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# SHAPE PRESERVING PROPERTIES OF GENERALIZED BERNSTEIN OPERATORS ON EXTENDED CHEBYSHEV SPACES

J. M. ALDAZ, O. KOUNCHEV AND H. RENDER

ABSTRACT. We study the existence and shape preserving properties of a generalized Bernstein operator  $B_n$  fixing a strictly positive function  $f_0$ , and a second function  $f_1$  such that  $f_1/f_0$  is strictly increasing, within the framework of extended Chebyshev spaces  $U_n$ . The first main result gives an inductive criterion for existence: suppose there exists a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  with strictly increasing nodes, fixing  $f_0, f_1 \in U_n$ . If  $U_n \subset U_{n+1}$  and  $U_{n+1}$  has a non-negative Bernstein basis, then there exists a Bernstein operator  $B_{n+1} : C[a, b] \rightarrow U_{n+1}$  with strictly increasing nodes, fixing  $f_0$  and  $f_1$ . In particular, if  $f_0, f_1, \dots, f_n$  is a basis of  $U_n$  such that the linear span of  $f_0, \dots, f_k$  is an extended Chebyshev space over  $[a, b]$  for each  $k = 0, \dots, n$ , then there exists a Bernstein operator  $B_n$  with increasing nodes fixing  $f_0$  and  $f_1$ . The second main result says that under the above assumptions the following inequalities hold

$$B_n f \geq B_{n+1} f \geq f$$

for all  $(f_0, f_1)$ -convex functions  $f \in C[a, b]$ . Furthermore,  $B_n f$  is  $(f_0, f_1)$ -convex for all  $(f_0, f_1)$ -convex functions  $f \in C[a, b]$ .

## 1. INTRODUCTION

Given  $n \in \mathbb{N}$ , the space of polynomials generated by  $\{1, x, \dots, x^n\}$  on  $[a, b]$  is basic in approximation theory and numerical analysis, so generalizations and modifications abound. However, from a numerical point of view it is a well known fact that the Bernstein bases functions  $p_{n,k} = x^k (1-x)^{n-k}$  behave much better and provide optimal stability, see [13]. The associated Bernstein operator  $B_n : C[0, 1] \rightarrow U_n$ , defined by

$$(1) \quad B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

has been the object of intensive research. As is well known, the polynomials  $B_n f$  converge to  $f$  uniformly although the convergence might be very slow. More important is the fact

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that the Bernstein operator  $B_n$  reduces the variation and preserves the shape of  $f$ . In particular, if  $f$  is increasing then  $B_n f$  is increasing, while if  $f$  is convex then  $B_n f$  is convex, see e.g. [12]. And the derivative of  $B_n f$  of a function of class  $C^1$  converges uniformly to  $f'$ , cf. [20], pg. 25. For this reason Bernstein bases and operators are fundamental notions.

In Computer Aided Geometric Design (CADG) one is often interested, for instance, in rendering circumferences and other shapes not given by polynomial functions. It is thus natural to try to extend the preceding theory to more general spaces, containing not only  $1, x, \dots, x^n$ , but also, say, sine and cosine functions, while keeping as many of the good properties of Bernstein bases and operators as possible. If one generalizes the space of polynomials of degree at most  $n$  by retaining the bound on the number of zeros, one is lead to the notion of an *extended Chebyshev space* (or system)  $U_n$  of dimension  $n + 1$  over the interval  $[a, b]$ :  $U_n$  is an  $n + 1$  dimensional subspace of  $C^n([a, b])$  such that each  $f \in U_n$  has at most  $n$  zeros in  $[a, b]$ , counting multiplicities, unless  $f$  vanishes identically. Recently, a rich mathematical literature has emerged concerning *generalized Bernstein bases* in the framework of extended Chebyshev spaces, see [8], [9], [10], [11], [21], [22], [23], [24], [25], [26], [27], [28], [30].

It is well-known that extended Chebyshev spaces possess *non-negative Bernstein bases*, i.e. collections of non-negative functions  $p_{n,k}$ ,  $k = 0, \dots, n$ , in  $U_n$ , such that each  $p_{n,k}$  has a zero of order  $k$  at  $a$  and a zero of order  $n - k$  at  $b$ , for  $k = 0, \dots, n$ . Assuming that  $U_n$  has a non-negative Bernstein basis  $p_{n,k}$ ,  $k = 0, \dots, n$  over the interval  $[a, b]$ , it is natural to ask whether one may associate a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  with properties analogous to the classical operator defined in (1). We consider operators  $B_n$  of the form

$$(2) \quad B_n(f) = \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} p_{n,k}$$

where the nodes  $t_{n,0}, \dots, t_{n,n}$  belong to the interval  $[a, b]$ , and the weights  $\alpha_{n,0}, \dots, \alpha_{n,n}$  are positive. But it is not obvious how the nodes and weights should be defined. Recall that the classical Bernstein operator reproduces the constant function 1 and the identity function  $x$ . We mimic this feature by requiring that  $B_n$  fix two functions  $f_0, f_1 \in U_n$ , i.e. that

$$(3) \quad B_n(f_0) = f_0 \text{ and } B_n(f_1) = f_1,$$

where throughout the paper it is assumed that  $f_0 > 0$  and that  $f_1/f_0$  is strictly increasing, unless we explicitly state otherwise. We show in Section 2 that after choosing  $f_0$  and  $f_1$  in  $U_n$ , the requirements  $B_n(f_0) = f_0$  and  $B_n(f_1) = f_1$ , if they can be satisfied, uniquely determine the location of the nodes and the values of the coefficients; in other words, there is at most one Bernstein operator  $B_n$  of the form (2) satisfying (3).

The question of existence of a Bernstein operator in the above sense is studied in [1] and [2]. Here we present a new, inductive criterion for the existence of  $B_n$ , making this paper for the most part self-contained. Let  $f_0, \dots, f_n \in C^n[a, b]$  and assume that for each

$k = 0, \dots, n$ , the linear space  $U_k := \langle f_0, \dots, f_k \rangle$ , generated by  $f_0, \dots, f_k$ , is an extended Chebyshev space of dimension  $k + 1$ . Then, for every  $k = 1, \dots, n$ , there exists a Bernstein operator  $B_k : C[a, b] \rightarrow U_k$  fixing  $f_0$  and  $f_1$ , whose sequence of nodes is strictly increasing and interlaces with the nodes of  $B_{k-1}$ , cf. Corollary 7.

Sections 3 and 4 deal with the shape preserving properties of the generalized Bernstein operator  $B_n$ . We shall utilize a generalized notion of convexity,  $(f_0, f_1)$ -convexity, which, according to [16], p. 376, is originally due to Hopf, in 1926, and was later extensively developed by Popoviciu, specially in the context of Chebyshev spaces. Ordinary convexity corresponds to  $(1, x)$ -convexity.

Assume there exists a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  fixing  $f_0$  and  $f_1$ . We shall show that if  $f \in C[a, b]$  is  $(f_0, f_1)$ -convex, then

$$B_n f \geq f,$$

thus generalizing the same inequality for the standard polynomial Bernstein operator acting on convex functions. Assume next that  $B_n$  has strictly increasing nodes, that  $U_n \subset U_{n+1}$ , and that the latter space has a non-negative Bernstein basis. From the results in Section 2 we know that there exists a Bernstein operator  $B_{n+1} : C[a, b] \rightarrow U_{n+1}$  fixing  $f_0$  and  $f_1$ . In Section 3 we show that

$$B_n f \geq B_{n+1} f \geq f$$

for all  $(f_0, f_1)$ -convex functions  $f \in C[a, b]$ , generalizing once more the corresponding result for the standard polynomial Bernstein operator. And in Section 4 we prove that under the preceding hypotheses,  $B_n$  preserves  $(f_0, f_1)$ -convexity, i.e.,  $B_n f$  is  $(f_0, f_1)$ -convex for all  $(f_0, f_1)$ -convex functions  $f \in C[a, b]$ . A similar result is obtained for so-called  $f_0$ -monotone functions  $f$ . For the last results we shall employ general results from the theory of totally positive bases and their shape preserving properties.

In Section 5 we present an example showing that even in extended Chebyshev spaces with a totally positive normalized Bernstein basis it might not be possible to define a Bernstein operator in the above sense. The example also shows that the assumption of increasing nodes is needed for the preservation of convexity.

This paper is essentially self-contained. For simplicity, we consider only real valued functions.

## 2. BERNSTEIN OPERATORS FOR EXTENDED CHEBYSHEV SPACES.

We now introduce the concept of a *Bernstein basis* and of a *non-negative Bernstein basis* for a linear subspace  $U_n \subset C^n[a, b]$  of dimension  $n + 1$ . In the literature, the expressions “Bernstein like basis” or “B-basis” are often used instead of “Bernstein basis”.

**Definition 1.** Let  $U_n \subset C^n[a, b]$  be a linear subspace of dimension  $n + 1$ . A *Bernstein basis* (resp. *non-negative Bernstein basis*) for  $U_n$  is a collection of functions (resp. non-negative functions)  $p_{n,k}, k = 0, \dots, n$ , in  $U_n$ , such that each  $p_{n,k}$  has a zero of exact order  $k$  at  $a$  and a zero of exact order  $n - k$  at  $b$ , for  $k = 0, \dots, n$ .

It is easy to verify that a Bernstein basis is indeed a *basis* of the linear space  $U_n$ , and that the basis functions are unique up to a non-zero factor, see e.g. Lemma 19 and Proposition 20 in [19].

As we indicated in the introduction, extended Chebyshev spaces always have non-negative Bernstein bases. To make this paper as self-contained as possible, we briefly indicate the reason: Let  $\{h_0, \dots, h_n\}$  be a basis for  $U_n$ . To obtain a nonzero function  $p_{n,k}$  with (at least)  $k$  zeros at  $a$  and (at least)  $n - k$  zeros at  $b$ , write  $p_{n,k} := a_0 h_0 + \dots + a_n h_n$ . We impose the condition of having  $k$  zeros at  $a$  (which leads to  $k$  equations) and  $n - k$  zeros at  $b$  (which gives  $n - k$  additional equations). Having  $n + 1$  variables at our disposal, there is always a non-trivial solution. The assumption that  $U_n$  is an extended Chebyshev space guarantees that  $p_{n,k}$  has no more than  $n$  zeros, so it has exactly  $k$  zeros at  $a$  and  $n - k$  zeros at  $b$ . In particular,  $p_{n,k}$  is either strictly positive or strictly negative on  $(a, b)$ . Multiplying by  $-1$  if needed, we obtain a non-negative  $p_{n,k}$ .

In Proposition 3.2 in [24] it is shown that  $U_n \subset C^n[a, b]$  possesses a Bernstein basis  $p_{n,k}$ ,  $k = 0, \dots, n$  if and only if every non-zero  $f \in U_n$  vanishes at most  $n$  times on the set  $\{a, b\}$  (and not on the interval  $[a, b]$ ).

The existence of a Bernstein basis in a space  $U_n \subset C^n[a, b]$  is a rather weak property; e.g. it does not imply the non-negativity of the basis functions  $p_{n,k}$ ,  $k = 0, \dots, n$ , nor the existence of Bernstein bases on subintervals  $[\alpha, \beta]$  of  $[a, b]$ , cf. the proof of Theorem 24 in Section 5.

The next two results are essential tools and standard techniques in CAGD in the context of degree elevation.

**Proposition 2.** *Assume that the linear subspaces  $U_n \subset U_{n+1} \subset C^{n+1}[a, b]$  possess Bernstein bases  $p_{n,k}$ ,  $k = 0, \dots, n$ , and  $p_{n+1,k}$ ,  $k = 0, \dots, n + 1$ . Then*

$$(4) \quad p_{n,k} = \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} p_{n+1,k} + \frac{p_{n,k}^{(n-k)}(b)}{p_{n+1,k+1}^{(n-k)}(b)} p_{n+1,k+1}$$

for each  $k = 0, \dots, n$ .

*Proof.* Since  $p_{n,k} \in U_{n+1}$ , the function  $p_{n,k}$  is a linear combination of the basis functions  $p_{n+1,k}$ ,  $k = 0, \dots, n + 1$ . Using the fact that  $p_{n,k}$  has exactly  $k$  zeros at  $a$  and  $n - k$  zeros at  $b$ , we see that  $p_{n,k} = \alpha p_{n+1,k} + \beta p_{n+1,k+1}$  for some  $\alpha, \beta \in \mathbb{R}$ . Then  $p_{n,k}^{(k)} = \alpha p_{n+1,k}^{(k)} + \beta p_{n+1,k+1}^{(k)}$  and inserting  $x = a$  yields

$$\alpha = \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)}.$$

Similarly,  $p_{n,k}^{(n-k)} = \alpha p_{n+1,k}^{(n-k)} + \beta p_{n+1,k+1}^{(n-k)}$  and inserting  $x = b$  implies that

$$\beta = \frac{p_{n,k}^{(n-k)}(b)}{p_{n+1,k+1}^{(n-k)}(b)}.$$

□

**Lemma 3.** *Under the hypotheses of the preceding proposition, assume additionally that the functions in the Bernstein bases are non-negative. Then*

$$(5) \quad \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} > 0 \quad \text{and} \quad \frac{p_{n,k}^{(n-k)}(b)}{p_{n+1,k+1}^{(n-k)}(b)} > 0$$

for each  $k = 0, \dots, n$ .

*Proof.* If  $k = 0$  or  $k = n$  the assertion is obvious. If  $1 \leq k \leq n$ , then the first inequality in (5) can be obtained from (4): Divide both sides by  $p_{n+1,k}(x)$ , and then let  $x \downarrow a$ . The second inequality follows in an analogous way. Alternatively, (5) can be derived, without using (4), from the well known and elementary fact that if  $f \in C^{(k)}(I)$  has a zero of order  $k$  at  $c$ , then

$$(6) \quad k! \cdot \lim_{x \rightarrow c} \frac{f(x)}{(x-c)^k} = f^{(k)}(c).$$

Of course, the same formula holds for one side limits. □

Let  $U_n \subset C^n[a, b]$  be a linear subspace of dimension  $n + 1$  possessing a non-negative Bernstein basis  $p_{n,k}$ ,  $k = 0, \dots, n$ . Unless otherwise stated, we assume that  $f_0 \in U_n$  is strictly positive on  $[a, b]$  (i.e.,  $f_0(x) > 0$  for all  $x \in [a, b]$ , or more concisely,  $f_0 > 0$ ) and that  $f_1 \in U_n$  is such that the function  $f_1/f_0$  is strictly increasing on  $[a, b]$ . The terms increasing and decreasing are understood in the non-strict sense. For a constant  $c$  positive means  $c > 0$ , while for a function  $f$  it means  $f \geq 0$ . We now introduce the concept of a Bernstein operator:

**Definition 4.** *We say that a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  exists for the pair  $(f_0, f_1)$  if there are points  $t_{n,0}, \dots, t_{n,n} \in [a, b]$  and coefficients  $\alpha_{n,0}, \dots, \alpha_{n,n} > 0$  such that the operator  $B_n : C[a, b] \rightarrow U_n$  defined by*

$$(7) \quad B_n f = \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} p_{n,k}$$

has the property that

$$(8) \quad B_n f_0 = f_0 \quad \text{and} \quad B_n f_1 = f_1.$$

We say that the nodes  $t_{n,0}, \dots, t_{n,n} \in [a, b]$  are strictly increasing if

$$t_{n,0} < t_{n,1} < \dots < t_{n,n}.$$

Observe that the preceding notion of a Bernstein operator imposes no restrictions on the nodes (save that they belong to  $[a, b]$ ). For a natural example of a Bernstein operator without strictly increasing nodes, see Example 10 below. On the other hand, the strict

positivity of the coefficients  $\alpha_{n,k}$  for  $k = 0, \dots, n$ , is included in our definition of Bernstein operator.

Two natural questions arise: when is the existence of a Bernstein operator guaranteed? and, is the Bernstein operator unique? It turns out that existence depends on *additional* properties of the space  $U_n$ , while uniqueness is easy to establish.

We will consistently use the following notation. Assume that  $p_{j,k}$ ,  $k = 0, \dots, j$ , is a Bernstein basis of the space  $U_j$ , then given  $f_0, f_1 \in U_j$  there exist coefficients  $\beta_{j,0}, \dots, \beta_{j,j}$  and  $\gamma_{j,0}, \dots, \gamma_{j,j}$  such that

$$(9) \quad f_0(x) = \sum_{k=0}^j \beta_{j,k} p_{j,k}(x) \quad \text{and} \quad f_1(x) = \sum_{k=0}^j \gamma_{j,k} p_{j,k}(x).$$

The next lemma answers the question of uniqueness positively. Remember that  $f_0$  is taken to be strictly positive on  $[a, b]$ , and  $f_1/f_0$ , strictly increasing.

**Lemma 5.** *Suppose that the linear subspace  $U_n \subset C^n[a, b]$ , where  $n \geq 1$ , possesses a non-negative Bernstein basis  $\{p_{n,k}\}_{k=0}^n$ . Let  $\beta_{n,k}$ ,  $k = 0 \dots, n$ , be the coefficients of the expansion of  $f_0$  in terms of the basis  $\{p_{n,k}\}_{k=0}^n$ , and let  $\gamma_{n,k}$ ,  $k = 0 \dots, n$  be the corresponding coefficients for  $f_1$ . If there exists a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  fixing the functions  $f_0, f_1 \in U_n$ , then  $\beta_{n,k} > 0$  for all  $k = 0 \dots, n$ . Moreover, the nodes of  $B_n$  are defined, for  $k = 0$  and  $k = n$ , by  $t_{n,0} = a$  and  $t_{n,n} = b$ , and in general, for  $k = 0 \dots, n$ , by*

$$(10) \quad t_{n,k} := \left( \frac{f_1}{f_0} \right)^{-1} \left( \frac{\gamma_{n,k}}{\beta_{n,k}} \right).$$

Furthermore, the coefficients of  $B_n$  are defined, for  $k = 0 \dots, n$ , by

$$(11) \quad \alpha_{n,k} := \frac{\beta_{n,k}}{f_0(t_{n,k})}.$$

In particular,

$$(12) \quad \alpha_{n,0} = \frac{1}{p_{n,0}(a)} \quad \text{and} \quad \alpha_{n,n} = \frac{1}{p_{n,n}(b)}.$$

*Proof.* Since  $B_n(f_0) = f_0$  and

$$(13) \quad f_0(x) = \sum_{k=0}^n \beta_{n,k} p_{n,k}(x),$$

we have

$$\sum_{k=0}^n f_0(t_{n,k}) \alpha_k p_{n,k} = \sum_{k=0}^n \beta_{n,k} p_{n,k}.$$

This entails that  $f_0(t_{n,k})\alpha_{n,k} = \beta_{n,k}$ , since  $\{p_{n,k}\}_{k=0}^n$  is a basis, and now (11) follows. Similarly, from  $B_n f_1 = f_1$  and

$$(14) \quad f_1(x) = \sum_{k=0}^n \gamma_k p_{n,k}(x)$$

we obtain  $f_1(t_{n,k})\alpha_k = \gamma_{n,k}$ . Using  $f_0 > 0$  and  $\alpha_{n,k} > 0$  we see that  $\beta_{n,k} > 0$ . Dividing by  $f_0(t_{n,k})\alpha_{n,k} = \beta_{n,k}$ , we find that  $t_{n,k}$  satisfies

$$(15) \quad \frac{f_1(t_{n,k})}{f_0(t_{n,k})} = \frac{\gamma_{n,k}}{\beta_{n,k}},$$

and now, since  $f_1/f_0$  is injective, its inverse exists and we get (10). Next, inserting  $x = a$  in (13) and in (14) we obtain  $f_0(a) = \beta_0 p_{n,0}(a)$  and  $f_1(a) = \gamma_0 p_{n,0}(a)$ . Thus

$$\frac{f_1(a)}{f_0(a)} = \frac{\gamma_{n,0}}{\beta_{n,0}},$$

and it follows by injectivity that  $t_{n,0} = a$ . An entirely analogous argument shows that  $t_{n,n} = b$ . Since  $f_0(a) = \beta_{n,0} p_{n,0}(a) = f_0(a)\alpha_{n,0} p_{n,0}(a)$  and  $f_0(b) = \beta_{n,n} p_{n,n}(b) = f_0(b)\alpha_{n,n} p_{n,n}(b)$ , (12) follows.  $\square$

Lemma 5 tells us that to obtain a Bernstein operator  $B_n$  fixing  $f_0$  and  $f_1$ , the nodes  $t_{n,k}$  must be the ones given by equation (10), and the coefficients  $\alpha_{n,k}$  by (11). A simple algebraic manipulation then shows that  $B_n$  does fix  $f_0$  and  $f_1$ . The difficulty to construct  $B_n$  lies in showing that for  $k = 0, \dots, n$ , the numbers

$$\frac{\gamma_{n,k}}{\beta_{n,k}}$$

belong to the image of  $[a, b]$  under  $f_1/f_0$ , so that we can define at all the corresponding node  $t_{n,k}$ . Even if this is the case, it does not follow in general that the nodes are increasing, cf. Theorem 24 in Section 5. It seems to be a non-trivial task to characterize those spaces  $U_n \subset C^n[a, b]$  such that there exists a Bernstein operator for given  $f_0, f_1 \in U_n$ , cf. [2].

It is not our aim to promote the idea that Bernstein operators with non-increasing nodes are of special interest. To the contrary, we prefer simple conditions guaranteeing the existence of Bernstein operators with increasing nodes as we shall do in Corollary 7 below. However, it should be noted that the assumption of an extended Chebyshev space over an interval  $[a, b]$  is not sufficient for the existence of Bernstein operators, cf. Section 5.

From a proof-technical point of view the following new criterion depending on an inductive argument is very useful: Existence at level  $n$  with increasing nodes entails existence at level  $n + 1$  with increasing nodes, and as a by-product we obtain strict interlacing property of the nodes at level  $n + 1$  and  $n$ .

**Theorem 6.** *Suppose that the linear subspaces  $U_n \subset U_{n+1} \subset C^{n+1}[a, b]$ , where  $n \geq 1$ , possess non-negative Bernstein bases  $p_{n,k}, k = 0, \dots, n$ , and  $p_{n+1,k}, k = 0, \dots, n+1$  respectively. If there exists a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  fixing the functions  $f_0, f_1 \in U_n$ , with strictly increasing nodes  $a = t_{n,0} < t_{n,1} < \dots < t_{n,n} = b$ , then there exists a Bernstein operator  $B_{n+1} : C[a, b] \rightarrow U_{n+1}$  fixing  $f_0, f_1$ , with strictly increasing and strictly interlacing nodes  $t_{n+1,0}, \dots, t_{n+1,n+1}$ , that is,*

$$(16) \quad a = t_{n+1,0} = t_{n,0} < t_{n+1,1} < t_{n,1} < t_{n+1,2} < t_{n,2} < \dots < t_{n+1,n} < t_{n,n} = t_{n+1,n+1} = b.$$

*Proof.* Let us write  $f_0 = \sum_{k=0}^{n+1} \beta_{n+1,k} p_{n+1,k}$  and  $f_1 = \sum_{k=0}^{n+1} \gamma_{n+1,k} p_{n+1,k}$ . By the preceding lemma, if the Bernstein operator  $B_{n+1} : C[a, b] \rightarrow U_{n+1}$  for  $(f_0, f_1)$  exists, then it has the form

$$B_{n+1}f := \sum_{k=0}^{n+1} f(t_{n+1,k}) \alpha_{n+1,k} p_{n+1,k},$$

where the positive coefficients  $\alpha_{n+1,k}$  are given by (11) (with  $n+1$  replacing  $n$ ), and the increasing nodes  $t_{n+1,k}$ , are given by  $t_{n+1,0} = a$ , by  $t_{n+1,n+1} = b$ , and in general, by (10) when  $k = 0 \dots, n+1$ . Thus, we need to show, first, that  $\beta_{n+1,0}, \dots, \beta_{n+1,n+1} > 0$ , in order to get the positivity of the coefficients  $\alpha_{n+1,k}$ , and second, that

$$(17) \quad \frac{\gamma_{n,k-1}}{\beta_{n,k-1}} < \frac{\gamma_{n+1,k}}{\beta_{n+1,k}} < \frac{\gamma_{n,k}}{\beta_{n,k}} \text{ for } k = 1, \dots, n,$$

to obtain the (strict) interlacing property of nodes; note that  $\gamma_{n,0}/\beta_{n,0} = \gamma_{n+1,0}/\beta_{n+1,0}$ , since both quantities equal  $f_1(a)/f_0(a)$ , and similarly  $\gamma_{n,n}/\beta_{n,n} = \gamma_{n+1,n+1}/\beta_{n+1,n+1}$  (since both quantities equal  $f_1(b)/f_0(b)$ ).

At level  $n$ , by assumption the Bernstein operator is defined via the coefficients  $\alpha_{n,k} > 0$ . Now the argument runs as follows: From the numbers  $\alpha_{n,k}$  we obtain the  $\beta_{n,k}$ , and from these the  $\beta_{n+1,k}$ , which in turn give us the  $\alpha_{n+1,k}$ .

Since  $\beta_{n,k} = f_0(t_{n,k})\alpha_{n,k}$ , it follows that  $\beta_{n,k} > 0$ . From  $f_0(a) = \beta_{n+1,0}p_{n+1,0}(a)$  and  $f_0(b) = \beta_{n+1,n+1}p_{n+1,n+1}(b)$  we see that  $\beta_{n+1,0} > 0$  and  $\beta_{n+1,n+1} > 0$ . We show next that for  $k = 1, \dots, n$ ,

$$(18) \quad \beta_{n+1,k} = \beta_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} + \beta_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)},$$

from which the positivity of  $\beta_{n+1,1}, \dots, \beta_{n+1,n}$  follows by Lemma 3. Applying the index raising formula given by Proposition 2 to  $f_0 = \sum_{k=0}^n \beta_{n,k} p_{n,k}$ , we see that

$$\begin{aligned} f_0 &= \beta_{n,0} \frac{p_{n,0}(a)}{p_{n+1,0}(a)} p_{n+1,0} + \sum_{k=1}^n \left[ \beta_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} + \beta_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)} \right] p_{n+1,k} \\ &\quad + \beta_{n,n} \frac{p_{n,n}(b)}{p_{n+1,n+1}(b)} p_{n+1,n+1}, \end{aligned}$$

and we obtain (18).

Regarding the interlacing property of nodes, another application of the index raising formula from Proposition 2, this time to  $f_1 = \sum_{k=0}^n \gamma_{n,k} p_{n,k}$ , yields

$$(19) \quad \gamma_{n+1,k} = \gamma_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} + \gamma_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)}$$

for  $k = 1, \dots, n$ . To show that  $\frac{\gamma_{n,k-1}}{\beta_{n,k-1}} < \frac{\gamma_{n+1,k}}{\beta_{n+1,k}}$ , or equivalently, that  $\gamma_{n,k-1} \beta_{n+1,k} < \gamma_{n+1,k} \beta_{n,k-1}$ , we use formulas (18) and (19) to rewrite the latter inequality as

$$(20) \quad \gamma_{n,k-1} \left( \beta_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} + \beta_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)} \right) < \left( \gamma_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} + \gamma_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)} \right) \beta_{n,k-1}.$$

Simplifying and using  $\frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} > 0$  (by Lemma 3), inequality (20) is easily seen to be equivalent to  $\frac{\gamma_{n,k-1}}{\beta_{n,k-1}} < \frac{\beta_{n,k}}{\gamma_{n,k}}$ , which is true by (10) together with the assumptions that  $f_1/f_0$  is increasing and that  $t_{n,k-1} < t_{n,k}$ .

Inequality  $\frac{\gamma_{n+1,k}}{\beta_{n+1,k}} < \frac{\gamma_{n,k}}{\beta_{n,k}}$  is proven in the same way.  $\square$

For the next corollary we *do not* a priori assume that  $f_0 > 0$  and  $f_1/f_0$  is strictly increasing, since multiplying by  $-1$  if needed, these properties can be obtained from the other assumptions.

**Corollary 7.** *Let  $f_0, \dots, f_n \in C^n[a, b]$ , and assume that the linear spaces  $U_k$  generated by  $f_0, \dots, f_k$  are extended Chebyshev spaces of dimension  $k+1$  for  $k = 0, \dots, n$ . Then for every  $k = 1, \dots, n$ , there exists a Bernstein operator  $B_k : C[a, b] \rightarrow U_k$  fixing  $f_0$  and  $f_1$ , with strictly increasing nodes and strictly interlacing with those of  $B_{k-1}$ .*

*Proof.* Since  $U_0$  is an extended Chebyshev space over  $[a, b]$ , the function  $f_0$  has no zeros. Multiplying by  $-1$  if needed, we may assume that  $f_0 > 0$ . Since  $U_1 = \langle f_0, f_1 \rangle$  is an extended Chebyshev space over  $[a, b]$  it is easy to see that that  $f_1/f_0$  is either strictly increasing or strictly decreasing. By multiplying  $f_1$  by  $-1$  if needed one may assume that  $f_1/f_0$  is strictly increasing. Let  $\{p_{1,0}, p_{1,1}\}$  be a non-negative Bernstein basis for  $U_1$ . We define

$$(21) \quad B_1 f := \alpha_{1,0} f(a) p_{1,0} + \alpha_{1,1} f(b) p_{1,1},$$

where  $\alpha_{1,0} = 1/p_{1,0}(a)$  and  $\alpha_{1,1} = 1/p_{1,1}(b)$ . Since both functions  $(B_1 f_0 - f_0) \in U_1$  and  $(B_1 f_1 - f_1) \in U_1$  have a zero at  $a$  and another zero at  $b$ , and  $U_1$  is an extended Chebyshev space we see that these functions are zero, so  $B_1$  fixes  $f_0$  and  $f_1$ . And now the result follows by inductively applying Theorem 6 to each  $U_{k+1}$  in the chain  $U_1 \subset U_2 \subset \dots \subset U_n$ .  $\square$

It is well known that given an extended Chebyshev space  $U_n$ , one can find functions  $f_0, \dots, f_n \in C^n[a, b]$  such that the linear spaces  $U_k$  generated by  $f_0, \dots, f_k$  are extended

Chebyshev spaces of dimension  $k + 1$  for  $k = 0, \dots, n$ , see e.g. Proposition 2.8 of [24] (cf. also Definition 2.4 in [24]). The functions  $f_0, \dots, f_n$  can be constructed in the following way: at first one shows that there exists a strictly larger interval  $[a, \beta] \supset [a, b]$  such that  $U_n$  is an extended Chebyshev system over  $[a, \beta]$  (cf. [24, p. 351]). Take now  $n$  different points  $\xi_1, \dots, \xi_n$  in the open interval  $(b, \beta)$ . For each  $k = 0, \dots, n$  define a non-zero function  $f_k \in U_n$  which vanishes on  $\xi_1, \dots, \xi_{n-k}$ . Then the linear spaces  $U_k$  generated by  $f_0, \dots, f_k$  are extended Chebyshev spaces over  $[a, b]$ . The disadvantage of this procedure is that the choice of the functions  $f_0$  and  $f_1$  does depend on the space  $U_n$ .

Thus, Corollary 7 implies the next result:

**Corollary 8.** *Let  $n \geq 1$  and let  $U_n$  be an extended Chebyshev space over  $[a, b]$ . Then it is possible to find functions  $f_0, f_1 \in U_n$  (with  $f_0$  strictly positive and  $f_1/f_0$  strictly increasing) and a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  with strictly increasing nodes, such that  $B_n$  fixes  $f_0$  and  $f_1$ .*

**Remark 9.** *Observe that the hypothesis of Corollary 8 is weaker than that of Corollary 7, and so is the conclusion, since  $f_0$  and  $f_1$  are chosen a posteriori, cf. also the discussion at the beginning of Section 5.*

We finish this section with an example illustrating our methods. The article [18] exhibits a sequence of positive linear operators converging to the identity on  $C[0, 1]$  and fixing 1 and  $x^2$ . This sequence is obtained by replacing  $x$  in (1) with a suitably chosen function  $r_n(x)$  such that  $\lim_n r_n(x) = x$ . It is also possible to fix 1 and  $x^2$  by using the generalized Bernstein operators considered here. As a matter of fact, it is possible to fix  $f_0(x) = 1$  and  $f_1(x) = x^j$  for any  $j \geq 1$  we wish (of course  $j = 1$  gives the classical case). From Lemma 5 we know how to determine the nodes and the coefficients, i.e., how  $B_n$  must be constructed.

On the other hand, we *cannot* use Corollary 7 to conclude that such a Bernstein operator  $B_{n,0,j}$  exists (the subscripts 0 and  $j$  refer to the exponents of the functions being fixed) since whenever  $j > 1$ , the space  $U_1 = \langle 1, x^j \rangle$  is not an extended Chebyshev space over the *closed* interval  $[0, 1]$ :  $x^j$  has a zero of order  $j$ . And unlike the situation considered in Theorem 6, the sequence of nodes we obtain is not strictly increasing: As noted below, given  $1 < j \leq n$ , we get  $t_{n,0} = \dots = t_{n,j-1} = 0$  by counting zeros at  $a = 0$ .

**Example 10.** *Fix  $j > 1$ , and let  $U_n$  be the space of polynomials over  $[0, 1]$  of degree at most  $n$ . For every  $n \geq j$ , there exists a Bernstein operator  $B_{n,0,j} : C[0, 1] \rightarrow U_n$  that fixes 1 and  $x^j$ , and converges in the strong operator topology to the identity, as  $n \rightarrow \infty$ . The operator  $B_{n,0,j}$  is explicitly given by*

$$B_{n,0,j}f(x) = \sum_{k=0}^n f \left( \left[ \frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)} \right]^{1/j} \right) \binom{n}{k} x^k (1-x)^{n-k}.$$

*Proof.* For the purposes of this argument we set  $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$  (this differs from the notation used in the introduction for the classical Bernstein polynomials, but

it is more convenient here). The condition  $1 = B_{n,0,j}1(x) = \sum_{k=0}^n \alpha_{n,k} p_{n,k}$  entails that  $\alpha_{n,k} = 1$  for all  $n, k$ . We use the equality  $x^j = B_{n,0,j}x^j$  to determine the nodes  $t_{n,k}$ . Writing  $x^j = \sum_{k=0}^n \gamma_{n,k} p_{n,k}$ , by (10) we have  $t_{n,k} = \gamma_{n,k}^{1/j}$ . It is immediate, from the order of the zeros at  $a = 0$ , that  $\gamma_i = 0$  whenever  $0 \leq i < j$ . Computing a few more coefficients leads to the conjecture that

$$(22) \quad \gamma_{n,k} = \frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}.$$

Instead of proving (22) by explicitly solving the general equation, it is easier to verify by substitution that these are the correct coefficients:

$$\begin{aligned} \sum_{k=0}^n \gamma_{n,k} p_{n,k} &= \sum_{k=j}^n \frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)} \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=j}^n \binom{n-j}{k-j} x^k (1-x)^{n-k} \\ &= x^j \sum_{k=0}^{n-j} \binom{n-j}{k} x^k (1-x)^{n-j-k} = x^j (x+1-x)^{n-j} = x^j. \end{aligned}$$

For  $l = 1, \dots, j-1$ , the inequalities

$$\frac{k-j+1}{n} < \frac{k-l}{n-l} < \frac{k}{n}$$

can be checked by simplifying and inspection. It follows that  $((k-j+1)/n)^j < t_{n,k}^j = \gamma_{n,k} < (k/n)^j$ , or equivalently, that  $0 < k/n - t_{n,k} < (j-1)/n$ . Thus,  $B_{n,0,j}x^m$  converges uniformly to  $x^m$  for  $m = 0, 1, 2$ , and by Korovkin's Theorem,  $B_{n,0,j}f \rightarrow f$  uniformly for all  $f \in C[0, 1]$ .  $\square$

### 3. GENERALIZED CONVEXITY

Let  $B_n$  denote the classical Bernstein operator defined in (1). W.B. Temple showed in [29] that for a convex function  $f$  the following monotonicity property

$$(23) \quad B_n f(x) \geq B_{n+1} f(x)$$

holds for all  $x \in [0, 1]$ . In [3] O. Aramă proved that

$$B_n f(x) - B_{n+1} f(x) = \frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} \left[ \frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n} \right] (f) \cdot \binom{n-1}{k} x^k (1-x)^{n-1-k}$$

where  $\left[ \frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n} \right] (f)$  is the divided difference of second order, thus providing a simple proof of Temple's result. A similar formula (see Theorem 7.5 in [17]) is due to Averbach. We obtain analogous results for the generalized Bernstein operators considered here. These generalized Bernstein operators  $B_n$  fix  $f_0$  and  $f_1$  instead of 1 and  $x$ , so rather than  $(1, x)$ -convexity, which is equivalent to standard convexity, the adequate notion for

our purposes is  $(f_0, f_1)$ -convexity, to be defined next. We shall see that for  $(f_0, f_1)$ -convex functions  $f$ , the following holds:  $B_n f \geq B_{n+1} f \geq f$ .

**Definition 11.** ([17, p. 280]) *Let  $E \subset \mathbb{R}$ . A function  $f : E \rightarrow \mathbb{R}$  is called convex on  $E$  with respect to  $(g_0, g_1)$  if for all  $x_0, x_1, x_2$  in  $E$  with  $x_0 < x_1 < x_2$ , the determinant*

$$(24) \quad D_{x_0, x_1, x_2}(f) := \det \begin{pmatrix} g_0(x_0) & g_0(x_1) & g_0(x_2) \\ g_1(x_0) & g_1(x_1) & g_1(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{pmatrix}$$

*is non-negative. We shall also use the shorter expression “ $(g_0, g_1)$ -convex”. Likewise, we say that  $f$  is  $(g_0, g_1)$ -concave if  $-f$  is  $(g_0, g_1)$ -convex, and  $(g_0, g_1)$ -affine if  $f \in \langle g_0, g_1 \rangle$ .*

**Remark 12.** *Note that the condition  $D_{x_0, x_1, x_2}(f) \geq 0$  for all  $x_0, x_1, x_2$  in  $E$  with  $x_0 < x_1 < x_2$  is equivalent to the same requirement but with  $x_0 \leq x_1 \leq x_2$ . Of course, in the degenerate case  $x_i = x_{i+1}$  the determinant is zero, so it makes no difference whether or not this possibility is included in the definition. In other words, only the ordering of the points  $x_0, x_1, x_2$  actually matters.*

One of the standard definitions of convexity stipulates that the graph of  $f$  must lie below the segment joining any two given points on the graph. It is well known that an analogous characterization holds for  $(f_0, f_1)$ -convex functions, but with affine functions being replaced by  $(f_0, f_1)$ -affine functions. More precisely,

**Proposition 13.** *Denote by  $\psi_{x_0, x_2}^f$  the unique function in  $U_1 := \langle f_0, f_1 \rangle$  that interpolates  $f$  at the points  $x_0 < x_2$ ,  $x_0, x_2 \in [a, b]$ , i.e.,  $\psi_{x_0, x_2}^f(x_0) = f(x_0)$  and  $\psi_{x_0, x_2}^f(x_2) = f(x_2)$ . Then  $f$  is  $(f_0, f_1)$ -convex if and only if for all  $x_0, x, x_2$  such that  $a \leq x_0 < x < x_2 \leq b$ ,*

$$(25) \quad f(x) \leq \psi_{x_0, x_2}^f(x),$$

*and in this case, for all  $y \in [a, b] \setminus [x_0, x_2]$ ,*

$$(26) \quad f(y) \geq \psi_{x_0, x_2}^f(y).$$

*Proof.* Let  $D_{x_0, x, x_2}(f)$  be as defined in (24). Observe that

$$\begin{aligned} D_{x_0, x, x_2}(f) &= D_{x_0, x, x_2}(f - \psi_{x_0, x_2}^f) \\ &= - (f(x) - \psi_{x_0, x_2}^f(x)) (f_1(x_2) f_0(x_0) - f_0(x_2) f_1(x_0)). \end{aligned}$$

Since  $f_1/f_0$  is strictly increasing,  $f_1(x_2) f_0(x_0) - f_0(x_2) f_1(x_0) > 0$ , so  $D_{x_0, x, x_2}(f) \geq 0$  is equivalent to  $f(x) \leq \psi_{x_0, x_2}^f(x)$ .

Next, assume that (25) holds for all  $x_0, x, x_2$  such that  $a \leq x_0 < x < x_2 \leq b$ . Suppose that for some  $u \in [a, b] \setminus [x_0, x_2]$  we have  $f(u) < \psi_{x_0, x_2}^f(u)$ . Without loss of generality we may assume that  $x_2 < u$ . We interpolate between  $x_0$  and  $u$  to obtain a contradiction:  $\psi_{x_0, x_2}^f(x_0) = \psi_{x_0, u}^f(x_0)$ , while  $\psi_{x_0, x_2}^f(u) > \psi_{x_0, u}^f(u)$ . Since  $\psi_{x_0, x_2}^f - \psi_{x_0, u}^f$  has exactly one zero, it follows that  $\psi_{x_0, x_2}^f > \psi_{x_0, u}^f$  on  $(x_0, u]$ . In particular,  $f(x_2) = \psi_{x_0, x_2}^f(x_2) > \psi_{x_0, u}^f(x_2)$ , which is impossible by (25) applied to  $f$  and  $\psi_{x_0, u}^f$ .  $\square$

Note that by inequality (25), convexity is the same as  $(1, x)$ -convexity. The next result generalizes to  $(f_0, f_1)$ -convex functions, the familiar inequality  $B_n f \geq f$  for the classical Bernstein operator acting on convex functions. Here it is not assumed that  $B_n$  is defined via an increasing sequence of nodes; it is enough to know that  $t_{n,k} \in [a, b]$ .

**Theorem 14.** *Assume that for some  $n \geq 1$ , there is a Bernstein operator  $B_n$  fixing  $f_0$  and  $f_1$ . Then for every  $(f_0, f_1)$ -convex function  $f \in C[0, 1]$  we have  $B_n f \geq f$ .*

*Proof.* Suppose  $B_n$  exists for some  $n \geq 1$ , and let  $\varepsilon > 0$ . We show that for an arbitrary  $x \in [a, b]$ ,  $B_n f(x) \geq f(x) - \varepsilon$ . Assume that  $x \in (a, b)$  (the cases  $x = a$  and  $x = b$  can be proven via obvious changes in the notation, or just by using continuity). First, select  $\delta > 0$  such that  $B_n \delta < \varepsilon$ . Next, by continuity of  $f$ , choose  $h > 0$  so small that  $[x - h, x + h] \subset [a, b]$  and  $\psi_{x-h, x+h}^f < f + \delta$  on  $[x - h, x + h]$ . Then  $\psi_{x-h, x+h}^f < f + \delta$  on  $[a, b]$  by (26), so

$$B_n f(x) > B_n \left( \psi_{x-h, x+h}^f - \delta \right) (x) = B_n \psi_{x-h, x+h}^f (x) - B_n \delta (x) > \psi_{x-h, x+h}^f (x) - \varepsilon \geq f(x) - \varepsilon,$$

where for the last inequality we have used (25).  $\square$

We shall use below the following characterization of  $(f_0, f_1)$ -convexity, due to M. Bessenyei and Z. Páles (cf. Theorem 5, p. 388 of [4]). While the result is stated there for open intervals, it also holds for compact intervals.

**Theorem 15.** *Let  $I := (f_1/f_0)([a, b])$ . Then  $f \in C[a, b]$  is  $(f_0, f_1)$ -convex if and only if  $(f/f_0) \circ (f_1/f_0)^{-1} \in C(I)$  is convex in the standard sense.*

**Example 16.** *Consider the Bernstein operator  $B_{n,0,j}$  from Example 10, defined on  $C[0, 1]$  and fixing 1 and  $x^j$ . It is easy to see from Theorem 15 that for  $s \in (0, j)$ , the function  $x^s$  is  $(1, x^j)$ -concave, while if  $s \in (j, \infty)$ ,  $x^s$  is  $(1, x^j)$ -convex. Therefore, by Theorem 14, for all  $x \in [0, 1]$  we have  $B_{n,0,j} x^s \leq x^s$  if  $s \in (0, j)$  and  $B_{n,0,j} x^s \geq x^s$  when  $s \in (j, \infty)$ .*

Our next objective is to obtain an analog of Aramă's result (presented at the beginning of this section) for generalized Bernstein operators  $B_n$ . Here the interlacing property of nodes is used in an essential way.

**Proposition 17.** *Let  $s_k, s_{k+1}, s_{k+2} \in [a, b]$  be such that  $s_k < s_{k+1} < s_{k+2}$ , and assume that  $G_k : C[c, d] \rightarrow \mathbb{R}$  is a functional of the form*

$$G_k(f) = a_k f(s_k) + b_k f(s_{k+1}) + c_k f(s_{k+2}),$$

*satisfying  $G_k(f_0) = G_k(f_1) = 0$ . Then  $b_k \geq 0$  if and only if  $G_k(f) \leq 0$  for all  $(f_0, f_1)$ -convex functions  $f \in C[t_k, t_{k+2}]$ .*

*Proof.* Let  $f$  be  $(f_0, f_1)$ -convex, and let  $\psi_{s_k, s_{k+2}}^f$  be the function in  $\langle f_0, f_1 \rangle$  that interpolates  $f$  at the points  $s_k$  and  $s_{k+2}$ . By (25),

$$G_k(f) = G_k \left( f - \psi_{s_k, s_{k+2}}^f \right) = b_k \left( f - \psi_{s_k, s_{k+2}}^f \right) (s_{k+1}) \leq 0$$

if and only if  $b_k \geq 0$ .  $\square$

**Theorem 18.** *Under the same hypotheses and with the same notation as in Theorem 6, let the linear functionals  $G_k, k = 1, \dots, n-1$ , be defined by*

$$G_k(f) = f(t_{n,k}) \alpha_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} - f(t_{n+1,k}) \alpha_{n+1,k} + f(t_{n,k-1}) \alpha_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)}.$$

Then

$$B_n f - B_{n+1} f = \sum_{k=1}^n G_k(f) \cdot p_{n+1,k}.$$

In particular, if  $f$  is  $(f_0, f_1)$ -convex then  $B_n f - B_{n+1} f \geq 0$ .

*Proof.* Recall that

$$B_n f = \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} p_{n,k} \text{ and } B_{n+1} f = \sum_{k=0}^{n+1} f(t_{n+1,k}) \alpha_{n+1,k} p_{n+1,k}$$

where  $t_{n,0} = t_{n+1,0} = a$ ,  $t_{n,n} = t_{n+1,n+1} = b$ , and  $t_{n,k-1} < t_{n+1,k} < t_{n,k}$  for  $k = 1, \dots, n$ . Using Proposition 2 we obtain

$$\begin{aligned} B_n f - B_{n+1} f &= \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} p_{n+1,k} \\ &\quad + \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} \frac{p_{n,k}^{(n-k)}(b)}{p_{n+1,k+1}^{(n-k)}(b)} p_{n+1,k+1} - \sum_{k=0}^{n+1} f(t_{n+1,k}) \alpha_{n+1,k} p_{n+1,k}. \end{aligned}$$

It follows from (12) that the first summands (corresponding to  $k = 0$ ) of the first and the last sum are the same, so they cancel out. Likewise, the  $n$ -th summand of the second sum and the  $(n+1)$ -th summand of the last sum cancel out. Thus

$$B_n f - B_{n+1} f = \sum_{k=1}^n p_{n+1,k} \left[ f(t_{n,k}) \alpha_{n,k} \frac{p_{n,k}^{(k)}(a)}{p_{n+1,k}^{(k)}(a)} - f(t_{n+1,k}) \alpha_{n+1,k} + f(t_{n,k-1}) \alpha_{n,k-1} \frac{p_{n,k-1}^{(n+1-k)}(b)}{p_{n+1,k}^{(n+1-k)}(b)} \right]$$

Finally, let  $f$  be  $(f_0, f_1)$ -convex. Taking  $s_k = t_{n,k-1}$ ,  $s_{k+1} = t_{n+1,k}$ , and  $s_{k+2} = t_{n,k}$  in Proposition 17, we get  $B_n f - B_{n+1} f \geq 0$ .  $\square$

A very natural question, not touched upon here, is under which conditions a sequence of Bernstein operators for  $(f_0, f_1)$  converges to the identity. It follows from Theorems 14 and 18 that if  $f$  is  $(f_0, f_1)$ -convex, then the sequence  $\{B_n f\}_{n=1}^{\infty}$  monotonically converges to some function  $g \geq f$  (assuming that a sequence of functions  $f_0, f_1, f_2, \dots$  are given such that  $\langle f_0, \dots, f_n \rangle$  is an extended Chebyshev space of dimension  $n+1$  for each  $n \in \mathbb{N}$ ). But we have not determined which conditions will ensure that  $g = f$ . In this regard, we expect

the strict interlacing property of nodes to be useful, since it entails, in a qualitative sense, that the sampling of functions is not “too biased”.

#### 4. TOTAL POSITIVITY AND GENERALIZED CONVEXITY

Let  $B_n : C[a, b] \rightarrow U_n$  be a Bernstein operator for the pair  $(f_0, f_1)$ . In Section 3 we proved that  $B_n f \geq f$  for all  $(f_0, f_1)$ -convex functions  $f \in C[a, b]$ . This did not require an increasing sequence of nodes; it was enough to know that  $t_{n,k} \in [a, b]$ .

In this section we show that  $B_n f$  is  $(f_0, f_1)$ -convex for every  $(f_0, f_1)$ -convex function  $f \in C[a, b]$ , provided that the nodes  $t_{n,0}, \dots, t_{n,n}$  are *increasing* and  $U_n$  is an extended Chebyshev space over  $[a, b]$ , and a similar result holds for so-called  $g$ -monotone functions. These statements will follow directly from more general results presented in [17] concerning shape preserving properties of linear transformations with totally positive kernels. The connection between total positivity and shape preserving properties of bases is a classical subject and it has been widely described, see e.g. [17] or the more recent survey [7].

The following definitions come from [17]. Let  $X$  and  $Y$  be subsets of  $\mathbb{R}$ . A function  $K : X \times Y \rightarrow \mathbb{R}$  is called *sign-consistent of order  $m$*  if there exists an  $\varepsilon_m \in \{-1, 1\}$  such

$$(27) \quad \varepsilon_m \det \begin{pmatrix} K(x_1, y_1) & K(x_1, y_2) & \dots & K(x_1, y_m) \\ K(x_2, y_1) & K(x_2, y_2) & \dots & K(x_2, y_m) \\ \dots & \dots & \dots & \dots \\ K(x_m, y_1) & \dots & \dots & K(x_m, y_m) \end{pmatrix} \geq 0$$

for all  $x_1 < x_2 < \dots < x_m$  in  $X$  and  $y_1 < y_2 < \dots < y_m$  in  $Y$ . If  $\varepsilon_m = 1$  we shall call  $K$  *positive of order  $m$* . A function  $K$  is *totally positive* if it is positive of all orders  $m$  with  $m \in \mathbb{N}, m \geq 1$ . Similarly, if one has strict positivity in (27) then  $K$  is called *strictly sign-consistent of order  $m$* , and if in addition  $\varepsilon_m = 1$  then  $K$  is *strictly positive of order  $m$* . Strict total positivity means that  $K$  is strictly positive of all orders  $m \in \mathbb{N}, m \geq 1$ .

The following result is well-known, see e.g. [24, p. 358], or the proof presented in [11, pp. 342–344]:

**Theorem 19.** *Let  $U_n \subset C^n[a, b]$  be an extended Chebyshev space over  $[a, b]$  and let  $p_{n,k}, k = 0, \dots, n$ , be a non-negative Bernstein basis for  $[a, b]$ . Then  $K : [a, b] \times \{0, \dots, n\} \rightarrow \mathbb{R}$  defined by*

$$(28) \quad K(x, k) := p_{n,k}(x)$$

*is totally positive, and  $K$  is strictly totally positive on  $(a, b) \times \{0, \dots, n\}$ .*

Following the notation of [8], [9], [21], we can deduce from the previous result that a non-negative Bernstein basis of an extended Chebyshev system over  $[a, b]$  is totally positive on  $[a, b]$  and so a B-basis.

We cite from [17, p. 284] the following result (specialized to the case of two functions  $F_0, F_1$  instead of a family  $F_1, \dots, F_m$ ).

**Theorem 20.** *Let  $X$  and  $Y$  be subsets of  $\mathbb{R}$ , let  $F_0, F_1$  be functions on  $Y$  and let  $K : X \times Y \rightarrow \mathbb{R}$  be continuous, and positive of order 3. Let  $\mu$  be a non-negative sigma-finite measure and  $B_K : C(Y) \rightarrow C(X)$  be defined*

$$B_K(F)(x) := \int_Y K(x, y) F(y) d\mu(y).$$

*If  $F$  is  $(F_0, F_1)$ -convex then  $B_K(F)$  is  $(B_K F_0, B_K F_1)$ -convex.*

From this we conclude:

**Theorem 21.** *Let  $U_n$  be an extended Chebyshev space over  $[a, b]$ . Assume there exists a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  fixing  $f_0$  and  $f_1$ , with increasing nodes  $t_{n,0} \leq \dots \leq t_{n,n}$ . If  $f \in C[a, b]$  is  $(f_0, f_1)$ -convex, then  $B_n(f)$  is  $(f_0, f_1)$ -convex.*

*Proof.* Put  $X = [a, b]$  and  $Y := \{0, \dots, n\}$ . Define the function  $\varphi : Y \rightarrow X$  by  $\varphi(k) := t_{n,k}$ , for  $k = 0, \dots, n-1$ . Observe that  $\varphi$  is monotone increasing (though perhaps not strictly), so it is order preserving. Next, set  $\mu := \sum_{k=0}^n \alpha_{n,k} \delta_k$ , where the  $\alpha_{n,k}$  are the positive coefficients defining  $B_n$  and  $\delta_k$  is the Dirac measure at the point  $k \in \{0, \dots, n\}$ . With  $K(x, k) := p_{n,k}(x)$ , we obtain, for every  $F \in C(Y)$ ,

$$B_K(F)(x) := \int_Y K(x, y) F(y) d\mu(y) = \sum_{k=0}^n F(k) \alpha_{n,k} p_{n,k}(x).$$

Now let  $f \in C(X)$ , and define  $F := f \circ \varphi \in C(Y)$ . Then

$$(29) \quad B_K(f \circ \varphi)(x) = \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} p_{n,k}(x) = B_n(f)(x).$$

If  $f \in C(X)$  is  $(f_0, f_1)$ -convex, then  $F = f \circ \varphi$  is  $(f_0 \circ \varphi, f_1 \circ \varphi)$ -convex, since  $\varphi$  preserves order (cf. Remark 12). Putting  $F_j = f_j \circ \varphi$  for  $j = 0, 1$ , an application of Theorem 20 shows that  $B_K(F)$  is  $(B_K F_0, B_K F_1)$ -convex. By formula (29) and the property that  $B_n$  fixes  $f_0$  and  $f_1$  one obtains

$$B_K F_j = B_n(f_j) = f_j$$

for  $j = 0, 1$ . Thus  $B_K(F) = B_n(f)$  is  $(f_0, f_1)$ -convex.  $\square$

In a similar way one might derive generalized monotonicity properties of the Bernstein operator. Here we need the following concept:

**Definition 22.** *Let  $g > 0$ . We say that  $f$  is  $g$ -increasing on  $[a, b]$  if  $f/g$  is increasing on  $[a, b]$ , i.e. that*

$$\frac{f(x_0)}{g(x_0)} \leq \frac{f(x_1)}{g(x_1)}$$

*for all  $x_0 < x_1$  in  $[a, b]$ .*

It is easy to see that a function  $f : [a, b] \rightarrow \mathbb{R}$  is  $g$ -increasing on  $[a, b]$  if for all  $x_0, x_1$  in  $[a, b]$  with  $x_0 < x_1$  the determinant

$$D_{x_0, x_1}(f) := \det \begin{pmatrix} f(x_0) & f(x_1) \\ g(x_0) & g(x_1) \end{pmatrix}$$

is non-negative. Using again Theorem 3.3 in [17, p. 284] and the proof of Theorem 21 one obtains:

**Theorem 23.** *Let  $U_n$  be an extended Chebyshev space over  $[a, b]$ . Assume that a Bernstein operator  $B_n : C[a, b] \rightarrow U_n$  exists for the pair  $(f_0, f_1)$  with increasing nodes  $t_{n,0} \leq \dots \leq t_{n,n}$ . If  $g \in \langle f_0, f_1 \rangle$  is positive and if  $f \in C[a, b]$  is  $g$ -monotone then  $B_n(f)$  is  $g$ -monotone.*

As one of the referees has pointed out to us, one might prove Theorem 23 and 21 without referring to Theorem 3.3 in [17, p. 284], and by using elementary shape preserving properties of totally positive bases as described in the surveys [7] or [15]. Moreover, Theorem 23 and 21 for the special case  $f_0 = 1$  and  $f_1(x) = x$  are direct consequences of Corollary 3.7 and 3.8 proved by T. Goodman in [15, p. 162].

## 5. NORMALIZED BERNSTEIN BASES AND EXISTENCE OF BERNSTEIN OPERATORS

Let  $b_{n,k}, k = 0, \dots, n$ , be a basis of a given subspace  $U_n \subset C[a, b]$  of dimension  $n + 1$ . The basis  $b_{n,k}, k = 0, \dots, n$  is *totally positive* if the kernel  $K(x, k) := b_{n,k}(x)$  is totally positive (in particular the functions  $b_{n,k}$  are non-negative). Suppose now that  $U_n$  contains the constant function 1. Then the basis  $b_{n,k}, k = 0, \dots, n$  is called *normalized* if

$$1 = \sum_{k=0}^n b_{n,k}(x)$$

for all  $x \in [a, b]$ . Normalized totally positive bases are important in geometric design due to their good shape preserving properties. J.-M. Carnicer and J.-M. Peña have shown that a normalized totally positive Bernstein basis is optimal, see [5], [6], [15]. Moreover it was shown in [8], and independently in [24], that a subspace  $U_n$  of  $C^n[a, b]$  of dimension  $n + 1$  containing the constant function possesses a normalized totally positive Bernstein basis provided that  $U_n$  and the space of all derivatives  $U'_n := \{f' : f \in U_n\}$  are extended Chebyshev systems over  $[a, b]$ .

From this point of view it is natural to conjecture that one might define well-behaved Bernstein operators (with increasing nodes) under the assumption that  $U_n$  possesses a normalized totally positive Bernstein basis and  $\langle f_0 \rangle, \langle f_0, f_1 \rangle$  are extended Chebyshev systems. However, we shall show by a counterexample that this is not true. We refer to [2] for a more detailed discussion under which conditions there might exist a Bernstein operator for given  $f_0, f_1 \in U_n$ .

In the following we consider the linear space  $U_3$  generated by the functions

$$1, x, \cos x, \sin x$$

over the interval  $[0, b]$  with  $b > 0$  which has been considered by several authors, see the references in [8] or [21]. Note that  $U_3$  and the space  $U_3'$  of all derivatives are extended Chebyshev spaces over  $[0, b]$  for every  $b \in (0, 2\pi)$ . Thus  $U_3$  possesses a normalized totally positive Bernstein basis for every  $b \in (0, 2\pi)$ . By [8] this entails that the critical length of  $U_3$  for design purposes is  $2\pi$ . However, we show in Theorem 24 that for  $b$  sufficiently close to  $2\pi$  (say,  $b \geq 4.5$ ) there is no Bernstein operator on  $U_3$  fixing 1 and  $x$ .

The obstruction for employing Corollary 7 is due to the fact that neither  $\langle 1, x, \cos x \rangle$  nor  $\langle 1, x, \sin x \rangle$  are extended Chebyshev spaces over  $[0, b]$  for all  $b < 2\pi$  (for instance,  $\sin x - x$  has a zero of order 3 at 0) so the chain of nested spaces cannot be continued beyond  $U_1 = \langle 1, x \rangle$ . By Corollary 8, it is nevertheless possible to construct a Bernstein operator fixing some pair of functions  $g_0, g_1 \in U_3' = \langle 1, \cos x, \sin x \rangle$ , with  $g_0 > 0$  and  $g_1/g_0$  strictly increasing, and hence, by Theorem 6 there is a corresponding Bernstein operator on  $U_3$ , fixing  $g_0$  and  $g_1$ , with strictly interlacing nodes.

**Theorem 24.** *Let  $\rho_0$  be the first positive root of  $b \mapsto \sin b - b \cos b$ , ( $\rho_0 \approx 4.4934$ ). Let  $U_3 = \langle 1, x, \cos x, \sin x \rangle$  and  $f_0 = 1$  and  $f_1(x) = x$ . Then for any  $b \in (0, \rho_0)$  there exists a Bernstein operator for  $(f_0, f_1)$  over  $[0, b]$ . The nodes  $t_0(b), t_1(b), t_2(b), t_3(b)$  satisfy the following inequalities:*

$$\begin{aligned} 0 = t_0(b) < t_1(b) < t_2(b) < t_3(b) = b & \text{ for } b \in (0, \pi) \\ 0 = t_0(b) < t_1(b) = t_2(b) < t_3(b) = b & \text{ for } b = \pi \\ 0 = t_0(b) < t_2(b) < t_1(b) < t_3(b) = b & \text{ for } b \in (\pi, \rho_0). \end{aligned}$$

*For  $b \in (\rho_0, 2\pi)$  there does not exist a Bernstein operator for  $(f_0, f_1)$ . The Bernstein operator preserves convex functions for any  $b \in (0, \pi)$ , but not for  $b \in (\pi, \rho_0)$ .*

In order to hold computations simple we shall present at first two general propositions and the following definition: We say that a subspace  $U_n \subset C[a, b]$  is *symmetric* if  $f \in U_n$  implies that the function  $F$  defined by  $F(x) := f(a + b - x)$  is in  $U_n$ .

Assume that  $U_n \subset C^n[a, b]$  is a symmetric, extended Chebyshev space over  $[a, b]$ . Let  $p_{n,k}, k = 0, \dots, n$  be a non-negative Bernstein basis of  $U_n$ , and let  $\beta_0, \dots, \beta_n$  and  $\gamma_0, \dots, \gamma_n$  be constants such that  $1 = \sum_{k=0}^n \beta_k p_{n,k}(x)$  and  $x = \sum_{k=0}^n \gamma_k p_{n,k}(x)$  for all  $x \in [a, b]$ .

**Proposition 25.** *Suppose that  $U_n \subset C^n[a, b]$  is a symmetric, extended Chebyshev space over  $[a, b]$  containing the constant function  $f_0 = 1$  and the identity function  $f_1(x) = x$ . If there exists a Bernstein operator  $B_n$  fixing  $f_0$  and  $f_1$ , then the following equalities hold for the coefficients  $\beta_k$  and the nodes  $t_{n,k}$ , whenever  $k = 0, \dots, n$ :*

$$(30) \quad \beta_k = \beta_{n-k} \quad \text{and} \quad t_{n,k} + t_{n,n-k} = a + b.$$

*Proof.* Let  $p_{n,k}, k = 0, \dots, n$  be a non-negative Bernstein basis of  $U_n$ . By suitably rescaling we may assume that

$$(31) \quad p_{n,k} \left( \frac{a+b}{2} \right) = 1$$

for all  $k = 0, \dots, n$ . Note that while this assumption will change the size of the constants  $\beta_k$  and  $\gamma_k$ , the ratio  $\gamma_k/\beta_k$  is invariant under any such rescaling, and hence so is the location of the nodes. Now  $q_{n,k}(x) := p_{n,k}(a+b-x)$  is in  $U_n$  and since  $q_{n,k}$  has a zero of order  $n-k$  at  $a$  and a zero of order  $k$  at  $x=b$ , there exists a nonzero constant  $C_k$  such that  $p_{n,k}(a+b-x) = C_k p_{n,n-k}(x)$ . It follows from (31) that  $C_k = 1$ , so

$$(32) \quad p_{n,k}(a+b-x) = p_{n,n-k}(x).$$

Replacing  $x$  by  $a+b-x$  in the expressions for 1 and  $x$ , from (32) we get

$$1 = \sum_{k=0}^n \beta_k p_{n,k}(a+b-x) = \sum_{k=0}^n \beta_k p_{n,n-k}(x) = \sum_{k=0}^n \beta_{n-k} p_{n,k}(x),$$

so  $\beta_{n-k} = \beta_k$  for all  $k = 0, \dots, n$ , and

$$a+b-x = \sum_{k=0}^n \gamma_k p_{n,k}(a+b-x) = \sum_{k=0}^n \gamma_k p_{n,n-k}(x) = \sum_{k=0}^n \gamma_{n-k} p_{n,k}(x).$$

Now

$$0 = x + a + b - x - (a+b)1 = \sum_{k=0}^n [\gamma_{n-k} + \gamma_k - (a+b)\beta_k] p_{n,k}(x),$$

so  $(a+b)\beta_k = \gamma_k + \gamma_{n-k}$ . Dividing by  $\beta_k$  (since  $\beta_k > 0$ , cf. Lemma 5) we obtain

$$\frac{\gamma_{n-k}}{\beta_{n-k}} = \frac{\gamma_{n-k}}{\beta_k} = a+b - \frac{\gamma_k}{\beta_k}.$$

But  $f_1(x)/f_0(x) = x$ , so by (10) we have  $t_{n,k} = \gamma_k/\beta_k$ , and thus  $t_{n,k} + t_{n,n-k} = a+b$ .  $\square$

**Remark 26.** Consider the Bernstein operators  $B_{n,0,2k+1}$  fixing 1 and  $x^{2k+1}$  in the classical polynomial spaces, but this time over the interval  $[-1, 1]$  rather than  $[0, 1]$ . Arguing as in the preceding proposition, with  $x^{2k+1}$  instead of  $x$  and  $[-a, a]$  instead of  $[a, b]$ , we see that the nodes for  $B_{n,0,2k+1}$  over  $[-1, 1]$  are obtained by reflecting about zero the nodes in  $[0, 1]$  (cf. Example 10). So the situation is very different from the classical case where 1 and  $x$  are fixed. There the nodes in different intervals are automatically obtained by an affine change of coordinates.

**Proposition 27.** Let  $p_{n,k}, k = 0, \dots, n$ , be a Bernstein basis, and for  $f \in U_n$  let  $\beta_0, \dots, \beta_n$  be the coefficients in the expression

$$(33) \quad f = \sum_{k=0}^n \beta_k p_{n,k}.$$

Then  $p_{n,n}(b)\beta_n = f(b)$  and

$$(34) \quad f'(b) = \beta_{n-1} p'_{n,n-1}(b) + \beta_n p'_{n,n}(b)$$

*Proof.* For the first statement insert  $x = b$  in (33), for the second take the derivative of  $f$  in (33) and then insert  $x = b$ .  $\square$

Now we turn to the proof of Theorem 24:

*Proof.* Let  $b > 0$ . We define now four functions in the space  $U_3$  by

$$(35) \quad \begin{aligned} p_{3,3}(x) &= x - \sin x, \\ p_{3,2}(x) &= (b - \sin b)(1 - \cos x) - (1 - \cos b)(x - \sin x), \\ p_{3,1}(x) &= p_{3,2}(b - x) \text{ and } p_{3,0}(x) = p_{3,3}(b - x). \end{aligned}$$

Clearly  $p_{3,3}$  has a zero of exact order 3 at 0, and it is strictly positive over  $(0, \infty)$ . The function  $p_{3,2}$  has a zero of order 2 at 0 and clearly  $p_{3,2}(b) = 0$ . In order that  $p_{3,2}$  has a zero of exact order 1 at  $b$  one has to require that

$$p'_{3,2}(b) = b \sin b - 2 + 2 \cos b$$

is non-zero. On the other hand, if  $p'_{3,2}(b) \neq 0$  then it follows from the symmetry of the basis that the system  $p_{3,k}$ ,  $k = 0, \dots, 3$  is indeed a Bernstein basis on  $[0, b]$ . Thus for all  $b > 0$  with  $p'_{3,2}(b) \neq 0$  there does exist a Bernstein basis on  $[0, b]$ . The function  $p_{3,2}$  is non-negative on  $[0, b]$  if  $b < 2\pi$  since in this case  $\langle 1, x, \cos x, \sin x \rangle$  is an extended Chebyshev space  $[0, b]$ . For  $b > 2\pi$  the function  $p_{3,2}$  might attain also negative values.

Recall that Bernstein bases are unique up to multiplicative constants. So to prove that a Bernstein operator does not exist, it is sufficient to consider the preceding basis. On the other hand, to prove that a Bernstein operator does exist, we need to exhibit nodes  $t_k$  in  $[0, b]$  and positive coefficients  $\alpha_k$ , for  $k = 0, 1, 2, 3$ . Now let  $1 = \sum_{k=0}^3 \beta_k p_{3,k}$  and  $x = \sum_{k=0}^3 \gamma_k p_{3,k}$ . Lemma 5 tells us what the nodes and coefficients must be if  $B_3$  exists. By Proposition 27 we have

$$\beta_3 = \frac{1}{b - \sin b} \text{ and } \gamma_3 = \frac{b}{b - \sin b},$$

so  $t_3(b) := \gamma_3/\beta_3 = b$ . It follows from (34) that

$$(36) \quad \beta_2 p'_{3,2}(b) = -\beta_3 p'_{3,3}(b) \text{ and } \gamma_2 p'_{3,2}(b) = 1 - \gamma_3 p'_{3,3}(b).$$

Thus

$$t_2(b) := \frac{\gamma_2}{\beta_2} = \frac{-\gamma_3 p'_{3,3}(b)}{-\beta_3 p'_{3,3}(b)} + \frac{1}{-\beta_3 p'_{3,3}(b)} = b - \frac{b - \sin b}{1 - \cos b}.$$

Now (30) implies that

$$t_1(b) = \frac{b - \sin b}{1 - \cos b}.$$

We see that  $t_2(b) - t_1(b) > 0$  if and only if  $g(b) := 2 \sin b - b \cos b - b > 0$ . Elementary calculus shows that  $g > 0$  on  $(0, \pi)$ ,  $g(\pi) = 0$ , and  $g < 0$  (at least) on  $(\pi, 3\pi/2)$ . If  $b = \pi$ , then  $t_1(\pi) = t_2(\pi) = \pi/2$ . Furthermore,  $t_2(b) < 0$  whenever  $\sin b - b \cos b < 0$ , so by Lemma 5, for  $b \in (\rho_0, 2\pi)$  there does not exist a Bernstein operator. To see that such operator exists when  $b \in (0, \rho_0)$ , note that since  $f_0 \equiv 1$ , by (11) we have  $\alpha_k = \beta_k$ , so it is

enough to show that  $\beta_k > 0$  for  $k = 0, 1, 2, 3$ . Since  $\beta_0 = \beta_3$  and  $\beta_1 = \beta_2$  by Proposition 25, and  $\beta_3 > 0$ , it suffices to prove that  $\beta_2 > 0$ . Now from equation (36) we get

$$\beta_2 = - \left( \frac{1 - \cos b}{b - \sin b} \right) \frac{1}{b \sin b - 2 + 2 \cos b},$$

so  $\beta_2 > 0$  if and only if  $b \sin b - 2 + 2 \cos b < 0$ . Elementary calculus shows that this is the case for every  $b \in (0, \rho_0)$ .

Regarding the convexity assertions, if  $b \in (0, \pi]$  then the Bernstein operator  $B_3$  preserves convexity by Theorem 21. Next, fix  $b \in (\pi, \rho_0)$ , write  $t_1 = t_1(b)$ ,  $t_2 = t_2(b)$ , and consider the convex function  $f(x) = (x - t_1)(x - t_2)$ . Since  $t_1 + t_2 = b$ , we have  $f(0) = f(b) = t_1(b - t_1)$ . By Proposition 25,  $\beta_0 = \beta_3$ , so

$$B_3 f(x) = f(0) \beta_0 p_{3,0}(x) + f(b) \beta_3 p_{3,3}(x) = \beta_0 f(0) (p_{3,0}(x) + p_{3,3}(x)).$$

Using  $\beta_0 f(0) > 0$  we see that  $B_3 f$  is convex if and only if  $F := p_{3,0} + p_{3,3}$  is convex. A direct computation shows that  $F(x) = b - \sin(b - x) - \sin x$ , so

$$F''(x) = \sin(b - x) + \sin x.$$

Thus  $F''(0) = \sin b < 0$ , since  $b \in (\pi, 2\pi)$ . By continuity,  $F''(x) < 0$  for all  $x$  in a small neighborhood of 0, so  $F$  is not convex..  $\square$

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