,p} & \delta_{k-1} I_q \end{bmatrix} \left( \begin{bmatrix} Z_0^p \\ 0_{q,1} \end{bmatrix} + \begin{bmatrix} 0_{p,1} \\ Z_{-1}^q \end{bmatrix} \right)$$

and by using (17), (18)

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n} (J_p(k+n)^{\overline{n}}) (I_p - J_p) Z_0^p & 0_{p,q} \\ 0_{q,p} & \delta_{k-1} I_q \end{bmatrix} Q^{-1} (Y_0 + Y_{-1}).$$

Then for  $k = -1$  we obtain

$$Y_{-1} = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} 0_{p,p} & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix} Q^{-1}(Y_0 + Y_{-1})$$

and if we assume

$$Q^{-1} = \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix},$$

where  $Q_p^{-1} \in \mathbb{R}^{p \times m}$ ,  $Q_q^{-1} \in \mathbb{R}^{q \times m}$ , then from  $Q^{-1}Q = I_m$  we have that the matrix  $Q_q^{-1}$  is the left inverse of the matrix  $Q_q$ , i.e.  $Q_q^{-1}Q_q = I_q$ . Hence

$$Y_{-1} = \begin{bmatrix} 0_{m,p} & Q_q \end{bmatrix} \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix} (Y_0 + Y_{-1}),$$

or, equivalently,

$$Y_{-1} = Q_q Q_q^{-1} (Y_0 + Y_{-1}).$$

By multiplying from the left with  $Q_q^{-1}$  we get

$$Q_q^{-1}Y_0 = 0_{q,1},$$

or, equivalently,

$$Y_0 \in N_r \text{colspan} Q_q^{-1}.$$

The proof is completed.

**Theorem 3.1.** Let  $Y_0$  be a consistent condition of system (1) and  $Y_0^*$  an inconsistent. If there exist solutions for (1),  $Q$  is the a matrix as defined in (2), (3) and orthogonal, then

$$Y_0 = \text{proj}_{\text{colspan} Q_p}(Y_0 + Y_0^*), \quad (21)$$

i.e.  $Y_0$  is the orthogonal projection of  $Y_0 + Y_0^*$  on the set  $\text{colspan} Q_p$  and

$$Y_0^* = \text{proj}_{\text{colspan} Q_q}(Y_0 + Y_0^*), \quad (22)$$

i.e.  $Y_0^*$  is the orthogonal projection of  $Y_0 + Y_0^*$  on the set  $\text{colspan} Q_q$ .

**Proof.** As we already stated in (5)

$$\text{colspan} Q_p \oplus \text{colspan} Q_q = \mathbb{R}^m.$$

While  $Y_0$  is a consistent condition, from Proposition 2.1 we have that

$$Y_0 \in \text{colspan} Q_p$$

and while  $Y_0^*$  is an inconsistent condition, from (9) we have that

$$Y_0^* \in \text{colspan} Q_q.$$

Furthermore

$$Y_0 + Y_0^* \in \mathbb{R}^m.$$

Let  $Q$  be an orthogonal matrix, then  $Q^T Q = I_m$ , where  $Q^T$  is the transposed matrix of  $Q$ . If we assume

$$Q^{-1} = \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix},$$

where  $Q_p^{-1} \in \mathbb{R}^{p \times m}$ ,  $Q_q^{-1} \in \mathbb{R}^{q \times m}$ , then from  $Q^{-1}Q = I_m$  we have that  $Q_p^{-1} = Q_p^T$ ,  $Q_q^{-1} = Q_q^T$  and from Proposition 3.2

$$Y_0 \in N_r \text{colspan} Q_q^{-1},$$

i.e.

$$Y_0 \in N_r \text{colspan} Q_q^T,$$

or, equivalently,

$$\text{colspan} Q_q^T = (\text{colspan} Q_p)^\perp.$$

But  $\text{colspan} Q_q^T = \text{rowspan} Q_q$  and thus

$$\text{rowspan} Q_q = (\text{colspan} Q_p)^\perp. \quad (23)$$

From (9)

$$Y_0^* \in \text{colspan}Q_q,$$

or, equivalently,

$$(Y_0^*)^T \in \text{rowspan}Q_q.$$

Then from (5), (9), (23), Proposition 2.1 and Theorem 2.2

$$Y_0 = \text{proj}_{\text{colspan}Q_p}(Y_0 + Y_0^*)$$

and thus we proved (21). From Proposition 3.1 we have

$$Y_0^* \in N_r \text{colspan}Q_p^{-1},$$

i.e.

$$Y_0^* \in N_r \text{colspan}Q_p^T,$$

or, equivalently,

$$\text{colspan}Q_p^T = (\text{colspan}Q_q)^\perp.$$

But  $\text{colspan}Q_p^T = \text{rowspan}Q_p$  and thus

$$\text{rowspan}Q_p = (\text{colspan}Q_q)^\perp. \quad (24)$$

From Proposition 2.1

$$Y_0 \in \text{colspan}Q_p,$$

or, equivalently,

$$Y_0^T \in \text{rowspan}Q_p.$$

Then from (5), (9), (24), Proposition 2.1 and Theorem 2.2

$$Y_0^* = \text{proj}_{\text{colspan}Q_q}(Y_0 + Y_0^*)$$

and thus we proved (22). The proof is completed.

#### 4. Numerical Example

We assume the system (1) with

$$F = \frac{2}{3} \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ -2 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{4}{3} & -\frac{2}{3} & \frac{4}{3} \\ 2 & -2 & 1 \end{bmatrix}.$$

Then  $\det(sF - G) = s(s - \frac{1}{2})$  and the pencil is regular. Hence, from Theorem 2.1 there exists a solution for system (1). By calculating the eigenvectors of the finite and infinite eigenvalues we get the matrices

$$Q_p = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}, Q_q = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},$$

respectively.

*Example 4.1.*

We will begin with a simple example to justify the results of Theorem 3.1. We assume the IC

$$Y_0 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, Y_0^* = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

It is easy to observe that  $Y_0 \in \text{colspan}Q_p$  (consistent IC),  $Y_0^* \in \text{colspan}Q_q$  (inconsistent IC) and  $(Y_0 + Y_0^*)^T = [1 \quad -1 \quad 5]$ . Then  $\forall \alpha \in \mathbb{R}$  such that  $u_1 = \alpha Y_0 \in \text{colspan}Q_p$ , we have

$$\text{proj}_{\text{colspan}Q_p}(Y_0 + Y_0^*) = \frac{(Y_0 + Y_0^*)^T u_1}{\|u_1\|_2^2} u_1 = \frac{[1 \quad -1 \quad 5] \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}}{18} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = Y_0,$$

which justifies (21). In addition,  $\forall u_2 \in \text{colspan}Q_q$  we have

$$\text{proj}_{\text{colspan}Q_q}(Y_0 + Y_0^*) = \frac{(Y_0 + Y_0^*)^T u_2}{\|u_2\|_2^2} u_2 = \frac{\begin{bmatrix} 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = Y_0^*,$$

which justifies (22).

*Example 4.2.*

We assume now the IC

$$Y_0^* = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}.$$

It is easy to observe that  $Y_0^* \in \text{colspan}Q_q$ , i.e. the IC are inconsistent. We will use Theorem 3.1 to seek a consistent IC  $Y_0$  such that system (1) will have a unique solution. Let

$$Y_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad x, y, z \in \mathbb{R}.$$

From (22) and  $\forall u \in \text{colspan}Q_q$  we have

$$Y_0^* = \text{proj}_{\text{colspan}Q_q}(Y_0 + Y_0^*) = \frac{(Y_0 + Y_0^*)^T u}{\|u\|_2^2} u,$$

or, equivalently,

$$\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \frac{1}{9}(2x - 2y + z + 18) \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

or, equivalently,

$$2x - 2y + z = 0.$$

Hence

$$Y_0 \in \left\langle \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = \text{colspan}Q_p.$$

## 5. Conclusions

In this article we studied the relation between two different types of IC of a class of singular nabla fractional discrete time systems. We proved that these vectors are related to the column vector spaces of the finite and the infinite eigenvalues respectively and also that a consistent initial value (and an inconsistent initial value) can be viewed as the orthogonal projection of the sum of a consistent with an inconsistent initial value over a certain subspace.

## Acknowledgments

I. Dassios is supported by Science Foundation Ireland (award 09/SRC/E1780). The author would like to thank very much the anonymous referees for their comments that clearly improved the article.

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