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# Existence of universal Taylor series for non-simply connected domains

Stephen J. Gardiner

## Abstract

It is known that, for any simply connected proper subdomain  $\Omega$  of the complex plane and any point  $\zeta$  in  $\Omega$ , there are holomorphic functions on  $\Omega$  that possess “universal” Taylor series expansions about  $\zeta$ ; that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in  $\mathbb{C} \setminus \Omega$  that have connected complement. This paper shows, for non-simply connected domains  $\Omega$ , how issues of capacity, thinness and topology affect the existence of holomorphic functions on  $\Omega$  that have universal Taylor series expansions about a given point.

## 1 Introduction

Throughout this paper  $\Omega$  denotes a proper subdomain of the complex plane  $\mathbb{C}$  and  $\zeta$  denotes a point of  $\Omega$ . A holomorphic function  $f$  on  $\Omega$  is said to belong to the collection  $U(\Omega, \zeta)$ , of functions with universal Taylor series expansions about  $\zeta$ , if the partial sums

$$S_N(f, \zeta)(z) = \sum_{n=0}^N \frac{f^{(n)}(\zeta)}{n!} (z - \zeta)^n$$

have the following property:

*for every compact set  $K \subset \mathbb{C} \setminus \Omega$  with connected complement and every function  $g$  which is continuous on  $K$  and holomorphic on  $K^\circ$ , there is a subsequence  $(S_{N_k}(f, \zeta))$  that converges to  $g$  uniformly on  $K$ .*

Nestoridis [19], [20] has shown that  $U(\Omega, \zeta) \neq \emptyset$  for any simply connected domain  $\Omega$  and any point  $\zeta \in \Omega$ . (The corresponding result, in which  $K$  is

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required to be disjoint from  $\overline{\Omega}$ , had previously been established by Luh [14] and Chui and Parnes [4].) The same author further showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains  $\Omega$ ; that is,  $U(\Omega, \zeta)$  is a dense  $G_\delta$  subset of the space  $H(\Omega)$  of all holomorphic functions on  $\Omega$  endowed with the topology of local uniform convergence. (See also [16].)

Attention has more recently been directed to the following:

**General existence question:** *For which non-simply connected domains  $\Omega$  and points  $\zeta \in \Omega$  do we have  $U(\Omega, \zeta) \neq \emptyset$ ?*

Melas [15] has shown that  $U(\Omega, \zeta) \neq \emptyset$  for any  $\zeta \in \Omega$  whenever  $\mathbb{C} \setminus \Omega$  is compact and connected. Another result in the same paper tells us that  $U(\Omega, \zeta) \neq \emptyset$  for any  $\zeta \in \Omega$  whenever  $\mathbb{C} \setminus \Omega$  is a discrete set. In each of these cases  $U(\Omega, \zeta)$  was shown to be a dense  $G_\delta$  subset of  $H(\Omega)$ . On the other hand, Müller, Vlachou and Yavrian [17] have shown, for non-simply connected domains  $\Omega$ , that thinness of the complementary set  $\mathbb{C} \setminus \Omega$  at infinity is necessary for  $U(\Omega, \zeta)$  to be non-empty. They conjectured that this condition is also sufficient, but it has recently been shown [10] that  $U(\Omega, 0) = \emptyset$  when  $\Omega$  is a domain of the form  $\mathbb{C} \setminus (L \cup \{1\})$  and  $L$  is a non-degenerate continuum in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . Here  $\mathbb{D} = D(0, 1)$  and  $D(\xi, r) = \{z \in \mathbb{C} : |z - \xi| < r\}$ . A selection of other results concerning universal Taylor series for non-simply connected domains may be found in [2], [3], [5], [6], [7], [12], [21], [23], [24], [25], [26].

The main purpose of this paper is to show how issues of (logarithmic) capacity, thinness and Euclidean topology affect the existence of functions in  $U(\Omega, \zeta)$  when  $\Omega$  is non-simply connected. Each of our results about universal Taylor series will appear as a corollary of a more general result. We recall that a set  $A \subset \mathbb{C}$  is said to be *thin* at a point  $\xi$  if either  $\xi \notin \overline{A \setminus \{\xi\}}$  or there is a subharmonic function  $u$  on a neighbourhood of  $\xi$  such that

$$\limsup_{z \rightarrow \xi, z \in A} u(z) < u(\xi).$$

The classical Wiener criterion (Theorem 5.4.1 of [22]) provides a characterization of thinness in terms of capacity. By a *polar set* we mean a set of capacity zero.

We will abbreviate  $S_N(f, \zeta)$  to  $S_N$  where no confusion can arise.

**Theorem 1** *Let  $f \in H(\Omega)$  and  $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$ , and suppose that  $\mathbb{C} \setminus \Omega$  is thin at a point  $\xi \in \partial\Omega \cap \partial D(\zeta, r)$ . If some subsequence  $(S_{N_k})$  of  $(S_N(f, \zeta))$  is uniformly bounded on a non-polar subset of  $\mathbb{C} \setminus \overline{D}(\zeta, r)$ , then  $(S_{N_k})$  converges uniformly on a neighbourhood of  $\xi$ ; in particular,  $f$  has an analytic continuation to a neighbourhood of  $\xi$ .*

**Corollary 1** *Let  $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$ . If  $\mathbb{C} \setminus \Omega$  is thin at a point  $\xi \in \partial\Omega \cap \partial D(\zeta, r)$  and  $\mathbb{C} \setminus (\Omega \cup \overline{D}(\zeta, r))$  is non-polar, then  $U(\Omega, \zeta) = \emptyset$ .*

Thus, in contrast to the above result of Müller, Vlachou and Yavrian, we now see that *non-thinness* of  $\mathbb{C} \setminus \Omega$  at all points of  $\partial\Omega$  that are at the *minimum* distance  $r$  from  $\zeta$  is a necessary condition for  $U(\Omega, \zeta)$  to be non-empty (unless  $\Omega \cup \overline{D}(\zeta, r)$  has polar complement). However, it is not a sufficient condition, as can be seen from either Corollary 4 below or the example given at the end of Section 2.

Since polar sets are thin everywhere, we immediately obtain the following special case.

**Corollary 2** *If  $D(\zeta, R) \setminus \Omega$  is non-empty and polar for some  $R > 0$ , and if  $\mathbb{C} \setminus \Omega$  is non-polar, then  $U(\Omega, \zeta) = \emptyset$ .*

The result of [10], that  $U(\Omega, 0) = \emptyset$  when  $\Omega$  is a domain of the form  $\mathbb{C} \setminus (L \cup \{1\})$  and  $L$  is a non-degenerate continuum in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , is clearly a particular case of Corollary 2. This corollary also allows us to write down an example of a domain  $\Omega$  with bounded complement that admits no holomorphic functions with universal Taylor series expansions about even a single point:

**Example 1** *Let  $\Omega = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup E)$ , where*

$$E = \bigcup_{k=1}^{\infty} \left\{ \left(1 + 2^{-k}\right) e^{m2^{1-k}\pi i} : 1 \leq m \leq 2^k \right\}.$$

*Then  $U(\Omega, \zeta) = \emptyset$  for every  $\zeta \in \Omega$ .*

Theorem 1 also reveals a link between thinness and the existence of Hadamard-Ostrowski gaps in power series. We recall that a power series  $\sum a_n z^n$ , with radius of convergence 1, is said to possess *Hadamard-Ostrowski gaps* if there exist natural numbers  $p_1 < q_1 \leq p_2 < q_2 \leq \dots$  such that

$$\inf_k \frac{q_k}{p_k} > 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty, n \in \cup_k (p_k, q_k)} |a_n|^{1/n} < 1.$$

**Corollary 3** *Let  $\sum a_n z^n$  be a power series that converges on  $\mathbb{D}$  and is analytically continuable to a domain  $\Omega$ , but not to a neighbourhood of a given point  $\xi \in \partial\mathbb{D}$ . If  $\mathbb{C} \setminus \Omega$  is thin at  $\xi$ , then the series cannot possess Hadamard-Ostrowski gaps.*

Next, we consider the situation where a subsequence of the partial sums of a Taylor series is uniformly bounded on a disc  $\overline{D}(\xi, r) \subset \mathbb{C} \setminus \{\zeta\}$  that is

not contained in the disc of convergence. It turns out that an important role is played by the quantity

$$\rho_0 = \sqrt{|\zeta - \xi|^2 - r^2},$$

which has the property that  $\partial D(\zeta, \rho_0)$  and  $\partial D(\xi, r)$  meet at right angles.

**Theorem 2** *Let  $f \in H(\Omega)$  and  $\overline{D}(\xi, r) \subset \mathbb{C} \setminus \{\zeta\}$ , and suppose that the set  $D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$  is polar, where  $|\zeta - \xi| - r < \rho \leq \rho_0$ . If  $(S_{N_k}(f, \zeta))$  is uniformly bounded on  $\overline{D}(\xi, r)$ , then it is locally uniformly convergent either on  $D(\zeta, \rho) \setminus \overline{D}(\xi, r)$  or on a domain containing  $D(\zeta, |\zeta - \xi| - r) \cup D(\xi, r)$ .*

In the case where  $\Omega$  is the complement of a closed disc  $\overline{D}(\xi, r)$  we know from the above-cited work of Melas that  $U(\Omega, \zeta)$  is a dense  $G_\delta$  subset of  $H(\Omega)$ . Costakis and Vlachou [7] subsequently showed that there is a dense  $G_\delta$  subset of functions  $f$  in  $H(\Omega)$  with the stronger property that:

*for every function  $g$  which is continuous on  $\overline{D}(\xi, r)$  and holomorphic on  $D(\xi, r)$ , there is a subsequence  $(S_{N_k}(f, \zeta))$  that converges to  $g$  uniformly on  $\overline{D}(\xi, r)$  and to  $f$  locally uniformly on  $D(\zeta, |\zeta - \xi| + r) \setminus \overline{D}(\xi, r)$ .*

Theorem 2 shows that, for every function  $f$  in  $U(\Omega, \zeta)$ , where  $\Omega = \mathbb{C} \setminus \overline{D}(\xi, r)$ , subsequences of the partial sums of the Taylor series that converge on  $\overline{D}(\xi, r)$  necessarily also overconverge to  $f$  on  $D(\zeta, \rho_0) \setminus \overline{D}(\xi, r)$ . It also yields a stronger version of Corollary 2 for a certain family of domains:

**Corollary 4** *Let  $\Omega \subset \mathbb{C} \setminus \overline{D}(\xi, r)$  and  $0 < \rho \leq \rho_0$ . If  $D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$  is non-empty and polar, then  $U(\Omega, \zeta) = \emptyset$ .*

This last result brings us much closer to a complete solution of the existence question for universal Taylor series in the particular case where  $\mathbb{C} \setminus \Omega$  consists of a disc and an additional point. To summarise the current state of knowledge, let

$$\Omega = \mathbb{C} \setminus (\overline{D}(\xi, r) \cup \{\xi_0\}), \text{ where } 0 \in \Omega \text{ and } \xi_0 \notin \overline{D}(\xi, r). \quad (1)$$

If  $|\xi_0| > |\xi| + r$ , then we know from [7] that  $U(\Omega, 0)$  is a dense  $G_\delta$  subset of  $H(\Omega)$ . Recent work of Tsirivas and Vlachou [24] shows that this remains the case when  $|\xi_0| = |\xi| + r$ . In contrast, the above corollary shows that  $U(\Omega, 0)$  is empty when  $|\xi_0| < \sqrt{|\xi|^2 - r^2}$ . So we are left with the following question.

**Open problem** *If  $\Omega$  is given by (1) and  $\sqrt{|\xi|^2 - r^2} \leq |\xi_0| < |\xi| + r$ , is  $U(\Omega, 0)$  non-empty?*

Subharmonic functions are an important tool in this paper. Their role will become clearer in Theorem 3 below. This result itself yields, as a corollary, a necessary topological condition on  $\Omega$  for the existence of universal Taylor series. Theorems 1 and 2 will then be proved in the remaining two sections, along with their respective corollaries. The possibility that Theorem 2 might hold was initially suggested by earlier work of the author on the extension of superharmonic functions (see [8], or Chapter 6 of [9]). For the necessary potential theoretic background we refer to the books [22] and [1].

## 2 A preliminary result

We denote the extended complex plane  $\mathbb{C} \cup \{\infty\}$  by  $\widehat{\mathbb{C}}$ . Let  $G_\omega(\cdot, \xi)$  denote the Green function of an open set  $\omega \subset \widehat{\mathbb{C}}$  with pole at  $\xi \in \omega$ , when it exists, that is, when the complement of  $\omega$  is non-polar. The function  $G_\omega(\cdot, \xi)$  should be interpreted as 0 outside the connected component of  $\omega$  that contains  $\xi$ . The following theorem, which strengthens and extends Theorem 1 of Gehlen [11], will be useful in proving our main results. It captures information about the behaviour of a subsequence of the partial sums of a Taylor series in terms of the subharmonicity of a certain function, which we can later exploit.

**Theorem 3** *Let  $f \in H(\Omega)$ , where  $0 \in \Omega$ , let  $(S_{N_k})$  be a subsequence of  $(S_N(f, 0))$ , and let  $U$  be the largest domain containing 0 on which  $(S_{N_k})$  is locally uniformly convergent. Further, suppose that  $\Omega \setminus U \neq \emptyset$  and that  $(S_{N_k})$  is uniformly bounded on a compact set  $K$  (possibly empty) contained in  $\mathbb{C} \setminus U$ . Then*

- (i)  $U$  is bounded and simply connected;
- (ii)  $(\overline{U})^\circ \cap \Omega = U \cap \Omega$ ; and
- (iii) the upper semicontinuous regularization  $v^*$  of the function

$$v(z) = \begin{cases} -G_U(z, 0) & (z \in U) \\ G_V(z, \infty) & (z \in V) \\ 0 & (\text{elsewhere in } \mathbb{C}) \end{cases},$$

where  $V = \widehat{\mathbb{C}} \setminus (\overline{U} \cup K)$ , is subharmonic on  $\Omega \cup (\mathbb{C} \setminus \partial U)$  and continuously vanishes on  $\partial U$ .

**Proof.** If the domain  $U$  were unbounded, then there would exist a closed set  $E \subset U$ , non-thin at infinity, such that  $\limsup_{k \rightarrow \infty} |S_{N_k}|^{1/N_k} \leq 1$  on  $E$ . By Theorem 1 of Müller and Yavrian [18], this would then imply that  $\Omega \subseteq U$ . Since we have assumed that  $\Omega \setminus U \neq \emptyset$ , the domain  $U$  must be bounded. Also, it follows from the maximum modulus principle that  $U$  is simply connected, so part (i) of the theorem is established. Since both  $U$  and  $\widehat{\mathbb{C}} \setminus \overline{U}$  are regular for the Dirichlet problem, the function  $v$  is continuous

at points of  $\partial U$ , and so  $v^*$  continuously vanishes there. In the light of this it remains only to prove the subharmonicity in part (iii), for (ii) will then follow by the maximum principle. Since  $f$  has an analytic continuation to  $U \cup \Omega$ , we may assume from now on that  $U \subset \Omega$ .

The sequence  $(S_{N_k})$  is uniformly bounded on some closed disc centred at 0, so  $(|S_{N_k}|^{1/N_k})$  is locally uniformly bounded on  $\mathbb{C}$ , by a simple case of Bernstein's lemma (Theorem 5.5.7 in [22]). Since

$$|S_{N_k} - f|^{1/N_k} \leq 2^{1/N_k} \max \left\{ |S_{N_k}|^{1/N_k}, |f|^{1/N_k} \right\} \quad \text{on } \Omega$$

by the triangle inequality, we see that  $(|S_{N_k} - f|^{1/N_k})$  is locally uniformly bounded in  $\Omega$ . It follows that the upper semicontinuous regularization  $u^*$  of the function

$$u = \limsup_{k \rightarrow \infty} \log |S_{N_k} - f|^{1/N_k}$$

is subharmonic on  $\Omega$  (see Theorem 3.4.3(a) of [22]), and

$$u \leq \left( \limsup_{k \rightarrow \infty} \log |S_{N_k}|^{1/N_k} \right)^+ \quad \text{on } \Omega. \quad (2)$$

Clearly  $u^* \leq 0$  on  $U$ . On  $\Omega$  we have

$$\log |S_{N_k}(z) - f(z)|^{1/N_k} = \log |z| + w_k(z),$$

where  $w_k$  is a subharmonic function on  $\Omega$ . Hence  $u^*(z) = \log |z| + w(z)$  on  $\Omega$ , where  $w$  is a subharmonic function on  $\Omega$ . Since  $G_U(\cdot, 0)$  is the least nonnegative superharmonic function on  $U$  of the form  $-\log |\cdot| - w$ , where  $w$  is subharmonic on  $U$ , and  $u^* \leq 0$  on  $U$ , we conclude that

$$u^* \leq -G_U(\cdot, 0) \quad \text{on } U. \quad (3)$$

Next, let  $(L_n)$  be an exhaustion of  $U$  by compact sets, and let  $\omega_n = \widehat{\mathbb{C}} \setminus (L_n \cup K)$ . By Bernstein's lemma,

$$\left( \frac{|S_{N_k}|}{\sup_{L_n \cup K} |S_{N_k}|} \right)^{1/N_k} \leq e^{G_{\omega_n}(\cdot, \infty)} \quad \text{on } \mathbb{C} \quad (n \geq 1; k \geq 1).$$

Since  $(S_{N_k})$  is uniformly bounded on  $L_n \cup K$ , we see from (2) that  $u \leq G_{\omega_n}(\cdot, \infty)$  on  $\Omega$  for all  $n$ . Hence

$$u \leq \lim_{n \rightarrow \infty} G_{\omega_n}(\cdot, \infty) = G_V(\cdot, \infty) \quad \text{on } \Omega \quad (4)$$

and, in particular,  $u \leq 0$  on  $\Omega \setminus V$ . (One way of seeing the equality in (4) is to use Theorems 5.7.3(iv) and 5.7.4(iii) in [1] together with the fact that  $U$  is non-thin at each point of  $\partial U$ ). It follows from (3) and (4) that

$$u^* \leq v^* \quad \text{on } \Omega.$$

Let  $\xi \in \Omega \cap \partial U$  and  $\overline{D}(\xi, r) \subset \Omega$ . Then  $u^*(\xi) = 0$ , for otherwise  $u^*(\xi) < 0$  and we would arrive at the contradictory conclusion that  $S_{N_k} \rightarrow f$  uniformly on a neighbourhood of  $\xi$  and so  $\xi \in U$ . Hence

$$v^*(\xi) = 0 = u^*(\xi) \leq A(u^*; \xi, r) \leq A(v^*; \xi, r),$$

where  $A(w; \xi, r)$  denotes the area mean value of a function  $w$  over  $D(\xi, r)$ . Since  $v^*$  is clearly subharmonic on  $\mathbb{C} \setminus \partial U$ , we conclude that  $v^*$  is subharmonic on  $\Omega \cup (\mathbb{C} \setminus \partial U)$ , as required. ■

**Corollary 5** *Let  $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$  and suppose that  $\Omega \setminus \overline{D}(\zeta, r)$  has a bounded component  $\omega$ . If  $U(\Omega, \zeta) \neq \emptyset$ , then  $(\omega \cup \overline{D}(\zeta, r))^\circ \cap \Omega$  is simply connected.*

Before proving Corollary 5 we note that it yields another family of domains  $\Omega$  such that  $U(\Omega, \zeta) = \emptyset$  for certain  $\zeta$ . The example below illustrates this, and also the remark we made following Corollary 1.

**Example 2** *Suppose that  $\mathbb{C} \setminus \Omega$  consists of  $\partial D(3/2, 1) \setminus \mathbb{D}$  and  $\overline{D}(3/2, 1/2)$ . Then  $U(\Omega, 0) = \emptyset$ .*

**Proof of Corollary 5.** Without loss of generality we may suppose that  $\zeta = 0$ . If  $\Omega$  is simply connected, the conclusion is trivial, so we assume otherwise. The existence of functions in  $U(\Omega, 0)$  then implies that  $\Omega$  is unbounded. Let  $r$  and  $\omega$  be as in the statement of the corollary, let  $R > \sup\{|z| : z \in \omega\}$  and  $K = \overline{D}(0, R) \setminus \Omega$ . Then  $K$  has connected complement. Let  $f \in U(\Omega, 0)$  and  $q$  be an arbitrary polynomial. Then we can find a subsequence  $(S_{N_k})$  of  $(S_N(f, 0))$  that converges uniformly to  $q$  on  $K$ . Let  $U$  and  $v^*$  be as in the statement of Theorem 3 and let  $W = (\omega \cup \overline{D}(0, r))^\circ \cap \Omega$ . Then  $W$  is connected,  $D(0, r) \subset U$  and so  $\partial W \subset \overline{U} \cup K$ . Thus  $v^* < 0$  on  $W$  by Theorem 3 and the maximum principle, and so  $W \subset U$ . If  $\mathbb{C} \setminus W$  had a bounded component  $L$ , then  $L \subset U$  by the simple connectedness of  $U$  and so  $(S_{N_k})$  would converge uniformly on  $L$  to a limit function that is uniquely determined by the analytic continuation of  $f$  to  $U$ . This contradicts the fact that  $(S_{N_k})$  was chosen to converge to the arbitrary polynomial  $q$  on  $K$ , which intersects  $L$ . ■

### 3 Proof of Theorem 1 and Corollaries

**Proof of Theorem 1.** Let  $f \in H(\Omega)$  and  $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$ , and let  $\mathbb{C} \setminus \Omega$  be thin at a point  $\xi \in \partial \Omega \cap \partial D(\zeta, r)$ . Further, suppose that there is a subsequence  $(S_{N_k})$  of  $(S_N(f, \zeta))$  that is uniformly bounded on some non-polar subset  $K$  of  $\mathbb{C} \setminus \overline{D}(\zeta, r)$ . Without loss of generality we may assume that  $\zeta = 0$  and  $r = 1$ , and that  $K$  is compact.

Let  $U$ ,  $V$  and  $v^*$  be as in Theorem 3. Then  $\mathbb{D} \subset U$  and clearly

$$v^*(z) = -G_U(z, 0) \leq -G_{\mathbb{D}}(z, 0) = \log |z| \quad (z \in \overline{\mathbb{D}}).$$

Further, since  $V \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , and  $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup V)$  contains the non-polar set  $K$ , we see from the maximum principle that

$$v^*(z) < G_{\mathbb{C} \setminus \overline{\mathbb{D}}}(z, \infty) = \log |z| \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}).$$

Let  $\Omega_0$  be the component of  $\Omega \cap [D(0, 2) \setminus \overline{D}(0, 1/2)]$  that meets  $\mathbb{D}$ , and let  $H_{v^*}^{\Omega_0}$  denote the solution to the Dirichlet problem on  $\Omega_0$  with boundary data  $v^*$ . Since the set  $\partial\Omega_0 \setminus \overline{\mathbb{D}}$  has positive harmonic measure for  $\Omega_0$ , we obtain a strictly positive harmonic function on  $\Omega_0$  by defining

$$w(z) = \log |z| - H_{v^*}^{\Omega_0}(z) \quad (z \in \Omega_0).$$

The thinness of  $\mathbb{C} \setminus \Omega$  at  $\xi$  tells us that  $\xi$  is an irregular boundary point for the Dirichlet problem on  $\Omega_0$  (see Theorems 6.6.7 and 7.5.1 of [1]), so by Theorem 7.5.5 of [1] there exists a set  $A$ , thin at  $\xi$ , and a number  $\ell \geq 0$ , such that

$$w(z) \rightarrow \ell \quad (z \rightarrow \xi, z \notin A).$$

Further, by the continuity of  $w$  on  $\Omega_0$ , we may assume that the set  $A$  is closed. We must have  $\ell > 0$ , for otherwise  $w$  would be a barrier for the open set  $\Omega_0 \setminus A$  at the irregular boundary point  $\xi$ , which is impossible (see Theorem 6.6.5 of [1]). By a well-known property of thin sets (Theorem 7.3.9 of [1]) we can choose  $r_0 \in (0, e^{\ell/2} - 1)$  such that  $\partial D(\xi, r_0) \subset \Omega_0 \setminus A$  and  $w \geq \ell/2$  on  $\partial D(\xi, r_0)$ . We now use the subharmonicity of  $v^*$  on  $\Omega$  (by Theorem 3) to see that

$$v^*(z) \leq H_{v^*}^{\Omega_0}(z) = \log |z| - w(z) \leq \log(1 + r_0) - \ell/2 < 0 \quad (z \in \partial D(\xi, r_0)),$$

whence  $\partial D(\xi, r_0) \subset U$ . Since  $U$  is simply connected, we conclude that  $(S_{N_k})$  converges uniformly on  $\overline{D}(\xi, r_0)$ , as required. ■

**Proof of Corollary 1.** Let  $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$ , suppose that  $\mathbb{C} \setminus \Omega$  is thin at a point  $\xi \in \partial\Omega \cap \partial D(\zeta, r)$ , and that  $\mathbb{C} \setminus (\Omega \cup \overline{D}(\zeta, r))$  is non-polar. Then there exists a compact non-polar subset  $K$  of  $\mathbb{C} \setminus (\Omega \cup \overline{D}(\zeta, r))$ , and we can arrange further that  $\mathbb{C} \setminus K$  is connected. If there exists  $f$  in  $U(\Omega, \zeta)$ , then we can find a subsequence  $(S_{N_k})$  of  $(S_N(f, \zeta))$  that is uniformly bounded on  $K$  and that converges to some arbitrarily chosen limit  $\ell$  at  $\xi$ . Theorem 1 would then tell us that  $f$  has an analytic continuation to a neighbourhood of  $\xi$  and that  $S_{N_k}(\xi) \rightarrow f(\xi)$ . This contradicts the arbitrary nature of  $\ell$ , so  $U(\Omega, \zeta) = \emptyset$ . ■

**Proof of Corollary 3.** Let  $\sum a_n z^n$  be a power series that converges on  $\mathbb{D}$  and is analytically continuable to a domain  $\Omega$ , but not to any neighbourhood

of a given point  $\xi \in \partial\mathbb{D}$ . If the series possesses Hadamard-Ostrowski gaps, then a classical theorem of Ostrowski tells us that there is a subsequence of  $(S_N(f, 0))$  that converges uniformly on a neighbourhood of each point of  $\Omega \cap \partial\mathbb{D}$  (cf. Theorem 16.7.1 in [13]). Theorem 1 would then yield the conclusion that  $\mathbb{C} \setminus \Omega$  cannot be thin at  $\xi$ . ■

## 4 Proof of Theorem 2 and Corollary 4

In proving Theorem 2 we may assume that  $\zeta = 0$ . Let  $f \in H(\Omega)$  and  $\overline{D}(\xi, r) \subset \mathbb{C} \setminus \{0\}$ , let  $\rho_0 = \sqrt{|\xi|^2 - r^2}$  and suppose that  $D(0, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$  is polar, where  $|\xi| - r < \rho \leq \rho_0$ . Further, suppose that  $(S_{N_k}(f, 0))$  is uniformly bounded on  $\overline{D}(\xi, r)$ , and let  $U$  be the largest domain containing 0 on which  $(S_{N_k}(f, 0))$  is locally uniformly convergent. Since  $D(0, |\xi| - r) \setminus \Omega$  is polar, and so everywhere thin, Theorem 1 shows that the Taylor series of  $f$  about 0 has radius of convergence at least  $|\xi| - r$ , and so  $D(0, |\xi| - r) \subset U$ .

If  $\overline{D}(\xi, r) \cap U \neq \emptyset$ , then  $D(\xi, r) \subset U$  by Montel's theorem, and there is nothing more to prove. If  $\Omega \subset U$ , then  $D(0, \rho) \setminus \overline{D}(\xi, r) \subset U$ , since the polar set  $[D(0, \rho) \setminus \overline{D}(\xi, r)] \setminus \Omega$  is totally disconnected and  $U$  is simply connected, and again there is nothing more to prove. So we suppose from now on that  $\overline{D}(\xi, r) \subset \mathbb{C} \setminus U$  and  $\Omega \setminus U \neq \emptyset$ .

Let  $V$  and  $v^*$  be as defined in Theorem 3, where  $K$  is taken to be the disc  $\overline{D}(\xi, r)$ . From that result we see that  $U$  is bounded. Let

$$R = \sup\{t : D(0, t) \subset U \cup \overline{D}(\xi, r)\}.$$

Then  $R \geq |\xi| - r$ . We suppose, for the sake of contradiction, that  $R < \rho$ , and choose a point  $\eta$  in the set  $\partial D(0, R) \cap \partial(U \cup \overline{D}(\xi, r))$ . By Theorem 3 the function  $v^*$  is subharmonic on  $D(0, \rho) \setminus \overline{D}(\xi, r)$  because polar sets are removable singularities for locally upper-bounded subharmonic functions.

We first consider the case where  $\eta \in \partial D(0, R) \cap \partial U \setminus \partial \overline{D}(\xi, r)$ . We define

$$w(z) = \begin{cases} -G_{D(0, R) \setminus \overline{D}(\xi, r)}(z, 0) & (|z| < R) \\ G_{\widehat{\mathbb{C}} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r))}(z, \infty) & (|z| > R) \\ 0 & (|z| = R) \end{cases}$$

and

$$h(z) = \begin{cases} -G_{D(0, R) \setminus \overline{D}(\xi, r)}(z, 0) & (|z| < R) \\ G_{D(0, R) \setminus \overline{D}(\xi, r)}(R^2/\bar{z}, 0) & (|z| > R) \\ 0 & (|z| = R) \end{cases}.$$

The latter function is harmonic on  $\mathbb{C} \setminus (\overline{D}(\xi, r) \cup \{0\})$  by the reflection principle (and a conformal mapping from the upper half-plane to  $D(0, R)$ ). Since

$$U \supset D(0, R) \setminus \overline{D}(\xi, r) \quad \text{and} \quad V \subset \widehat{\mathbb{C}} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r)),$$

we see that  $v^* \leq w$  on  $\mathbb{C}$ . Also, it follows from inversion in  $\partial D(0, R)$ , and the fact that  $R < \rho_0$ , that

$$G_{D(0,R) \setminus \overline{D}(\xi,r)}(R^2/\bar{z}, 0) > G_{\widehat{\mathbb{C}} \setminus (\overline{D}(0,R) \cup \overline{D}(\xi,r))}(z, \infty)$$

when  $z \in \mathbb{C} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r))$ , so  $w < h$  on that set. Now let  $\delta > 0$  be small enough that  $\overline{D}(\eta, \delta) \subset D(0, \rho) \setminus \overline{D}(\xi, r)$ . Since  $\eta \in \partial U$ , we arrive at the contradictory conclusion that

$$0 = v^*(\eta) \leq A(v^*; \eta, \delta) \leq A(w; \eta, \delta) < A(h; \eta, \delta) = h(\eta) = 0.$$

It remains to consider the more delicate case where  $\eta \in \partial D(0, R) \cap \partial U \cap \partial D(\xi, r)$ . We will use the following special case of the boundary Harnack principle, which can be deduced from Lemma 8.5.1 in [1] using inversion.

**Lemma 4** *There is an absolute constant  $C_0 > 1$  with the following property. If  $\eta \in \partial D(\xi, r)$  and  $\sigma \in (0, r]$ , and if  $h_0$  is a positive harmonic function on  $D(\eta, \sigma) \setminus \overline{D}(\xi, r)$  that vanishes continuously on  $D(\eta, \sigma) \cap \partial D(\xi, r)$ , then*

$$C_0^{-1} \frac{h_0(\eta_\sigma)}{\log(1 + \sigma/2r)} \leq \frac{h_0(z)}{\log(|z - \xi|/r)} \leq C_0 \frac{h_0(\eta_\sigma)}{\log(1 + \sigma/2r)} \quad (z \in D(\eta, \sigma/2) \setminus \overline{D}(\xi, r)),$$

where  $\eta_\sigma = \xi + (1 + \frac{\sigma}{2r})(\eta - \xi)$ .

We can write  $\eta = Re^{i\phi}$  and  $\xi - \eta = re^{i\theta}$  for some  $\phi$  and  $\theta$ . Then

$$|\xi|^2 = R^2 + r^2 + 2Rr \cos(\theta - \phi),$$

so

$$\rho_0^2 = R^2 + 2Rr \cos(\theta - \phi),$$

and since  $R < \rho_0$ , we see that  $\cos(\theta - \phi) > 0$ . It follows that  $U$  contains a sector of vertex  $\eta$  and angle exceeding  $\pi/2$ . We will assume for the present that  $0$ ,  $\eta$  and  $\xi$  are not collinear. This implies that, for small  $s$ , the set  $V \cap D(\eta, s)$  is contained in a sector of vertex  $\eta$  and angle less than  $\pi/2$ . More precisely, we can choose sectors of the form

$$S_j = \begin{cases} \{\eta + se^{it} : 0 < s < s_0, |t - \pi - (\theta + \phi)/2| < \alpha_j\} & (j = 1, 2) \\ \{\eta + se^{it} : 0 < s < s_0, |t - t_0| < \alpha_j\} & (j = 3, 4) \end{cases},$$

where  $s_0 \in (0, r]$ ,  $t_0 \in [0, 2\pi]$ ,

$$\alpha_1 > \alpha_2 > \frac{\pi}{4} > \alpha_3 > \alpha_4, \quad \overline{S}_1 \setminus U = \{\eta\}, \quad \overline{D}(\eta, s_0) \cap V \subset \overline{S}_4$$

and  $S_1 \cap S_3 = \emptyset$ . We can define a positive harmonic function on  $S_1$  by

$$p_1(\eta + se^{it}) = s^{\frac{\pi}{2\alpha_1}} \cos\left(\frac{\pi}{2\alpha_1} \left(t - \pi - \frac{\theta + \phi}{2}\right)\right).$$

Since  $\bar{S}_1 \setminus U = \{\eta\}$ , the function  $G_U(\cdot, 0)$  has a positive lower bound on the set

$$\left\{ \eta + s_0 e^{it} : \left| t - \pi - \frac{\theta + \phi}{2} \right| \leq \alpha_1 \right\},$$

so by the minimum principle there is a positive constant  $c_1$  such that

$$G_U(\cdot, 0) \geq c_1 p_1 \quad \text{on } S_1. \quad (5)$$

Similarly, we can define a positive harmonic function on  $S_3$  by

$$p_2(\eta + s e^{it}) = s^{\frac{\pi}{2\alpha_3}} \cos\left(\frac{\pi}{2\alpha_3}(t - t_0)\right).$$

This function has a positive lower bound on  $\partial D(\eta, s_0) \cap V$ , since the latter set is contained in  $\bar{S}_4$ , so by the maximum principle there is a positive constant  $c_2$  such that

$$c_2 p_2 \geq G_V(\cdot, \infty) \quad \text{on } S_3. \quad (6)$$

We assign  $p_1$  (respectively,  $p_2$ ) the value 0 on  $D(\eta, s_0) \setminus S_1$  (respectively, on  $D(\eta, s_0) \setminus S_3$ ).

Now let  $W_\sigma = D(\eta, \sigma) \setminus \bar{D}(\xi, r)$ , and let  $H_g^{W_\sigma}$  denote the solution to the Dirichlet problem on  $W_\sigma$  with given boundary data  $g$  on  $\partial D(\eta, \sigma) \setminus \bar{D}(\xi, r)$  and 0 elsewhere. If  $\sigma < \min\{s_0, \rho - R\}$ , then we see from (5) and (6) that

$$H_{v^*}^{W_\sigma} = H_{G_V(\cdot, \infty)}^{W_\sigma} - H_{G_U(\cdot, 0)}^{W_\sigma} \leq c_2 H_{p_2}^{W_\sigma} - c_1 H_{p_1}^{W_\sigma}.$$

Since

$$p_1(z) \geq |z - \eta|^{\frac{\pi}{2\alpha_1}} \cos\left(\frac{\pi\alpha_2}{2\alpha_1}\right) \quad \text{on } S_2$$

and

$$p_2(z) \leq |z - \eta|^{\frac{\pi}{2\alpha_3}} \quad \text{on } S_3,$$

we deduce that

$$H_{v^*}^{W_\sigma} \leq c_2 \sigma^{\frac{\pi}{2\alpha_3}} h_{2,\sigma} - c_1 \sigma^{\frac{\pi}{2\alpha_1}} h_{1,\sigma} \quad \text{on } W_\sigma,$$

where

$$h_{1,\sigma} = \cos\left(\frac{\pi\alpha_2}{2\alpha_1}\right) H_{\chi_{S_2}}^{W_\sigma}, \quad h_{2,\sigma} = H_{\chi_{S_3}}^{W_\sigma}$$

and  $\chi_A$  denotes the characteristic function of a set  $A$ . We note that  $h_{1,\sigma} > 0$  since  $\alpha_1 > \alpha_2$ . If  $|z - \eta| < \sigma/2$ , then we deduce, using Lemma 4, that

$$H_{v^*}^{W_\sigma}(z) \leq \left\{ C_0 c_2 \sigma^{\frac{\pi}{2\alpha_3}} h_{2,\sigma}(\eta_\sigma) - C_0^{-1} c_1 \sigma^{\frac{\pi}{2\alpha_1}} h_{1,\sigma}(\eta_\sigma) \right\} \frac{\log(|z - \xi|/r)}{\log(1 + \sigma/2r)}.$$

Since

$$H_{\chi_{S_2}}^{W_\sigma}(\eta_\sigma) \geq H_{\chi_{S_2}}^{S_2 \cap D(\eta, \sigma)}(\eta_\sigma)$$

and  $\alpha_2 > \pi/4$ , it follows by dilation that there is a constant  $c_3 > 0$ , independent of  $\sigma$ , such that  $h_{1,\sigma}(\eta_\sigma) \geq c_3$ . Also, trivially  $h_{2,\sigma}(\eta_\sigma) \leq 1$ . Hence

$$H_{v^*}^{W_\sigma}(z) \leq \sigma^{2\frac{\pi}{\alpha_1}} \left\{ C_0 c_2 \sigma^{\frac{\pi}{2}(\frac{1}{\alpha_3} - \frac{1}{\alpha_1})} - C_0^{-1} c_1 c_3 \right\} \frac{\log(|z - \xi|/r)}{\log(1 + \sigma/2r)} \quad \left( z \in D\left(\eta, \frac{\sigma}{2}\right) \right).$$

Since  $\alpha_3 < \alpha_1$ , we conclude from the subharmonicity of  $v^*$  on  $D(0, \rho) \setminus \overline{D}(\xi, r)$  that

$$v^* \leq H_{v^*}^{W_\sigma} < 0 \quad \text{on } D(\eta, \sigma/2) \setminus \overline{D}(\xi, r),$$

provided  $\sigma$  is chosen small enough. This implies that  $D(\eta, \sigma/2) \setminus \overline{D}(\xi, r) \subset U$ , which contradicts the fact that  $\eta \in \partial(U \cup \overline{D}(\xi, r))$ .

Finally, if  $0, \eta$  and  $\xi$  are collinear, then  $D(0, R)$  and  $D(\xi, r)$  are externally tangent. A similar argument applies in this case, except that *two* disjoint sectors with small aperture and vertices at  $\eta$  are now required to cover  $D(\eta, s_0) \cap V$ . The resulting contradiction shows that we cannot have  $R < \rho$ , so Theorem 2 is established.

**Proof of Corollary 4.** Let  $\Omega \subset \mathbb{C} \setminus \overline{D}(\xi, r)$  and suppose that the set  $D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$  is non-empty and polar, where  $0 < \rho \leq \rho_0$ . If  $\rho \leq |\zeta - \xi| - r$ , then  $D(\zeta, \rho) \setminus \Omega$  is non-empty and polar, and the result follows from Corollary 2. We now assume, for the sake of contradiction, that  $|\zeta - \xi| - r < \rho \leq \rho_0$  and that there exists  $f$  in  $U(\Omega, \zeta)$ . Let  $\xi_0 \in D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$  and  $\ell \in \mathbb{C} \setminus \{0\}$ . Then we can choose a subsequence  $(S_{N_k})$  of  $(S_N(f, \zeta))$  such that  $S_{N_k} \rightarrow 0$  uniformly on  $\overline{D}(\xi, r)$  and  $S_{N_k}(\xi_0) \rightarrow \ell$ . Since  $f \not\equiv 0$ , we see from Theorem 2 that  $f$  has an analytic continuation to  $D(\zeta, \rho) \setminus \overline{D}(\xi, r)$ , which is valued  $\ell$  at  $\xi_0$ . This contradicts the arbitrary nature of  $\ell$ . ■

We conclude with an example which relies on elements of both Theorem 2 and Corollary 5.

**Example 3** Let  $\Omega = \mathbb{C} \setminus (E \cup \{-5\})$ , where

$$E = \overline{D}(5, 4) \cup [\partial D(-5, 4) \setminus D(0, 3)].$$

Then  $U(\Omega, 0) = \emptyset$ .

To see this, suppose that there exists  $f$  in  $U(\Omega, 0)$ . Then we can choose a subsequence  $(S_{N_k})$  of  $(S_N(f, 0))$  such that  $S_{N_k} \rightarrow 0$  uniformly on  $E$  and  $S_{N_k}(-5) \rightarrow 1$ . We can apply Theorem 2, where  $\rho = \rho_0 = \sqrt{5^2 - 4^2} = 3$ , to see that  $(S_{N_k})$  is locally uniformly convergent on  $D(0, 3) \setminus \overline{D}(5, 4)$ , and then argue as in the proof of Corollary 5 to obtain the contradictory conclusion that the set

$$[D(-5, 4) \cup D(0, 3)] \setminus [\overline{D}(5, 4) \cup \{-5\}]$$

is simply connected.

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