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Title	On the Bernstein operator of S. Morigi and M. Neamtu
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Publication date	2009-10
Publication information	Kounchev, Ognyan, and Hermann Render. “On the Bernstein Operator of S. Morigi and M. Neamtu.” Springer, October 2009. https://doi.org/10.1007/s00025-008-0342-0.
Publisher	Springer
Item record/more information	http://hdl.handle.net/10197/5496
Publisher's statement	The final publication is available at www.springerlink.com
Publisher's version (DOI)	10.1007/s00025-008-0342-0

Downloaded 2026-05-02 01:16:19

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On the Bernstein operator of S. Morigi and M. Neamtu

O. Kounchev and H. Render

Abstract. We discuss a Bernstein type operator introduced by S. Morigi and M. Neamtu for $\mathcal{D}$-polynomials in the more general framework of exponential polynomials.

Mathematics Subject Classification (2000). Primary 41A35; Secondary 41A50.

Keywords. Bernstein polynomial, Bernstein operator, extended Chebyshev system, exponential polynomial.

1. Introduction

Let K be either the field of real or complex numbers, denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. Assume that U_n is a K-linear subspace of dimension $n+1$ of $C(I,K)$, the space of n-times continuously differentiable K-valued functions on an interval $I=[a,b]$. A system $p_{n,k}, k=0,...,n$, in U_n is called a *Bernstein basis* for $a<b\in I$, if each function $p_{n,k}$ has a zero of order k at a, and a zero of order $n-k$ at b for $k=0,...,n$. It is easily seen that a Bernstein basis is indeed a *basis* of the linear space U_n and that the basis functions $p_{n,k}$ are unique up to a non-zero factor, see e.g. the proof of Lemma 19 and Proposition 20 in [8]. The existence of Bernstein bases and their special properties have been discussed by several authors, see e.g. [3], [4], [5], [6], [9], [10], [11], [12], [14], [15], [16]. Let us recall that a K-linear subspace $U_n\subset C^n(I,K)$ of dimension $n+1$ is an *extended Chebyshev system (or space) for the subset* $A\subset I$ if each non-zero $f\in U_n$ vanishes at most n times in A, counting multiplicities. It is not difficult to prove that a Bernstein basis exists for $U_n\subset C^n(I,K)$ if and only if U_n is an extended Chebyshev system for the *set* $\{a,b\}$, see e.g. [3], [11] for the case $K=\mathbb{R}$.

The first author is partially supported by Grant MTM2006-13000-C03-03 of the D.G.I. of Spain. Both authors acknowledge support within the project "Institutes Partnership" with the Alexander von Humboldt Foundation, Bonn.

In this paper we want to discuss and compare a recent result of S. Morigi and M. Neamtu in [13] about the construction and convergence of a Bernstein operator for so-called $\mathcal{D}$-polynomials with our recent results in [1] for exponential polynomials. Let us recall that the space of *exponential polynomials* for given complex numbers $\lambda_0, ..., \lambda_n$ is defined by

$$E_{(\lambda_0,...,\lambda_n)} := \left\{ f \in C^{\infty}(\mathbb{R}, \mathbb{C}) : \left(\frac{d}{dx} - \lambda_0 \right) .... \left(\frac{d}{dx} - \lambda_n \right) f = 0 \right\}.$$

In the case that the exponents in $\lambda_0, ..., \lambda_n$ are *equidistant,* i.e., that there exists $\omega \in \mathbb{C}$ such that $\lambda_j = \lambda_0 + j\omega$ for $j = 0, ..., n$, the elements of $E_{(\lambda_0,...,\lambda_n)}$ are called $\mathcal{D}$-polynomials in [7, Remark 2.1]. Note that in the case $\omega = 0$ and $\lambda_0 = 0$, the set $E_{(\lambda_0,...,\lambda_n)}$ is the space of all polynomials of degree $\leq n$. Another important example is the class of scaled trigonometric polynomials, defined for even n by

$$\text{span}\left\{1, \sin(2x/n), \cos(2x/n), \sin(4x/n), \cos(4x/n) ....., \sin x, \cos x\right\}.$$

and $\text{span}\left\{\sin(x/n), \cos(x/n), \sin(3x/n), \cos(3x/n) ....., \sin x, \cos x\right\}$ for odd n.

In [1] we have discussed the existence of Bernstein bases for exponential spaces $E_{(\lambda_0,...,\lambda_n)}$ with complex exponents $\lambda_0, ..., \lambda_n$. We say that $E_{(\lambda_0,...,\lambda_n)}$ is *closed under complex conjugation* if $f \in E_{(\lambda_0,...,\lambda_n)}$ implies that the complex conjugate $\overline{f}$ is in $E_{(\lambda_0,...,\lambda_n)}$. If the space $E_{(\lambda_0,...,\lambda_n)}$ is closed under complex conjugation then $E_{(\lambda_0,...,\lambda_n)}$ is an extended Chebyshev system over any interval $[a, b]$ with $b - a < \pi/M_n$ where

$$M_n := \max\left\{|\text{Im}\lambda_j| : j = 0, ..., n\right\}, \tag{1.1}$$

see [1]. Therefore there exists under this assumption a Bernstein basis in $E_{(\lambda_0,...,\lambda_n)}$ for $\{a, b\}$. In particular, if $\lambda_0, ..., \lambda_n$ are real then M_n is just zero and one obtains the well known result that $E_{(\lambda_0,...,\lambda_n)}$ is an extended Chebyshev system over any interval $[a, b]$. In passing, we note that Bernstein bases for exponential spaces can be defined in a simple and recursive way, for details see [1].

In this paper we shall consider the case of equidistant exponents, i.e. $\lambda_j = \lambda_0 + j\omega_n$ for $j = 0, ..., n$. If $\omega_n \neq 0$ one may try to define a Bernstein basis for $[a, b]$ directly by the expression

$$p_{(\lambda_0,...,\lambda_n),k}(x) := \frac{e^{\lambda_0(x-a)}}{k!\omega_n^k} \left(e^{\omega_n(x-a)} - 1\right)^k \left(\frac{1 - e^{\omega_n(x-b)}}{1 - e^{\omega_n(a-b)}}\right)^{n-k}. \tag{1.2}$$

It is easy to see that $p_{(\lambda_0,...,\lambda_n),k}$ is indeed an element in the exponential space $E_{(\lambda_0,...,\lambda_n)}$: clearly $p_{(\lambda_0,...,\lambda_n),k}$ is a sum of elements of the form

$$A_{s,t} e^{\lambda_0 x} e^{s\omega_n x} e^{t\omega_n x} \tag{1.3}$$

where $A_{s,t}$ is a constant and $s \in \{0, ..., k\}$ and $t \in \{0, ..., n-k\}$; obviously elements of the form (1.3) are in $E_{(\lambda_0,...,\lambda_n)}$. Moreover $p_{(\lambda_0,...,\lambda_n),k}$ has a zero in a of order at least k and in b of order at least $n - k$. However, the definition of a Bernstein bases requires that $p_{n,k}$ has a zero in a of exact order k and in b of exact order

$n-k$. So in order to guarantee the existence of a Bernstein basis for equidistant exponents one has only to require that

$$e^{\omega_n(b-a)} \neq 1.$$

By Proposition 4 in [1], the existence of a Bernstein basis consisting of *real-valued* functions is equivalent to the property that $E_{(\lambda_0,\ldots,\lambda_n)}$ is closed under complex conjugation.

In [13] S. Morigi and M. Neamtu introduced a Bernstein operator based on $\mathcal{D}$-polynomials. At first they consider the null space $\mathcal{D}$ of a differential operator of the form

$$L := \frac{d^2}{dx^2} + \gamma \frac{d}{dx} + \delta \text{ with } \gamma, \delta \in \mathbb{R}.$$

Writing

$$L = \left(\frac{d}{dx} - \mu_0\right)\left(\frac{d}{dx} - \mu_1\right)$$

with complex numbers μ_0, μ_1 one sees that either μ_0, μ_1 are both real or $\mu_0 = \overline{\mu_1}$ and $\mu_0 \notin \mathbb{R}$. They introduce the function

$$d(x) = \begin{cases} \left(e^{\mu_1 x} - e^{\mu_0 x}\right) / \left(\mu_1 - \mu_0\right), & \text{for } \mu_0 \neq \mu_1 \\ x e^{\mu_0 x} & \text{for } \mu_0 = \mu_1, \end{cases}$$

and the space of $\mathcal{D}$-polynomials of degree $\leq n$ is defined by

$$\mathcal{D}_n := \operatorname{span}\left\{d^n\left((\cdot - t)/n\right) : t \in \mathbb{R}\right\}.$$

Note that the space $\mathcal{D}_n$ is closed under complex conjugation since $\overline{d}$ is in $\mathcal{D}_n$. For given μ_0, μ_1 define $\omega_n := \frac{1}{n}(\mu_1 - \mu_0)$ and the equidistant exponents

$$\lambda_j = \mu_0 + j\omega_n \text{ for } j = 0, ..., n. \tag{1.4}$$

Then it is not difficult to see that $\mathcal{P}_n$ is equal to $E_{(\lambda_0,\ldots,\lambda_n)}$ with exponents defined by (1.4). In the case that $\mu_1 = \mu_0$, so $\omega_n = 0$, one obtains the space of all polynomials of degree $\leq n$ multiplied by $e^{\lambda_0 x}$. In the sequel we shall assume that $\omega_n \neq 0$, since the case $\omega_n = 0$ is covered by the classical Bernstein operator for polynomials.

If $\mu_0 \neq \mu_1$ the *Bernstein operator of S. Morigi and M. Neamtu* is defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(a + \frac{k}{n}(b-a)\right) \frac{n!}{(n-k)!} \frac{\omega_n^k e^{-\lambda_0\left(\frac{k}{n}(b-a)\right)}}{\left(e^{\omega_n(b-a)} - 1\right)^k} p_{(\lambda_0,\ldots,\lambda_n),k}(x) \tag{1.5}$$

for $f \in C[a,b]$ where $p_{(\lambda_0,\ldots,\lambda_n),k}$ is defined in (1.2). According to Proposition 3.1 in [13] the Bernstein operator B_n has the interesting property that

$$B_n\left(e^{\mu_0 x}\right) = e^{\mu_0 x} \text{ and } B_n\left(e^{\mu_1 x}\right) = e^{\mu_1 x}.$$

Moreover it is shown that $B_n f$ converges uniformly to f for each $f \in C[a,b]$ provided that μ_0 and μ_1 are either real or $\overline{\mu_1} = \mu_0 \notin \mathbb{R}$ and $b - a < \pi / |\mathrm{Im}\mu_0|$.

In [1] we have shown that an analogous Bernstein operator B_n can be introduced in the general setting of exponential polynomials, or more recently, for extended Chebyshev spaces in [2]. Indeed, if we suppose that a space U_n of dimension $n+1$ possesses a Bernstein basis $p_{n,k}, k=0,...,n$, one may try to define a Bernstein operator as an operator of the form

$$B_n f = \sum_{k=0}^{n} f(t_k)\, \alpha_k p_{n,k} \tag{1.6}$$

with the property that two given function $f_0, f_1 \in U_n$ are fixed by B_n, i.e. that

$$B_n f_0 = f_0 \text{ and } B_n f_1 = f_1. \tag{1.7}$$

In the first section we shall survey some recent results concerning this question. In the second section we want to illustrate the results for the case that the exponents are equidistant arriving at the construction of Morigi and Neamtu. The main aim of the paper is to demonstrate that a sufficient criterion in [1] for the convergence of Bernstein operators to the identity (see Theorem 2.3) can be used for the case of equidistant exponents, giving an alternative proof of the convergence result of Morigi and Neamtu in the case that $\mu_0 \neq \mu_1$ are real.

2. Bernstein operators for exponential polynomials

The assumption in Theorem 2.1 below, taken from [1], namely that the length of the interval $[a,b]$ is smaller than π/M_n, implies that $E_{(\lambda_0,...,\lambda_n)}$, $E_{(\lambda_1,...,\lambda_n)}$ and $E_{(\lambda_0,\lambda_2...,\lambda_n)}$ are extended Chebyshev spaces over the interval $[a,b]$. In particular there exists a Bernstein basis $p_{(\lambda_0,...,\lambda_n),k}, k=0,...,n$, in $E_{(\lambda_0,...,\lambda_n)}$ for $\{a,b\}$ and we shall assume without loss of generality that it satisfies the condition

$$k! \lim_{x\to a, x>a} \frac{p_{(\lambda_0,...,\lambda_n),k}(x)}{(x-a)^k} = p^{(k)}_{(\lambda_0,....,\lambda_n),k}(a) = 1. \tag{2.1}$$

Similarly there exists a Bernstein basis $p_{(\lambda_1,...,\lambda_n),k}, k=0,...,n-1$ for the space $E_{(\lambda_1,...,\lambda_n)}$ and a Bernstein basis $p_{(\lambda_0,\lambda_2,...,\lambda_n),k}$ for the space $E_{(\lambda_0,\lambda_2...,\lambda_n)}$ with the corresponding norming condition. In the next result these bases are needed for defining the nodes $t_k \in [a,b]$. In the next section we shall see that the Bernstein operator of S. Morigi and M. Neamtu for real values $\mu_0 \neq \mu_1$ is a special case of the following construction.

Theorem 2.1. *Let $\lambda_0,...,\lambda_n$ be complex numbers with λ_0 and λ_1 real and $\lambda_0 < \lambda_1$. Suppose $E_{(\lambda_0,...,\lambda_n)}$ is closed under complex conjugation and $0 < b-a < \pi/M_n$, where M_n is defined in (1.1). Define inductively points $t_0,...,t_n$ by setting $t_0 = a$ and*

$$e^{(\lambda_0-\lambda_1)(t_k-t_{k-1})} = \lim_{x\to b} \frac{p_{(\lambda_0,\lambda_2,...,\lambda_n),k-1}(x)}{p_{(\lambda_1,...,\lambda_n),k-1}(x)}$$

for $k=1,2,...,n$. Then

$$a = t_0 < t_1 < .... < t_n = b.$$

Put $\alpha_0 = 1$, *and define numbers*

$$\alpha_k = e^{-\lambda_0 (t_k - a)} (-1)^k \prod_{l=0}^{k-1} \lim_{x \to b} \frac{\frac{d}{dx} p_{(\lambda_0, \ldots, \lambda_n), l}(x)}{p_{(\lambda_1, \ldots, \lambda_n), l}(x)} \tag{2.2}$$

for $k = 1, ..., n$. *Then* $\alpha_0, ..., \alpha_n > 0$ *and the operator* B_n *defined on* $C[a,b]$ *by*

$$B_n f = \sum_{k=0}^{n} f(t_k) \alpha_k p_{(\lambda_0, \ldots, \lambda_n), k}$$

*(*1.6*) satisfies the equations*

$$B_n \left(e^{\lambda_0 x} \right) = e^{\lambda_0 x} \text{ and } B_n \left(e^{\lambda_1 x} \right) = e^{\lambda_1 x}.$$

Next we recall from [1] a sufficient condition for the Bernstein operator B_n to converge to the identity. At first we need the following

Definition 2.2. For each $n \in \mathbb{N}$, let $\{a(n,k) : k = 0, ..., n\}$ be a triangular array of complex numbers. We say that $a(n,k)$ converges uniformly to c if for each $\varepsilon > 0$ there exists a natural number n_0 such that $|a(n,k) - c| < \varepsilon$, for all $n \geq n_0$ and all $k = 0, ..., n$.

Theorem 2.3. *Let* $\lambda_0, \lambda_1, \lambda_2$ *be distinct real numbers and let* $\Lambda_n = (\lambda_0, \lambda_1, ..., \lambda_n)$, *where for* $j = 3, ..., n$ *the complex numbers* λ_j *are allowed to vary. Suppose each* $E_{(\lambda_0, \ldots, \lambda_n)}$ *is closed under complex conjugation, and furthermore there exists a positive number* M *such that for every* $n \geq 2$ *and every* $j = 0, ..., n$, *we have* $|Im \lambda_j| \leq M$. *For each* $k \leq n$ *set*

$$a(n,k) = \lim_{x \to b} \frac{p_{(\lambda_0, \lambda_2, ..., \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, ..., \lambda_n), k}(x)}, \tag{2.3}$$

$$b(n,k) = \lim_{x \to b} \frac{p_{(\lambda_0, \lambda_1, \lambda_3, ..., \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, ..., \lambda_n), k}(x)}. \tag{2.4}$$

Let t_k, $k = 0, \ldots, n$, *be the uniquely determined points given by Theorem* 2.1. *Assume that*

$$\lim_{n \to \infty} t_k - t_{k-1} = 0 \tag{2.5}$$

uniformly in k, *and likewise, that*

$$\lim_{n \to \infty} \frac{\log b(n,k)}{t_k - t_{k+1}} = \lambda_2 - \lambda_0 \tag{2.6}$$

uniformly in k. *Then the Bernstein operator* B_n *defined in Theorem* 2.1, *converges to the identity operator on* $C([a,b], \mathbb{C})$ *with the uniform norm.*

Remark 2.4. It is not difficult to see that the assumptions (2.5) and (2.6) are equivalent to say that $a(n,k)$ and $b(n,k)$ converge uniformly to 1, and that

$$\frac{1 - b(n,k)}{1 - a(n,k)} \to \frac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0}. \tag{2.7}$$

3. A proof of the convergence result of Morigi and Neamtu

Let $\mu_0 \neq \mu_1$ be complex numbers. Define $\omega_n := \frac{1}{n}(\mu_1 - \mu_0)$ and the equidistant exponents $\lambda_j = \mu_0 + j\omega_n$ for $j = 0, ..., n$. Note that

$$\lambda_0 = \mu_0 \text{ and } \lambda_n = \mu_1.$$

Assuming that $e^{\omega_n(b-a)} \neq 1$ one can define a Bernstein basis for the equidistant exponents $\lambda_j = \mu_0 + j\omega_n$ for $j = 0, ..., n$ by

$$p_{(\lambda_0,...,\lambda_n),k}(x) := \frac{e^{\lambda_0(x-a)}}{k!\omega_n^k}\left(e^{\omega_n(x-a)} - 1\right)^k \left(\frac{1 - e^{\omega_n(x-b)}}{1 - e^{\omega_n(a-b)}}\right)^{n-k} \tag{3.1}$$

The factor in (3.1) ensures that condition (2.1) is fulfilled:

$$k! \lim_{x\to a} p_{(\lambda_0,...,\lambda_n),k}(a) / (x-a)^k = 1. \tag{3.2}$$

Since the exponents $\lambda_1, ..., \lambda_n$ are also equidistant with the same width ω_n one obtains a Bernstein basis for the space $E_{(\lambda_1,...,\lambda_n)}$ by

$$p_{(\lambda_1,...,\lambda_n),k}(x) = \frac{e^{\lambda_1(x-a)}}{k!\omega_n^k}\left(e^{\omega_n(x-a)} - 1\right)^k \left(\frac{1 - e^{\omega_n(x-b)}}{1 - e^{\omega_n(a-b)}}\right)^{n-1-k}.$$

Similarly we have

$$p_{(\lambda_0,....,\lambda_{n-1}),k}(x) = \frac{e^{\lambda_0(x-a)}}{k!\omega_n^k}\left(e^{\omega_n(x-a)} - 1\right)^k \left(\frac{1 - e^{\omega_n(x-b)}}{1 - e^{\omega_n(a-b)}}\right)^{n-1-k}.$$

A straightforward calculation shows that

$$d_{k-1} := \lim_{x\to b} \frac{\frac{d}{dx}p_{(\lambda_0,...,\lambda_n),k}(x)}{p_{(\lambda_1,...,\lambda_n),k}(x)} = -\frac{(n-k)\,\omega_n}{1 - e^{\omega_n(a-b)}} e^{(b-a)(\lambda_0-\lambda_1)} \tag{3.3}$$

$$D_{k-1} : \lim_{x\to b} \frac{\frac{d}{dx}p_{(\lambda_0,...\lambda_n),k}(x)}{p_{(\lambda_0,....,\lambda_{n-1}),k}(x)} = -\frac{(n-k)\,\omega_n}{1 - e^{\omega_n(a-b)}}. \tag{3.4}$$

Thus we see that

$$\lim_{x\to b} \frac{p_{(\lambda_0,....,\lambda_{n-1}),k-1}(x)}{p_{(\lambda_1,...,\lambda_n),k-1}(x)} = \frac{d_{k-1}}{D_{k-1}} = e^{(b-a)(\lambda_0-\lambda_1)}. \tag{3.5}$$

Next we want to apply Theorem 2.1 and for this reason we shall require that μ_0 and μ_1 are real. We remark that Theorem 2.1 could be generalized to the case of complex conjugates $\mu_1 = \overline{\mu_0}$ (compare Theorem 4.3) with the same type of calculations.

Proposition 3.1. *Assume that $\mu_0 \neq \mu_1$ are real numbers and $\omega_n := (\mu_1 - \mu_0)/n$. Let $\lambda_j = \lambda_0 + j\omega_n$ for $j = 0, ..., n$. Define $t_k := a + \frac{k}{n}(b-a)$. Then the operator defined by*

$$B_n f(x) = \sum_{k=0}^{n} f(t_k) \frac{n!}{(n-k)!} \frac{\omega_n^k e^{-\lambda_0\left(\frac{k}{n}(b-a)\right)}}{\left(e^{\omega_n(b-a)} - 1\right)^k} p_{(\lambda_0,...,\lambda_n),k}(x)$$

for $f \in C[a,b]$ *satisfies*

$$B_n\left(e^{\mu_0 x}\right) = e^{\mu_0 x} \text{ and } B_n\left(e^{\mu_1 x}\right) = e^{\mu_1 x}.$$

Proof. By Theorem 2.1, applied to the exponents $\lambda_0 = \mu_0$ and $\lambda_n = \mu_1$, the nodes t_k are defined by the equation

$$e^{(\lambda_0-\lambda_n)(t_k-t_{k-1})} = \lim_{x\to b} \frac{p_{(\lambda_0,\ldots,\lambda_{n-1}),k-1}(x)}{p_{(\lambda_1,\ldots,\lambda_n),k-1}(x)} = e^{(b-a)(\lambda_0-\lambda_1)}, \tag{3.6}$$

so we have

$$t_k - t_{k-1} = \frac{\lambda_0-\lambda_1}{\lambda_0-\lambda_n}(b-a) = \frac{1}{n}(b-a).$$

It follows that $t_k = a + \frac{k}{n}(b-a)$. According to (2.2) and (3.3) we have

$$\alpha_k = e^{-\lambda_0(t_k-a)}(-1)^k d_0....d_{k-1} = e^{-\lambda_0\left(\frac{k}{n}(b-a)\right)} \frac{e^{(a-b)k\omega_n}\omega_n^k}{\left(1-e^{\omega_n(a-b)}\right)^k} \frac{n!}{(n-k)!}.$$

□

The following lemma will be needed later:

Lemma 3.2. *Define* $F_{2n,k}(z) = (z\alpha-1)^k (z\beta-1)^{2n-k-1} = \sum_{l=0}^{2n-1} f_{2n,k}(l) z^l$. *Then for* $k \le n-1$

$$f_{2n,k}(n) = \frac{(-1)^{n-1}\beta^n k!}{n!} \frac{(2n-k-1)!}{(n-1)!} \sum_{l=0}^{k} \binom{n}{l}\binom{n-1}{k-l}\left(\frac{\alpha}{\beta}\right)^l$$

$$f_{2n,k}(n-1) = \frac{(-1)^n \beta^{n-1} k!}{n!} \frac{(2n-k-1)!}{(n-1)!} \sum_{l=0}^{k} \binom{n-1}{l}\binom{n}{k-l}\left(\frac{\alpha}{\beta}\right)^l.$$

Proof. Clearly the n-th coefficient $f_{2n,k}(n)$ of $F_{2k,n}$ is given by $F_{2n,k}^{(n)}(0)/n!$. According to the Leibniz rule we obtain

$$f_{2n,k}(n) = \frac{1}{n!}F_{2n,k}^{(n)}(0) = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}\frac{d^l}{dz^l}(z\alpha-1)^k \frac{d^{n-l}}{dz^{n-l}}(z\beta-1)^{2n-k-1}|_{z=0}.$$

Note that the summation over l is trivial for $l > k$, and one obtains that

$$f_{2n,k}(n) = \frac{1}{n!}\sum_{l=0}^{k}\binom{n}{l}\frac{k!}{(k-l)!}\alpha^l(-1)^{k-l}\frac{(2n-k-1)!}{(n-k-1+l)!}\beta^{n-l}(-1)^{n-k-1+l}.$$

The case $f_{2n,k}(n-1)$ is similar. □

Now we prove the result of S. Morigi and M. Neamtu for the case of real numbers $\mu_0 \neq \mu_1$. The result is also valid for complex conjugates $\mu_1 = \overline{\mu_0} \notin \mathbb{R}$ with the requirement that $b-a < \pi/|Im\mu_0|$, see [13, p. 137]. For technical reasons we consider only the case

$$\Lambda_{2n} = (\lambda_0, \lambda_1, ..., \lambda_{2n})$$

(so the last index is even) which guarantees that λ_n occurs as a component of Λ_{2n} for every n.

Theorem 3.3. *(S. Morigi, M. Neamtu) Let $\mu_0 \neq \mu_1$ be real numbers and define $\lambda_j = \mu_0 + j\frac{1}{2n}(\mu_1 - \mu_0)$ for $j = 0, ..., 2n$. Then $B_{(\lambda_0,...,\lambda_{2n})} f(x)$ converges uniformly to f for all $f \in C[a,b]$.*

Proof. 1. We apply Theorem 2.3 to the triple $\lambda_0 = \mu_0, \lambda_{2n} = \mu_1$ and $\lambda_n = \frac{1}{2}(\mu_1 + \mu_0)$ (instead of $\lambda_0, \lambda_1, \lambda_2$). Note that

$$t_{k+1}(2n) - t_k(2n) = \frac{1}{2n}(b-a)$$

converges uniformly in k to 0 for $n \to \infty$. By formula (3.6) this is equivalent to say that

$$a(n,k) := \lim_{x\to b} \frac{p_{(\lambda_0,,...,\lambda_{2n-1}),k}(x)}{p_{(\lambda_1,...,\lambda_{2n}),k}(x)} = e^{(\lambda_0-\lambda_{2n})(t_{k+1}-t_k)}$$

converges uniformly to 1.

2. By $\Lambda_{2n} \setminus \lambda_n$ we denote the vector where we have deleted the number λ_n. By Theorem 14 in [1] there exists a constant $C_k^{\lambda_0,\lambda_n} \neq 0$ such that

$$p_{\Lambda_{2n}\setminus\lambda_n,k}(x) - p_{(\lambda_1,\lambda_2,....,\lambda_{2n}),k}(x) = C_k^{\lambda_0,\lambda_n} p_{\Lambda_{2n},k+1}(x). \tag{3.7}$$

It follows that

$$b(n,k) := \lim_{x\to b} \frac{p_{\Lambda_{2n}\setminus\lambda_n,k}(x)}{p_{(\lambda_1,\lambda_2,....,\lambda_{2n}),k}(x)} = 1 + C_k^{\lambda_0,\lambda_n} \lim_{x\to b} \frac{p_{\Lambda_{2n},k+1}(x)}{p_{(\lambda_1,\lambda_2,....,\lambda_{2n}),k}(x)}.$$

Furthermore, by (1.2) and the definition of $p_{\Lambda_{2n},k+1}(x)$

$$\frac{p_{\Lambda_{2n},k+1}(x)}{p_{(\lambda_1,\lambda_2,....,\lambda_{2n}),k}(x)} = \frac{e^{(\lambda_0-\lambda_1)(x-a)}}{(k+1)\,\omega_{2n}} \left(e^{\omega_{2n}(x-a)} - 1\right).$$

Hence

$$b(n,k) = 1 + C_k^{\lambda_n,\lambda_0} \frac{1}{k+1} \frac{e^{-\omega_{2n}(b-a)}}{\omega_{2n}} \left(e^{\omega_{2n}(b-a)} - 1\right).$$

3. Now we determine $C_k^{\lambda_0,\lambda_n}$ from equation (3.7): we expand the functions occuring in (3.7) according to the basis $e^{\lambda_j x} \in E_{(\lambda_0,...,\lambda_{2n})}$ for $j = 0, ..., 2n$ and compare the coefficient for the basis function $e^{\lambda_n x}$. Since $p_{\Lambda_{2n}\setminus\lambda_n,k} \in E_{\Lambda_{2n}\setminus\lambda_n}$ it is clear that the coefficient of $p_{\Lambda_{2n}\setminus\lambda_n,k}$ is zero.

In order to keep the notation simpler, let us put

$$\alpha_n = e^{-a\omega_{2n}} \text{ and } \beta_n = e^{-\omega_{2n} b} \tag{3.8}$$

and consider the polynomial

$$F_{2n,k}(z) = (z\alpha_n - 1)^k (z\beta_n - 1)^{2n-k-1}$$

defined in Lemma 3.2. Then with $z = e^{\omega_{2n}x}$ we have

$$p_{\Lambda_{2n},k+1}(x) = \frac{e^{\lambda_0(x-a)}}{(k+1)!\omega_{2n}^{k+1}\left(1-e^{\omega_{2n}(a-b)}\right)^{2n-k-1}}(z\alpha_n - 1)F_{2n,k}(z)$$

$$p_{(\lambda_1,\lambda_2,\dots,\lambda_{2n}),k}(x) = \frac{e^{\lambda_0(x-a)}e^{-\omega_{2n}a}}{k!\omega_{2n}^{k}\left(1-e^{\omega_{2n}(a-b)}\right)^{2n-k-1}}zF_{2n,k}(z).$$

The n-th coefficient of $(z\alpha_n - 1)F_{2n,k}(z)$ is equal to $\alpha_n f_{2n,k}(n-1) - f_{2n,k}(n)$ where $f_{2n,k}(n)$ is defined in Lemma 3.2. Hence

$$C_k^{\lambda_0,\lambda_n} = -e^{-\omega_{2n}a}(k+1)\omega_{2n}\frac{f_{2n,k}(n-1)}{\alpha_n f_{2n,k}(n-1) - f_{2n,k}(n)}$$

and

$$1 - b(n,k) = e^{-\omega_{2n}b}\left(e^{\omega_{2n}(b-a)} - 1\right)\frac{f_{2n,k}(n-1)}{\alpha_n f_{2n,k}(n-1) - f_{2n,k}(n)}.$$

Note that $\alpha_n f_{2n,k}(n-1)$ and $f_{2n,k}(n)$ have opposite signs by the formulas in Lemma 3.2. Thus we obtain

$$|\alpha f_{2n,k}(n-1) - f_{2n,k}(n)| \geq \alpha |f_{2n,k}(n-1)|.$$

Then

$$|b(n,k) - 1| \leq e^{-\omega_{2n}b}\cdot\left|e^{\omega_{2n}(b-a)} - 1\right|\cdot\frac{1}{\alpha},$$

and we conclude that $b(n,k)$ converges uniformly to 1.

4. Next we consider

$$\frac{1-a(n,k)}{1-b(n,k)} = e^{\omega_{2n}b}\frac{1-e^{-\omega_{2n}(b-a)}}{e^{\omega_{2n}(b-a)} - 1}\left(\alpha_n - \frac{f_{2n,k}(n)}{f_{2n,k}(n-1)}\right). \tag{3.9}$$

The first two factors on the right hand side converge to 1. By (3.8) α_n converges to 1. Hence, if we prove that

$$\frac{f_{2n,k}(n)}{f_{2n,k}(n-1)} \tag{3.10}$$

converges to -1 then (3.9) converges to $2 = (\lambda_{2n} - \lambda_0)/(\lambda_n - \lambda_0)$. By Remark 2.4 and Theorem 2.3 we conclude that B_n converges to the identity operator. In order to show the convergence of (3.10) we define

$$g_k(r,s,x) := \sum_{l=0}^{k}\binom{r}{l}\binom{s}{k-l}x^l.$$

Then

$$\frac{f_{2n,k}(n)}{f_{2n,k}(n-1)} = -\beta\frac{g_k\left(n, n-1, \frac{\alpha}{\beta}\right)}{g_k\left(n-1, n, \frac{\alpha}{\beta}\right)}.$$

The following identity is elementary:

$$(1+xy)^r(1+y)^s = \sum_{k=0}^{\infty}g_k(r,s,x)y^k.$$

Now we consider for a fixed x the polynomial

$$Q_n(y) := (1+xy)^{n-1}(1+y)^{n-1} = \sum_{k=0}^{2n-2} q_k(x)\, y^k.$$

Clearly the coefficients $q_k(x)$ are non-negative. Since $g_k(n, n-1, x)$ is the k-th coefficient of $(1+xy)\, Q_n(y)$ it follows that

$$g_k(n, n-1, x) = q_k(x) + x q_{k-1}(x).$$

Similarly $g_k(n-1, n, x)$ is the k-th coefficient of $(1+y)\, Q_n(y)$, so we have

$$g_k(n-1, n, x) = q_k(x) + q_{k-1}(x).$$

Thus

$$\left| \frac{g_k(n, n-1, x)}{g_k(n-1, n, x)} - 1 \right| \leq \frac{|x-1|\, q_{k-1}(x)}{q_k(x) + q_{k-1}(x)} \leq |x-1|.$$

Since $x = \omega_{2n}$ converges to 1 (independent of k) we see that for $k \leq n-1$

$$\frac{1 - a(n,k)}{1 - b(n,k)} \to 2 = \frac{\lambda_{2n} - \lambda_0}{\lambda_n - \lambda_0}$$

uniformly. The case $n - 1 < k \leq 2n$ follows by a symmetry argument. □

4. Bernstein operators for Chebyshev spaces

In this secton we survey some results from [2]. We need the following notation: for a strictly positive function $f_0 \in U_n \subset C^n[a,b]$ we define the *space of derivatives modulo* f_0 by

$$D_{f_0} U_n := \left\{ \frac{d}{dx}\left(\frac{f}{f_0}\right) : f \in U_n \right\}$$

which is clearly a linear space of dimension n. A Bernstein basis $p_{n,k}, k = 0, ..., n$, in a subspace $U_n \subset C^n[a,b]$ is called *non-negative*, if $p_{n,k}(x) \geq 0$ for all $x \in [a,b]$ and $k = 0, ..., n$. It is easy to see that a non-negative Bernstein basis exists if U_n is an extended Chebyshev system over the *interval* $[a,b]$ which is closed under complex conjugation. The following result is proved in [2]:

Theorem 4.1. *Assume that U_n possesses a non-negative Bernstein basis $p_{n,k}, k = 0, ..., n$ for $\{a,b\} \subset I$, $a < b$. Let $f_0 \in U_n$ be strictly positive, suppose $f_1 \in U_n$ has the property that f_1/f_0 is strictly increasing, and assume that $D_{f_0}U_n$ possesses a non-negative Bernstein basis $q_{n-1,k}, k = 0, ..., n-1$. If the coefficients $w_k, k = 0, ..., n-1$, defined by*

$$\frac{d}{dx}\frac{f_1}{f_0} = \sum_{k=0}^{n-1} w_k q_{n-1,k} \tag{4.1}$$

are non-negative, then there exist unique points $t_0, ..., t_n \in [a, b]$ *with* $t_0 = a$ *and* $t_n = b$ *and unique positive coefficients* $\alpha_0, ..., \alpha_n$*, such that the operator*

$$B_n f = \sum_{k=0}^{n} f(t_k) \alpha_k p_{n,k} \tag{4.2}$$

satisfies $B_n f_0 = f_0$ *and* $B_n f_1 = f_1$.

It follows from the above construction that B_n defined in Theorem 4.1 is a *positive* operator. From the last theorem one may derive the following two results for Bernstein operators in the framework of exponential polynomials:

Theorem 4.2. *Let* λ_0, λ_1 *be real numbers and* $f_0(x) = e^{\lambda_0 x}$ *and* $f_1(x) = e^{\lambda_1 x}$ *if* $\lambda_1 \neq \lambda_0$ *and* $f_1(x) = xe^{\lambda_0 x}$ *if* $\lambda_1 = \lambda_0$. *Suppose that* $E_{(\lambda_2,\ldots,\lambda_n)}$ *is an extended Chebyshev space for* $[a, b]$ *closed under complex conjugation. Then there exist unique points* $t_0, ..., t_n \in [a, b]$, *and unique positive coefficients* $\alpha_0, ..., \alpha_n$, *such that the operator* $B_n : C[a, b] \to E_{(\lambda_0,\ldots,\lambda_n)}$ *defined by (*1.6*) satisfies the equations* $B_n f_0 = f_0$ *and* $B_n f_1 = f_1$.

Theorem 4.3. *Let* $\lambda_0, ..., \lambda_n$ *be complex numbers such that* λ_0 *is not real and* $\lambda_1 = \overline{\lambda_0}$. *Assume that the spaces* $E_{(\lambda_0,\ldots,\lambda_n)}$, $E_{(\lambda_2,\ldots,\lambda_n)}$ *and* $E_{(\lambda_0,\lambda_1)}$ *are extended Chebyshev spaces over* $[a, b]$ *closed under complex conjugation. Then there exist unique points* $t_0, ...t_n \in [a, b]$ *and unique positive constants* $\alpha_0, ..., \alpha_n$ *such that the operator* $B_n : C[a, b] \to U_n$ *defined by (*1.6*) satisfies the equations* $B_n\left(e^{\lambda_0 x}\right) = e^{\lambda_0 x}$ *and* $B_n\left(e^{\overline{\lambda_0} x}\right) = e^{\overline{\lambda_0} x}$.

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