

# Control of MEMS vibration modes with Pulsed Digital Oscillators: Part I — Theory

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**Abstract**—The aim of this paper is to show that it is possible to excite selectively different mechanical resonant modes of a MEMS structure using Pulsed Digital Oscillators (PDOs). This can be done by simply changing the working parameters of the oscillator, namely its sampling frequency or its feedback filter.

A set of iterative maps is formulated to describe the evolution of the spatial modes between two sampling events in PDOs. With this lumped model, it is established that under some circumstances PDO bitstreams related to only one of the resonances can be obtained, and that in the antioscillation regions of the PDO the mechanical energy is absorbed into the electrical domain on average. The possibility of selecting for a given resonant frequency the oscillation and antioscillation behaviour allows one to obtain oscillations at any given resonant mode of the MEMS structure.

**Index Terms**—Microelectromechanical systems (MEMS), microresonators, oscillators, sigma-delta modulation, multimode control, energy efficiency

## List of symbols

$\langle a \rangle$	$a$ Modulus 1, $a \in \mathbb{R}$
$[a]$	Largest integer less or equal to $a$ , $a \in \mathbb{R}$
$z^*$	Complex conjugate of $z \in \mathbb{C}$
$\text{Re}(z)$	Real part of $z \in \mathbb{C}$
$\text{Im}(z)$	Imaginary part of $z \in \mathbb{C}$
$g.c.d.(M, N)$	Greater common divider, $M, N \in \mathbb{N}$

## I. INTRODUCTION

There is a large set of applications based on MEMS working in resonance, such as the sensing of acceleration, pressure, mass change, etc. These resonant sensors often detect shifts of the resonant frequency or amplitude changes of MEMS structures put in resonance in response to an external stimulus [1]–[4]. In most cases, resonant MEMS exploit structures such as beams [3]–[6] or plates [2], [7] with additional supporting elements (springs, arms, etc.) that are excited in one of their mechanical modes.

With the development of the MEMS technology, a lot of efforts have been made to improve the sensitivity of resonant sensors (see ref. [8] and papers cited there). In most cases, the sensitivity of such sensors depends on the resonant frequency of the MEMS structure, which is usually excited in one of its mechanical resonant modes. Most of the methods that

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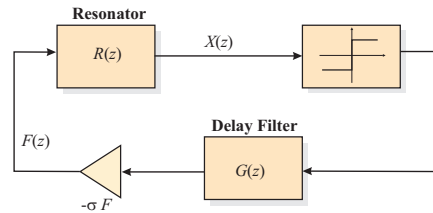


Fig. 1. General single feedback topology of the pulsed digital oscillator.

have been proposed to improve the sensitivity of gravimetric (or mass change) resonant sensors are based on increasing their operating frequency. For instance, parametric resonance amplification is an example of an efficient technique [9]–[12] that allows the excitation of the same mechanical structure at a higher frequency to improve sensitivity. In gravimetric sensors, the use of higher order modes generally serves to increase the sensitivity to mass changes, whereas it is desired in gyroscopes to avoid the activation of certain resonant modes of the inertial mass [13]. Moreover, the activation of higher vibration modes to increase performance has been also reported in atomic force microscopes [14], [15] and in piezoelectric sensors and actuators [16], where electrodes have been specifically designed to activate certain modes.

Thus, depending on the specific application, the selective activation of different spatial vibration modes of a mechanical resonator can be one way to improve performance for a large number of MEMS sensors [17]. The purpose of this paper is to show that Pulsed Digital Oscillators [18], [19], may be a tool to selectively excite self-sustained resonant modes in MEMS structures.

PDOs are simple circuits that allow linear resonators to maintain self-sustained oscillations. The PDO general single-feedback topology is depicted in fig. 1. The circuit topology consists of a resonator, a 1-bit quantizer (sign function), and a simple digital feedback filter,  $G(z)$ . The position of the MEMS resonator is evaluated at each sampling time, and short pulses of force are applied to the resonator. These circuits allow changes in the resonant frequency of the resonator to be monitored simply by processing the binary sequence generated at their output [20], [21]. The use of a pulsed actuation with constant amplitude and simplified position/velocity sensing requirements overcomes some of the nonlinearities usually found in MEMS with standard actuation techniques. This is clearly the case for the example of thermoelectric actuation,

where the applied force is proportional to the square of the applied voltage or current. The same approach can also be used to the electrostatic actuation case, if the variation of the electrode gap is negligible (small signal displacement).

In order to analyze the dynamics of PDOs working with MEMS resonators with more than one resonance it is convenient to obtain the iterative map of the dynamical system, using a lumped model for the MEMS structure. The dynamics of MEMS structures that display predominately one-dimensional behaviour, such as cantilevers or beams, are described through 1D partial differential equations (PDEs) [22]. A common strategy is to reduce distributed systems in the form of PDEs to lumped systems in the form of ordinary differential equations (ODEs) [23]. This way, instead of using a PDE, this device can be described by a set of mass-spring-damper coupled differential equations, the typical and yet effective way to model the dynamics of a variety of MEMS [7], [19], [20], [24]–[27]. This is an important result that will allow us to link the MEMS geometry with the iterative map of the PDO working with the resonator. We start with the partial differential equation (Sec. II) that describes the transversal deflections of a beam [22], [28]. Then a system of ordinary differential equations, coupled via the external driving, is obtained. Based on the pulsed nature of the actuation, in Sec. III a set of iterative maps is obtained to describe the evolution of the spatial modes between two sampling events.

Once the lumped model has been obtained, the iterative map of the PDO working with a resonator with several resonances is found in Sec. III. The objective of Sec. IV is to show that under some circumstances the PDO is able to excite only one of the resonances of the resonator. In this case the bitstream at the output of the oscillator is related exclusively to that resonance. It will be shown that this behaviour may depend on the initial condition of the resonator. This implies that in order to select a given resonance another mechanism is needed. In [25] it was shown that under some circumstances the actuation of PDOs on linear resonators may, on average, extract energy out of the resonator. This mechanism is for the first time analyzed and characterized in a general way in Sec. V. It will be used as a tool to generate selective oscillation in MEMS resonators in part II of the paper.

## II. ANALYSIS OF THE VIBRATION MODES OF ELASTIC BEAMS

In order to obtain a suitable lumped model to generate the iterative map of PDOs actuating resonators with more than one resonance, it is necessary to analyze the mechanical resonances of the structure. The general problem of transversal vibrations on elastic beams has been considered, for instance, in books [22], [28], including direct applications to MEMS in the latter book. A number of papers have studied beam vibrations in the context of MEMS [23], [29]–[33].

The aim of this section is to derive a reduced-order model of a specific MEMS cantilever with the given external driving using one of the techniques described in review [23] and successfully applied to MEMS in the above-mentioned papers. This is a required step in order to obtain the lumped model,

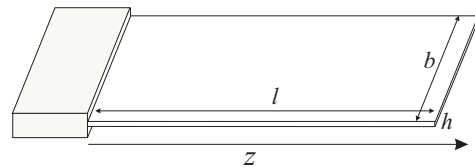


Fig. 2. A schematic drawing of the clamped-free beam in the PDO.

which later will allow the generation of the PDO iterative map actuating the MEMS resonator variables. We will start from a general PDE that describes transversal deflections of a cantilever and apply a strategy based on discretising the initial distributed system by means of a set of spatial eigenmode functions  $\psi_i(x, y, z)$ , which in the general case is infinite. As a result, we will obtain a set of ordinary differential equations that represent the dynamics of each spatial mode as the mass-spring-damper equation with appropriate parameters.

Let us emphasise that in contrast to Refs. [30]–[32] that examine MEMS devices with electrostatic actuation, we will focus on a system that is subjected to a different type of excitation (driving by short pulses) and discuss the impact of such type of driving on mechanical modes generated in beams.

According to this, we assume that the MEMS cantilever is a clamped-free beam shown in fig. 2 that is excited by the external force  $F(\xi, \tau)$ . The dimensionless equation and the boundary conditions that describe the transverse vibrations of the beam are as follows [28]:

$$\begin{aligned} \frac{\partial^4 u}{\partial \xi^4} + \frac{\partial^2 u}{\partial \tau^2} + \gamma \frac{\partial u}{\partial \tau} &= F(\xi, \tau), \\ u(0, \tau) = \frac{\partial u(0, \tau)}{\partial \xi} &= 0, \quad \frac{\partial^2 u(1, \tau)}{\partial \xi^2} = \frac{\partial^3 u(1, \tau)}{\partial \xi^3} = 0, \end{aligned} \quad (1)$$

where  $u(\xi, \tau)$  is the transverse displacement at the position  $\xi$  and time  $\tau$ . In (1), the dimensionless coordinate along the beam axis  $\xi$ , time  $\tau$  and the dissipation parameter due to viscous damping  $\gamma$  are defined through the dimensional ones  $z$ ,  $t$  and  $c$ :

$$\xi = z/l, \quad \tau = \frac{t}{l^2} \sqrt{\frac{EI}{\rho A}}, \quad \gamma = \frac{cl^2}{\sqrt{\rho A EI}}, \quad (3)$$

where  $E$  is the Young's modulus,  $I$  is the moment of inertia of the cross-section (for beams with a rectangular cross-section  $I = bh^3/12$ ),  $\rho$  is the density,  $A$  is the area of the cross-section,  $c$  is the dissipation coefficient,  $l$  is the length of the beam,  $b$  is its width and  $h$  is its thickness.

Next, the continuous equation (1) is discretized using the truncated set of linear mode shapes [23], [34]

$$u(\xi, \tau) = \sum_{i=1}^M x_i(\tau) \psi_i(\xi), \quad (4)$$

where  $x_i(\tau)$  are time-dependent functions and  $\psi_i(\xi)$  are the spatial mode shapes. For a clamped-free beam, these functions are [28]:

$$\psi_i(\xi) = \cosh(\Omega_i \xi) - \cos(\Omega_i \xi) + \nu_i (\sin(\Omega_i \xi) - \sinh(\Omega_i \xi)), \quad (5)$$

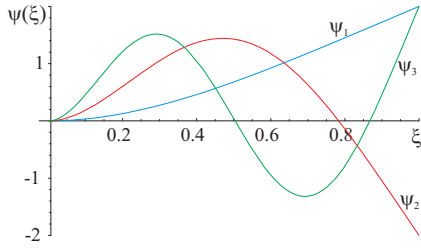


Fig. 3. Functions  $\psi_{1,2,3}(\xi)$  versus the axial coordinate  $\xi$ .

where  $\nu_i = (\cos \Omega_i + \cosh \Omega_i) / (\sin \Omega_i + \sinh \Omega_i)$ , and  $\Omega_i$  are the roots of the equation

$$\cos \Omega \cosh \Omega = -1. \quad (6)$$

The first three functions  $\psi_{1,2,3}(\xi)$  (calculated according to (5) for the first three solutions of (6)) are shown in fig. 3 as functions of the axial coordinate. Later in the paper, the vibrations that correspond to these functions will be called spatial modes with spatial frequencies  $\Omega_i$ .

Substituting (4) into (1) yields

$$\sum_{i=1}^M (\ddot{x}_i \psi_i + \gamma \dot{x}_i \psi_i + x_i \psi_i^{IV}) = F(\xi, \tau) + \varepsilon_r, \quad (7)$$

where the overdot notation denotes differentiation with respect to time, and the superscript ‘IV’ denotes the fourth derivative with respect to position. According the Galerkin procedure, we require that the residue term  $\varepsilon_r$  be orthogonal to each eigenmode  $\psi_i$ . Multiplying the latter equation by  $\psi_j$ , integrating over the beam domain and taking into account that

$$\int_0^1 \psi_i^2 d\xi = 1, \quad \int_0^1 \psi_i \psi_j d\xi = \delta_{ij}, \quad \int_0^1 \psi_i^{IV} \psi_j d\xi = \Omega_i^4 \delta_{ij}, \quad (8)$$

where  $\delta_{ij}$  is the Kronecker delta, we obtain the system of ordinary differential equations

$$\ddot{x}_i + \gamma \dot{x}_i + \Omega_i^4 x_i = F_i(\tau), \quad i = 1, \dots, M. \quad (9)$$

Note that the system (9) presents each spatial mode as the mass-spring-damper system with appropriate parameters such as the dissipation parameter  $\gamma$  and the temporal frequency  $\omega_{0i} = \Omega_i^2$ . Using that set of equations allows us to describe the evolution of spatial eigenmodes separately from each other and to obtain the overall solution of the problem in the form of a superposition of the modes. The functions  $F_i$  in the right hand of the equations are the ‘‘projections’’ of the external force onto each eigenmode. They define what quantity of the force is transmitted to excite the oscillations of a mode:

$$F_i(\tau) = \int_0^1 \psi_i(\xi) F(\xi, \tau) d\xi. \quad (10)$$

Using the expressions (3) that establish the correlations between dimension and dimensionless parameters, the radial and cyclic dimensional temporal frequencies in eq. (9) are

$$\omega_{0i} = \frac{\Omega_i^2}{l^2} \sqrt{\frac{EI}{\rho A}}, \quad f_{0i} = \frac{\Omega_i^2}{2\pi l^2} \sqrt{\frac{EI}{\rho A}}, \quad (11)$$

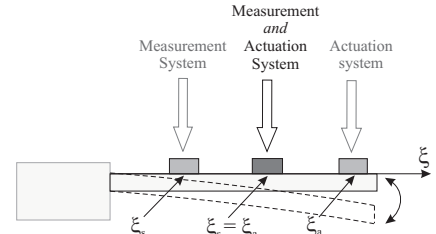


Fig. 4. Schematic drawing of the microbeam. The figure shows possible position of the measurement ( $\xi_s$ ) and actuation system ( $\xi_a$ ).

and they are proportional to squared spatial frequencies  $\Omega_i$ .

The first four frequencies  $\Omega_i$  calculated as the roots of eq. (6) are as follows:

$$\begin{aligned} \Omega_1 &= 1.8751, & \Omega_2 &= 4.69409, \\ \Omega_3 &= 7.85479, & \Omega_4 &= 10.9955. \end{aligned} \quad (12)$$

Now let us return to the PDO and rewrite the system of equations (9) for our specific case. To actuate the resonator, short pulses of force  $\{+F_0, -F_0\}$  are applied to it [18], with the sign of the pulses defined by the sign  $\sigma$  of the gain in the feedback loop and on the position of the resonator. (For the resonator of the paper, we assume that we actuate the system by delta-pulses.) Position sensing is required only at the fixed time instants defined by the sampling frequency  $f_s$ . At these moments, the sensing system defines whether the resonator position is above or below the steady-state position. The actuation and sensing points in the cantilever can be placed in the same place or separated (see for example fig. 4). Let us denote  $\xi_s$  as the coordinate along the beam axis at which the position of the resonator is measured by the sensing system and  $\xi_a$  as the coordinate at which the pulse of external driving is applied to the resonator by the actuating system.

Although we present the excitation in our model as a sequence of Dirac delta pulses with a constant amplitude, and this model has its limitations, we suppose that this approach could be applied to a wider class of realistic MEMS devices that utilises pulsed driving to excite or stabilise oscillation of a mechanical structure. Let us give some discussion here.

In practical realisations, force pulses  $F(t)$  always have finite width  $T_p$ . However, if  $f_0 \cdot T_p \ll 1$  and  $T_p \ll 1/\zeta$ , where  $f_0$  is the natural frequency of oscillations and  $\zeta$  is the normalised dissipation, then the displacement caused by this short pulse  $\Delta x \approx 0$  and the change of velocity  $\Delta v \approx \int_{\tau_n}^{\tau_n + T_p} F(t) dt$ . Therefore, mathematically this system can be reduced to one that is driven by Dirac delta pulses. Note that many MEMS are characterised by a relatively high frequency of oscillations and a high  $Q$ -factor, and these assumptions are typically valid.

Secondly, electrostatic short pulse actuation in the form  $F_e(t) = F_l(t) / (1 - \Delta x(t)/x_0)^2$ , where  $F_l$  is the linearised part of the force,  $\Delta x(t)$  is the displacement and  $x_0$  is the capacitor gap, can be modelled to some extent as constant amplitude pulses if oscillations of the resonator are small. In this case, we can neglect the nonlinear part of the force in that formula. In addition, in ref. [35], for an electrostatically driven system short pulses were used as a compensation to avoid instability. Both cases were studied, the linearised force

$F_l$  and the nonlinear  $F_e$ , and comparison of results were given there, showing that at high sampling frequencies the linearised approach is valid.

Thus, the external driving force for a beam embedded in the loop of a PDO can be written in the following form:

$$F(\xi, \tau) = -\sigma F_0 \sum_n \text{sgn}(u(\xi_s, (n-L-1)T_s)) \times Q(\xi - \xi_a) \delta(\tau - nT_s), \quad (13)$$

where  $u(\xi, \tau)$  is the position of the resonator,  $L+1$  is the number of delays in the feedback (see fig. 1),  $\sigma$  is the sign of the feedback loop,  $f_s = (f_s^*/2\pi l^2) \cdot \sqrt{EI/\rho A}$  is the dimensionless sampling frequency,  $T_s = 1/f_s$  is the dimensionless interval of time between two impulses,  $\delta(x)$  is the Dirac delta function and  $\text{sgn}(x)$  is the signum function. In the most general case, an impulse is applied to a finite area at the beam, and therefore the function  $Q(\xi)$  represents the ‘‘density distribution’’ of the actuating force. In the simplified statement of the problem, this function can be replaced by the Dirac delta function, i.e.  $Q(\xi) = \delta(\xi)$ . The latter means that the pulse is localised at the point  $\xi$ .

Let us denote  $\tau_n = nT_s$  as the  $n$ th instant of time at which we apply the force pulse. Using the general definition (10) and the specific form of the force (14), the functions  $F_i(\tau)$  may be expressed in the following way:

$$F_i(\tau) = -\sigma F_0 \sum_n \int_0^1 \psi_i(\xi) \text{sgn}(u(\xi_s, \tau_{n-L-1})) \times Q(\xi - \xi_a) \delta(\tau - \tau_n) d\xi = -\sigma F_{0i} \sum_n b_{n-L-1} \delta(\tau - \tau_n), \quad (14)$$

where the amplitude of the external driving for each mode is denoted by

$$F_{0i} = F_0 \int_0^1 \psi_i(\xi) Q(\xi - \xi_a) d\xi, \quad (15)$$

and the sequence of signs is introduced by

$$b_n = \text{sgn} \sum_{i=1}^M x_i(\tau_n) \psi_i(\xi_s). \quad (16)$$

Equation (15) emphasises that the  $i$ th spatial mode receives a portion of the external driving which is proportional to the value  $\psi_i(\xi_a)$ . Note that in the simple case  $Q(\xi - \xi_a) = \delta(\xi - \xi_a)$ , the expression (15) is reduced to

$$F_{0i} = F_0 \psi_i(\xi_a). \quad (17)$$

Equation (16) uses the truncated expansion (4) to express the sensed positions of the beam.

Now, expressions (9) and (14)–(16) can be used to specify the equations that will describe the dynamics of the first  $M$  spatial modes ( $i = 1, \dots, M$ ) in the PDO with  $L+1$  delay blocks in the feedback loop:

$$\ddot{x}_i + \gamma \dot{x}_i + \Omega_i^4 x_i = -\sigma F_{0i} \sum_n b_{n-L-1} \delta(\tau - \tau_n), \quad (18)$$

where the definitions of  $F_{0i}$  and  $b_n$  are given above. For simplicity, the expressions (16) can be replaced by the following one (on condition that  $\psi_1(\xi_s) > 0$ ):

$$b_n = \text{sgn}\{x_1(\tau_n) + \beta_2 x_2(\tau_n) + \dots + \beta_M x_M(\tau_n)\}, \quad (19)$$

where

$$\beta_i = \psi_i(\xi_s) / \psi_1(\xi_s). \quad (20)$$

Formula (20) introduces a new family of parameters in order to avoid always operating in terms of the spatial functions  $\psi_i$ . The parameters  $\beta_i$  now define the geometry of the sensing system and the contribution of the  $i$ th spatial mode to the overall position of the resonator.

In the same manner, we can introduce the parameters  $\mu_i$  that defines the geometry of the actuating system and the impact of the external driving on the  $i$ th mode excitation:

$$\mu_i = \psi_i(\xi_a) / \psi_1(\xi_a), \quad F_{0i} = \mu_i F_{01}. \quad (21)$$

In the case of two resonances, expressions (19), (20) and (21) can be simply written as  $b_n = \text{sgn}\{x_1(\tau_n) + \beta x_2(\tau_n)\}$  with  $\beta = \psi_2(\xi_s) / \psi_1(\xi_s)$  and  $F_{02} = \mu F_{01}$  with  $\mu = \psi_2(\xi_a) / \psi_1(\xi_a)$ .

Now the reduced-order model of the specific PDO system is defined by the set of eqs. (18)–(21). In the obtained model, only two parameters,  $\beta$  and  $\mu$ , fully describe all possible positions of the actuating and sensing of the PDO on the MEMS resonator.

### III. ITERATIVE SYSTEM FOR THE PDO

In the previous section we have converted the PDE to an ODE and obtained the system of equations (18) that describes the evolution of spatial modes in the PDO. In this section we discuss how the ordinary differential equations can be further reduced to discrete-time iterative equations (maps).

The system (9) is subjected to pulsed excitation given by the expressions (14). Between two sampling instants, the resonator moves freely, and at each sampling event its velocity is instantaneously changed by the applied delta-pulse. Thus, for the sequence of variables that we extract from the mass-spring-damper equations (9) at the sampling instants, we will be able to formulate iterative equations.

In order to do so, for each spatial mode with the index ‘‘ $i$ ’’ we introduce the normalised damping factor  $\rho_i = \gamma / (2\Omega_i^2)$  and the variable

$$v_i(\tau) = -\frac{1}{\sqrt{1-\rho_i^2}} \left( \rho_i x_i(\tau) + \frac{\dot{x}_i(\tau)}{\Omega_i^2} \right).$$

Now we can rewrite the ordinary differential equations of the second order (9) in the symmetrical form

$$\frac{d}{d\tau} \begin{pmatrix} x_i(\tau) \\ v_i(\tau) \end{pmatrix} = \Omega_i^2 \begin{pmatrix} -\rho_i & -\sqrt{1-\rho_i^2} \\ \sqrt{1-\rho_i^2} & -\rho_i \end{pmatrix} \begin{pmatrix} x_i(\tau) \\ v_i(\tau) \end{pmatrix},$$

$$v_i(\tau_n+) - v_i(\tau_n-) = \frac{\sigma F_{0i}}{\Omega_i^2 \sqrt{1-\rho_i^2}} b_{n-L-1}. \quad (22)$$

The solution of (22) is given by

$$\begin{pmatrix} x_i(\tau) \\ v_i(\tau) \end{pmatrix} = \exp(-\Omega_i^2 \rho_i \tau) \mathbf{R}(\Omega_i^2 \tau \sqrt{1 - \rho_i^2}) \begin{pmatrix} x_i(\tau_0) \\ v_i(\tau_0) \end{pmatrix}, \quad (23)$$

where  $\mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  is the rotation matrix by the angle  $\theta$ . If we define the sequence  $\{x_{i,n}, v_{i,n}\} = \{x_i(\tau_n), v_i(\tau_n)\}$  as the extracted values of  $x_i(\tau)$  and  $v_i(\tau)$  at the sampling instants  $\tau_n$ , the set of iterative equations will have the following form:

$$\begin{pmatrix} x_{i,n+1} \\ v_{i,n+1} \end{pmatrix} = \alpha_i \mathbf{R}(2\pi f_i) \begin{pmatrix} x_{i,n} \\ v_{i,n} \end{pmatrix} + \begin{pmatrix} 0 \\ \zeta_i \end{pmatrix} b_{n-L}, \quad (24)$$

for  $i = \overline{1, M}$ . Note that the values  $b_n$  are defined by (16). In the set (24), we introduce the following dimensionless parameters:

$$\begin{aligned} \rho_i &= \frac{\gamma}{2\Omega_i^2}, & f_i &= \frac{\Omega_i^2}{f_s} \sqrt{1 - \rho_i^2}, \\ \zeta_i &= \frac{\sigma F_{0i}}{\Omega_i^2 \sqrt{1 - \rho_i^2}}, & \alpha_i &= \exp\left(\frac{-2\pi \rho_i f_i}{\sqrt{1 - \rho_i^2}}\right), \end{aligned} \quad (25)$$

where  $\rho_i$  is the dimensionless damping parameter,  $f_i$  is the normalised sample ratio in terms of the paper [26] (or normalised frequency in terms of the paper [20], both terms will be used throughout the paper),  $\zeta_i$  is the normalised increment (note that it depends on the sign of the feedback loop) and  $\alpha_i$  is the contraction factor. In (24), the first letter of the index, i.e.  $i$ , refers to the number of a spatial mode, and the second letter, i.e.  $n$ , refers to the iteration number.

In a similar way to the previous section, we draw attention to the particular case of the first two spatial modes. It is very convenient to establish the correlation between the parameters that refer to the first and the second modes in explicit form. Therefore, we introduce the ratio of the spatial frequencies of the modes as follows:

$$\nu = \frac{\Omega_2}{\Omega_1}, \quad (26)$$

and apply it to connect the set of parameters for the modes:

$$\begin{aligned} \rho_2 &= \frac{\rho_1}{\nu^2}, \\ f_2 &= \nu^2 f_1 \frac{\sqrt{1 - \rho_1^2/\nu^4}}{\sqrt{1 - \rho_1^2}} \approx \nu^2 f_1, \\ \zeta_2 &= \frac{\mu \zeta_1}{\nu^2} \frac{\sqrt{1 - \rho_1^2}}{\sqrt{1 - \rho_1^2/\nu^4}} \approx \frac{\mu \zeta_1}{\nu^2}, \\ \alpha_2 &= \alpha_1 \sqrt{1 - \rho_1^2/\nu^4} / \sqrt{1 - \rho_1^2/\nu^4} \approx \alpha_1. \end{aligned} \quad (27)$$

Note that from the onset, our theory has included such parameters as the frequencies of the eigenmodes. As a consequence, the parameters used for formulation of the iterative system cannot be considered as independent ones, i.e. the values of controlling parameters have to be changed according the expressions (27). For instance, in the experimental set the sampling frequency is measured with respect to the frequency of the fundamental mode (i.e. the first spatial mode with the lowest frequency), and so the sampling ratio  $f_1$  is given as a ‘‘basic’’ parameter. The value of  $f_2$  can be obtained simply by re-calculating  $f_1$  with the ratio  $\nu$ .

Equations (24) introduce the iterative system as a set of 2D iterative maps and operate with variables such as the position of the resonator  $x_n$  and the variable  $v_n$  proportional to the velocity of the resonator. Results presented in these variables are easier to understand since  $x_n$  and  $v_n$  have an obvious physical interpretation. However, it is convenient to work on the complex plane with appropriate normalisation and introduce a set of complex 1D maps instead of 2D maps written in real variables. By defining the following complex variable

$$u_{i,n} = x_{i,n} - jv_{i,n},$$

the evolution of the system is described now by the complex map

$$u_{i,n+1} = p_i u_{i,n} - j\zeta_i b_{n-L}, \quad (28)$$

where  $p_i = \alpha_i \exp(-j2\pi f_i)$ , and for definition of  $\alpha_i$ ,  $f_i$  and  $\zeta_i$  see (25).

The system (24) contains two coupled subsets — iterative equations that describe the first spatial mode component by the variables  $x_{1,n}$  and  $v_{1,n}$  and the second spatial mode by the variables  $x_{2,n}$  and  $v_{2,n}$ . Later in the paper, dynamics that are displayed by the first subset of (24) will be referred as the ‘‘first spatial mode’’ (or the first mode, eigenmode or resonance), and dynamics that are displayed by the second subset will be referred as the ‘‘second spatial mode’’ (or the second mode). Note that the real position of the system is defined by the superposition of these two components according to the formula (4).

#### IV. CONTROL OF RESONANT MODES WITH PDOs

The objective of this section is to study the behaviour of PDOs working with resonators with more than one resonance. Now that we have established the possibility of extracting a lumped model of the PDE governing a given MEMS structure, it is possible to continue the line of work introduced in [20] to study the dynamics of PDOs. The first result will show that under some conditions, depending on the initial condition of the resonator, it is possible to obtain bitstreams at the output of the oscillator related only to one of the resonant frequencies of the resonator. It will be shown later that, at least in some cases, the oscillation frequency may generally depend on the initial condition of the resonator.

On the other hand, it has already been shown in [36] that, given a resonator with one resonance and a configuration of the oscillator, either the oscillator induces self-sustained oscillations on it, or, on the contrary, tends to cancel any oscillation by extracting energy until some limit cycle is reached near the origin. This behaviour will be used to selectively activate one of the resonances of a resonator with more than one resonance. The objective of Section V will be the study of this ‘anti-oscillation’ behaviour.

##### A. Iterative map with two resonances

Let us focus on the case with two resonances. We will assume that the feedback filter is of the form  $G(z) = z^{-L-1}$ . Using the general complex formula (28) to formulate the iterative equations we obtain the following equations:

$$u_{1,n+1} = pu_{1,n} - j\zeta_1 b_{n-L} \quad (29)$$

$$u_{2,n+1} = qu_{2,n} - j\zeta_2 b_{n-L} \quad (30)$$

$$b_n = \text{sgn}(\text{Re}(u_{1,n} + \beta u_{2,n})) \quad (31)$$

where  $\zeta_i$  are normalized forces (25),  $p = \alpha_1 \exp(-j2\pi f_1)$ ,  $q = \alpha_2 \exp(-j2\pi f_2)$  and  $\beta$  depends on the MEMS geometry. In the case of a cantilever it would be defined as in eq. (20).

The dynamical system associated with these equations may be defined in the following way:

$$\begin{aligned} \mathbb{C}^2 \times \mathcal{B}^L &\xrightarrow{\tau} \mathbb{C}^2 \times \mathcal{B}^L \\ (u_1, u_2, b_1, \dots, b_L) &\rightarrow (pu_1 - j\zeta_1 b_L, qu_2 - j\zeta_2 b_L, \\ &\quad \text{sgn}(\text{Re}(u_1 + \beta u_2)), b_1, \dots, b_{L-1}) \end{aligned} \quad (32)$$

where  $\mathcal{B} = \{-1, 1\}$ ,  $u_1, u_2 \in \mathbb{C}$ ,  $b_i \in \mathcal{B}$ ,  $1 \leq i \leq L$ . For  $L = 0$ :

$$\begin{aligned} \tau : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (u_1, u_2) &\rightarrow (pu_1 - j\zeta_1 \text{sgn}(\text{Re}(u_1 + \beta u_2)), \\ &\quad qu_2 - j\zeta_2 \text{sgn}(\text{Re}(u_1 + \beta u_2))) \end{aligned} \quad (33)$$

Now, following a similar approach to [20], we define:

*Definition 1:* The projections  $\pi_0 : \mathbb{C}^2 \times \mathcal{B}^L \rightarrow \mathbb{C}^2$  and  $\pi_i : \mathbb{C} \times \mathcal{B}^L \rightarrow \mathcal{B}$ ,  $1 \leq i \leq L$  are defined as  $\pi_0((u_1, u_2, b_1, \dots, b_L)) = (u_1, u_2)$ , and  $\pi_i((u_1, u_2, b_1, \dots, b_L)) = b_i$ .

We will consider that  $\tau^0(z)$ ,  $z \in \mathbb{C}^2 \times \mathcal{B}^L$ , is the identity and that  $\tau^n(z)$  is the result of  $n$  applications of the function  $\tau$  to  $z$ ,  $n > 0$ . In order to simplify notation we further define:

*Definition 2:* The function  $T_n(z) : \mathbb{C}^2 \times \mathcal{B}^L \rightarrow \mathbb{C}^2 \times \mathcal{B}^L$ ,  $n \geq 0$  is defined as  $\pi_0(\tau^n(z))$ .

*Definition 3:* The function  $s : \mathbb{C}^2 \rightarrow \mathcal{B}$  is defined as  $s(u_1, u_2) = \text{sgn}(\text{Re}(u_1 + \beta u_2))$ .

*Definition 4:* In the case  $L \geq 1$ , the functions  $b_n^i(z) : \mathbb{C} \times \mathcal{B}^L \rightarrow \mathcal{B}$  are defined as  $\pi_i(\tau^n(z))$ ,  $1 \leq i \leq L$ ,  $n \geq 0$ . If  $L = 0$ ,  $b_n^0(z)$  is defined as  $s(T_n(z))$ ,  $n \geq 0$ .

If  $L \geq 1$  the feedback loop of the oscillator ensures that  $b_n^i(z) = s(T_{n-i}(z))$ ,  $1 \leq i \leq L$ ,  $n-i \geq 0$ . If  $n-i < 0$ ,  $n \geq 0$ , then  $b_n^i(z) = \pi_{|n-i|}(z)$ .

Now, for a given binary sequence  $(\mathbf{b})$  we may define the sequence  $h_n(p, \mathbf{b})$  as:

$$h_n(p, \mathbf{b}) = -jp^{n-1} \sum_{i=0}^{n-1} p^{-i} b_{i-L}, \quad n > 0 \quad (34)$$

with  $h_0(p, \mathbf{b}) = 0$ ,  $p \in \mathbb{C}$ , and test for the admissibility of that binary sequence:

*Lemma 1:* Given a sequence  $(\mathbf{b})$  and  $z_0 = (u_{1,0}, u_{2,0}, b_{-1}, \dots, b_{-L}) \in \mathbb{C}^2 \times \mathcal{B}^L$ , if  $s(p^n u_{1,0} + \zeta_1 h_n(p, \mathbf{b}), q^n u_{2,0} + \zeta_2 h_n(q, \mathbf{b})) = b_n$ ,  $n > 0$ , then  $s(T_n(z)) = b_n$ ,  $n > 0$ .

*Proof.* It is obvious since in this case:

$$\begin{aligned} T_n(z_0) &= \left( p^n u_{1,0} - j\zeta_1 p^{n-1} \sum_{i=0}^{n-1} p^{-i} s(T_{n-L}(z_0)), \right. \\ &\quad \left. q^n u_{2,0} - j\zeta_2 q^{n-1} \sum_{i=0}^{n-1} q^{-i} s(T_{n-L}(z_0)) \right) = \\ &= (p^n u_{1,0} + \zeta_1 h_n(p, \mathbf{b}), q^n u_{2,0} + \zeta_2 h_n(q, \mathbf{b})) \end{aligned} \quad (35)$$

and  $s(T_n(z_0)) = b_n$ , for all  $n \geq 0$ .  $\square$

This is an important result because it is the mechanism we will use to prove that a given bitstream can be present at the output of a PDO. A similar result has been previously used in [20] to prove that under some circumstances the bitstream of a PDO, working with a resonator with only one resonance, is an  $S^3$  sequence at the resonant frequency of the resonator.

Due to the presence of two different resonances in the resonator, one might expect a bitstream related to a mixture of both frequencies, most probably rendering the bitstream useless, from the point of view of obtaining at least one of the resonant frequencies directly from it. However, it will be proved that under some conditions, the bitstream at the output of the PDO with two resonances may be related only to one of them.

We will principally focus on a certain group of frequencies, called 'tuned' frequencies which were first found in [20]. The main reason to focus on these frequencies is that the oscillator with low or no losses presents a regularity pattern that simplifies the analysis of its dynamics.

### B. Lossless resonator case

Now, we will obtain sufficient conditions for oscillation at one of the resonant frequencies of the resonator. This result will be valid for frequencies  $f_1 = \frac{M_1}{N_1}$  and  $f_2 = \frac{M_2}{N_2}$ , with  $\text{g.c.d.}(N_1, N_2) = 1$ , and  $\text{g.c.d.}(M_i, N_i) = 1$ . Furthermore, we will assume that frequency  $f_1$  is 'tuned' [20], i.e., for  $\text{sgn}(\zeta_1) = 1$ :

$$\begin{aligned} -N_1 \text{ even} : &\left( \lfloor \frac{N_1}{4} + \frac{1}{2} \rfloor - (L+1)M_1 \right) \bmod N_1 = 0. \\ -N_1 \text{ odd} : &\left( \lfloor \frac{N_1}{2} + \frac{1}{2} \rfloor - (L+1)2M_1 \right) \bmod 2N_1 = 0. \end{aligned} \quad (36)$$

In the case  $\text{sgn}(\zeta_1) = -1$ , the former conditions are:

$$\begin{aligned} -N_1 \text{ even} : &\left( \lfloor \frac{3N_1}{4} + \frac{1}{2} \rfloor - (L+1)M_1 \right) \bmod N_1 = 0 \\ -N_1 \text{ odd} : &\left( \lfloor \frac{3N_1}{2} + \frac{1}{2} \rfloor - (L+1)2M_1 \right) \bmod 2N_1 = 0 \end{aligned} \quad (37)$$

Under the above conditions it has been proved, [20], that if  $p = e^{-j2\pi f}$ , with  $f$  being a tuned frequency and  $b_n = \text{sgn}(\text{Re}(p^n u_0))$ , for  $u_0 \in \mathbb{C}$ , then:

$$\text{sgn}(\text{Re}(h_n(p, \mathbf{b}))) = b_n \quad (38)$$

for  $n \rightarrow \infty$ , which is a result greatly simplifying the use of Lemma 1 for sequences of the form of (b). Now, we may state the following theorem:

**Theorem 1:** If an oscillator with two lossless resonances,  $f_1 = \frac{M_1}{N_1}$ ,  $p = e^{-j2\pi f_1}$ , and  $f_2 = \frac{M_2}{N_2}$ ,  $q = e^{-j2\pi f_2}$ , with  $\text{g.c.d.}(M_i, N_i) = 1$ , is tuned to  $f_1$ , and  $u_{1,0}, u_{2,0} \in \mathbb{C}$  are such that  $|\text{Re}(p^n u_{1,0})| > |\beta u_{2,0}| + |\zeta_1|n \bmod N_1 + |\beta \zeta_2|N_1 N_2$ ,  $0 \leq n < N_1$ , then  $s(T_n(u_{1,0}, u_{2,0}, \text{sgn}(\text{Re}(p^{-1}u_{1,0})), \dots, \text{sgn}(\text{Re}(p^{-L}u_{1,0})))) = \text{sgn}(\text{Re}(p^n u_{1,0}))$ ,  $n \geq 0$ . (The proof is given in Appendix I.)

This last result shows us that under some circumstances it is possible to obtain a bitstream at the output of the PDO related to only one of the resonances of the resonator. Among the conditions needed to ensure this oscillation we find the initial condition in both resonators.

### C. Leaky resonator case

A similar result may be found for the leaky case.

**Theorem 2:** For any oscillator with two leaky resonances such that  $\langle f_1 \rangle = \frac{M_1}{N_1} = \frac{1}{4(L+1)}$ ,  $p = e^{-j2\pi f_1}$  and  $\langle f_2 \rangle = \frac{M_2}{N_2}$ ,  $q = e^{-j2\pi f_2}$ ,  $0 < \alpha_p, \alpha_q < 1$ , with  $\text{g.c.d.}(M_2, N_2) = 1$ ,  $\text{g.c.d.}(N_1, N_2) = 1$ , there exist a  $n_0 > 0$  and a  $\delta_1 > 0$  such that if  $1 - \delta_1 < \alpha_p < 1$  then there is a constant  $C_{\alpha_p} \in \mathbb{R}^+$  such that for  $u_{1,0}, u_{2,0} \in \mathbb{C}$ , with  $|\text{Re}(\alpha_p^n p^n u_{1,0})| > C_{\alpha_p} + |\beta u_{2,0}|$ ,  $0 \leq n < n_0$ , then  $s(T_n(u_{1,0}, u_{2,0}, \text{sgn}(\text{Re}(p^{-1}u_{1,0})), \dots, \text{sgn}(\text{Re}(p^{-L}u_{1,0})))) = \text{sgn}(\text{Re}(p^n u_{1,0}))$ ,  $n \geq 0$ . (The proof is given in Appendix II.)

This result extends the previous one on lossless resonances. It has been proved that under some conditions it is possible to obtain a bitstream related to only one of the resonances out of a PDO working with a leaky resonator.

### D. Discussion

These results provide some insight into the fact that the oscillation frequency of a PDO working with more than one resonance generally may depend on the initial condition of the resonator. Let us assume that we have a resonator with two tuned frequencies. The above results state that, depending on the initial conditions of the resonator, the oscillation frequency will be one or the other. This is reflected in Figures 5 and 6. As it can be observed, depending on the initial condition of the resonator, the oscillator may oscillate at the first or second resonance. This means that in order to control the selective activation of a given resonance we need a new mechanism: the ‘anti-oscillation’ mode.

The objective of the next Section is to show that, depending on the normalized frequency of the resonator and the feedback filter of the PDO, either the PDO on average tends to provide energy to the resonator, or, on the contrary, tends to extract it until a limit cycle is reached near the origin. This second behaviour is called the ‘anti-oscillation’ mode. By choosing an appropriate feedback filter and a sampling frequency, it will be possible to put a given resonance in the ‘oscillation’ mode and the others in the ‘anti-oscillation’ mode. This mechanism will allow the selective activation of the vibration modes of the structure.

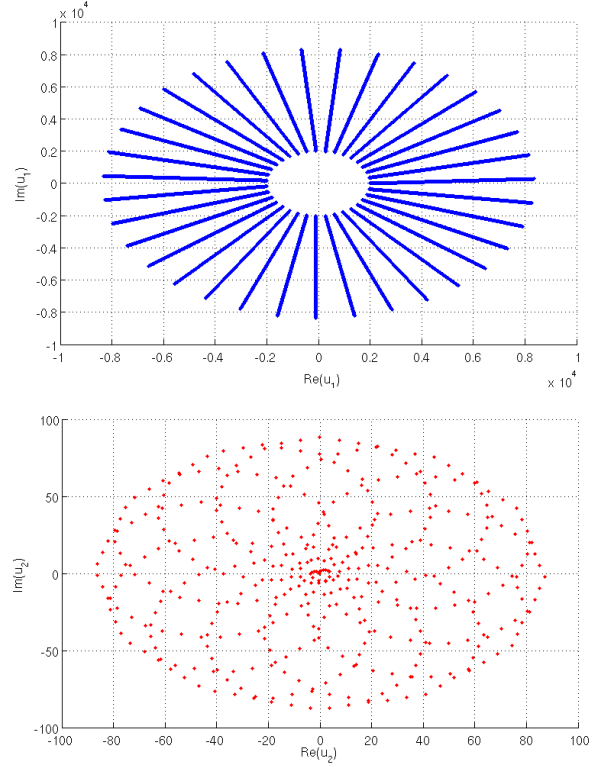


Fig. 5. Evolution of a PDO, with  $L = 2$ ,  $\sigma = 1$  and a lossless resonator with two tuned resonances,  $f_1 = 3/35$ ,  $f_2 = 1/12$ ,  $\beta = 1$ . The initial conditions of both resonances are  $u_{1,0} = 1000je^{j\pi/35}$ ,  $u_{2,0} = 10je^{j\pi/6}$ .

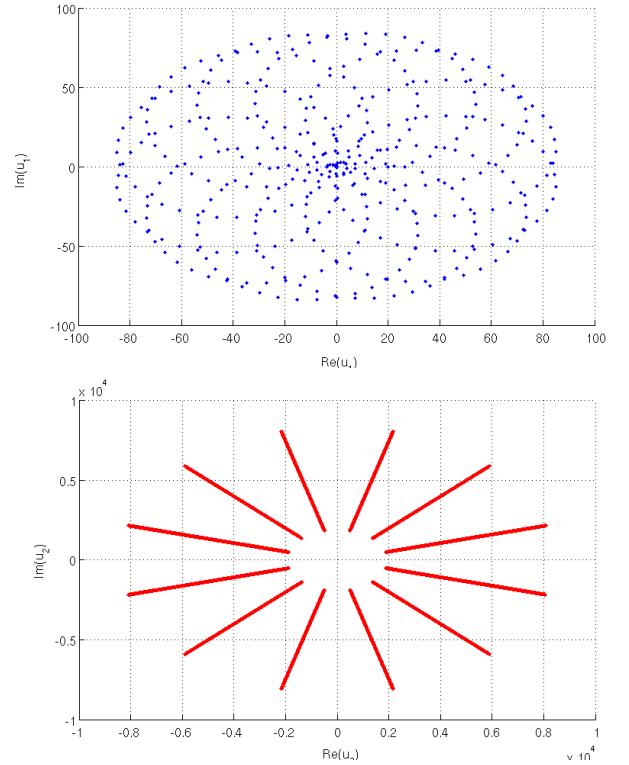


Fig. 6. Evolution of a PDO, with  $L = 2$ ,  $\sigma = 1$  and a lossless resonator with two tuned resonances,  $f_1 = 3/35$ ,  $f_2 = 1/12$ ,  $\beta = 1$ . The initial conditions of both resonances are  $u_{1,0} = 10je^{j\pi/35}$ ,  $u_{2,0} = 1000je^{j\pi/6}$ .



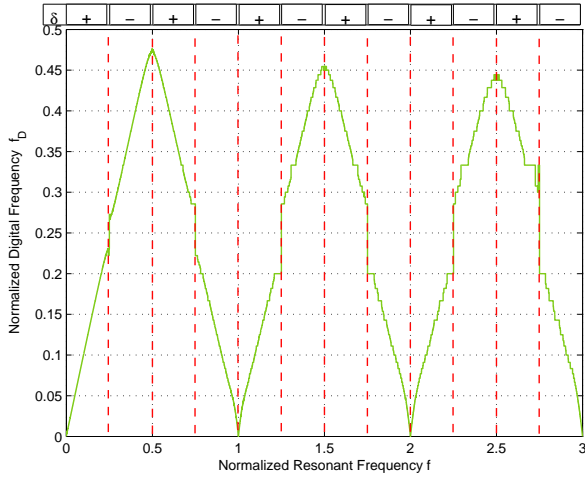


Fig. 9. Response of the oscillator,  $L = 1$ , as a function of the normalized resonant frequency when the feedback sign is changed in the antioscillation regions, for a resonator with  $\rho = 0.01$ .

PDO does not generate oscillations in the frequency segments  $\langle f(L+1) \rangle \in [0, 1/2)$  when  $\sigma > 0$  and  $\langle f(L+1) \rangle \in [1/2, 1)$  when  $\sigma < 0$ . The purpose of this section is to show that in these frequency segments there are no tuned frequencies and therefore it is not possible to find perfect  $S^3$  sequences at the resonant frequency of the resonator in the lossless case.

Now, let us assume that  $\sigma = +1$ , then if a frequency  $f = \frac{M}{N}$ ,  $\text{g.c.d.}(M, N) = 1$ , with  $N$  even, is tuned there is an integer  $m$  such that:

$$m = \frac{\lfloor \frac{N}{4} + \frac{1}{2} \rfloor - (L+1)M}{N} \quad (41)$$

which means that:

$$\langle (L+1) \frac{M}{N} \rangle = \frac{1}{4} + \frac{1}{2N} - \frac{\langle N/4 + 1/2 \rangle}{N} \quad (42)$$

and therefore  $\langle (L+1) \frac{M}{N} \rangle \notin (\frac{1}{2}, 1)$ . Now, for  $N$  odd,  $m$  is:

$$m = \frac{\lfloor \frac{N}{2} + \frac{1}{2} \rfloor - (L+1)2M}{2N} \quad (43)$$

and therefore:

$$\langle (L+1) \frac{M}{N} \rangle = \frac{1}{4} + \frac{1}{4N} \quad (44)$$

which means that  $\langle (L+1) \frac{M}{N} \rangle \in (\frac{1}{4}, \frac{1}{2}]$ . In both cases,  $N$  odd or even, we have that for any integer  $k$ :

$$\frac{M}{N} \notin \left( \frac{1}{2(L+1)} + \frac{k}{L+1}, \frac{1}{L+1} + \frac{k}{L+1} \right) \quad (45)$$

Following a similar procedure for the case  $\sigma = -1$ , it can be shown that  $\langle (L+1) \frac{M}{N} \rangle \in [\frac{1}{2}, 1]$  and therefore:

$$\frac{M}{N} \notin \left( \frac{k}{L+1}, \frac{1}{2(L+1)} + \frac{k}{L+1} \right) \quad (46)$$

Taking into account [20], this result tells us that oscillations giving rise to perfect  $S^3$  sequences at the resonant frequency of a lossless resonator are not possible in these regions. This is not a complete result because it does not provide any information regarding what happens in those forbidden zones,

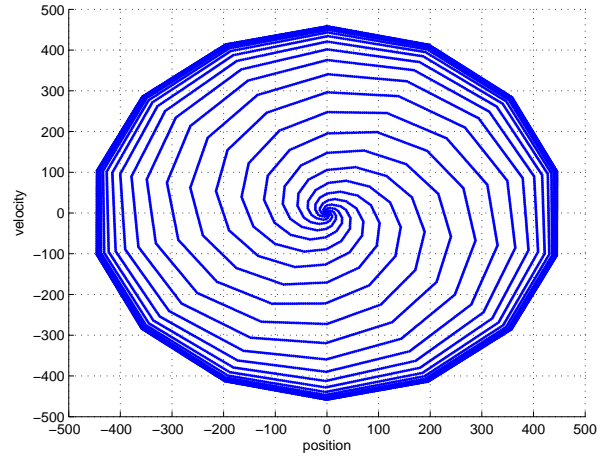


Fig. 10. Oscillation mode for a PDO with  $f = 1/7$ ,  $\rho = 0.0001$ ,  $\sigma = -1$ ,  $F = 1$ ,  $L = 3$  and initial condition of the resonator  $u_0 = 1 + j$ .

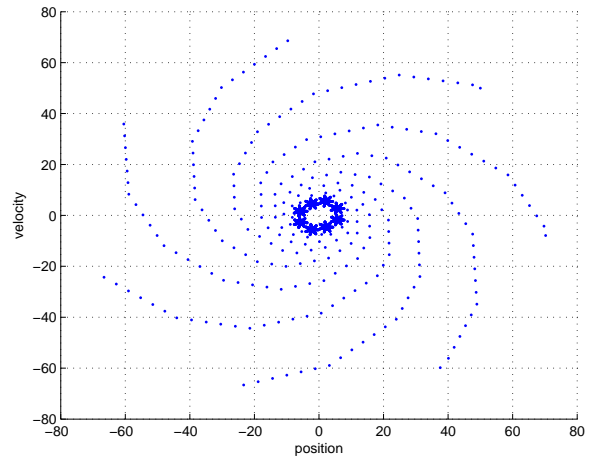


Fig. 11. Anti-oscillation mode for a PDO with  $f = 1/7$ ,  $\rho = 1e-4$ ,  $\sigma = 1$ ,  $F = 1$ ,  $L = 3$  and initial condition of the resonator  $u_0 = 50 + 50j$ .

although it is completely consistent with the antioscillation modes that are always observed in simulations.

The above result, together with Proposition 2 of [20], shows that there are frequency segments on which an  $S^3$  sequence at the resonant frequency of a lossless resonator is not possible.

### C. Energy considerations for PDOs in the Antioscillation mode

It has been shown in [38] that under some conditions  $S^3$  sequences at the resonant frequency of the resonator provide asymptotically a maximum in energy transfer to the resonator. The objective of this section is to show that, under some conditions, PDOs working in the antioscillation mode may provide maximum energy extraction from the resonator, i.e., absorption of the mechanical energy of the resonator into the electrical domain.

Each time a delta of force of amplitude  $P$  is applied to a resonator at time  $\tau = \tau_n$ , the energy delivered to the resonator

is (see the appendix of [19]):

$$\Delta E(\tau_n) = \frac{P^2}{2m} + Px'(\tau_n) \quad (47)$$

where  $m$  is the mass of the resonator and  $x'(\tau_n)$  is the resonator velocity at time  $\tau = \tau_n$ . This means that when a delta of force is applied to a resonator, depending on the sign and magnitude of both the excitation and the instantaneous velocity of the resonator, we may either increase or decrease its stored energy.

If we look again at the iterative map of a PDO with a lossless resonator, (39), if at a given time  $n$  we have that:

$$\text{sgn}(\text{Im}(pu_n)) = -\text{sgn}(\sigma)b_{n-L} \quad (48)$$

taking into account that  $\text{Im}(pu_n) = v_{n+1}^-/\omega_0$ , i.e., the imaginary part of  $pu_n$  is proportional to the instantaneous velocity of the resonator just before the delta is applied, then the energy transfer to the resonator at that sampling time has been maximum. In the case where  $\text{sgn}(\text{Im}(pu_n)) = \text{sgn}(\sigma)b_{n-L}$  the contrary happens, the energy extracted from the resonator has been maximum.

Let us first define the set of “anti-tuned” frequencies:

*Definition 5:* A rational frequency of the resonator,  $f = \frac{M}{N}$  is said to be an anti-tuned frequency of a PDO if either it obeys equation (36) and  $\sigma < 0$ , or it obeys equation (37) and  $\sigma > 0$ .

From the results of the previous section we know that no permanent oscillation at the resonant frequency is possible at any anti-tuned frequency. However it is possible to obtain a bitstream of the form  $b_n = \text{sgn}(\text{Re}(p^n u_0))$ , out of any PDO, at least for some clock cycles, depending on the initial condition of the resonator.

*Lemma 2:* Given a PDO with a resonator and an initial condition,  $u_0$ , such that  $|\text{Re}(p^n u_0)| > n\zeta$ ,  $|\text{Im}(p^n u_0)| > n\zeta$ , for  $0 \leq n < n_0$ , then  $\text{sgn}(\text{Re}(u_n)) = \text{sgn}(\text{Re}(p^n u_0))$  and  $\text{sgn}(\text{Im}(pu_n)) = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$ , for  $0 \leq n < n_0$ .

*Proof.* From the map of the oscillator, (39), we have that:

$$u_n = p^n u_0 - j\sigma\zeta p^{n-1} \sum_{i=0}^{n-1} p^{-i} b_{i-L} \quad (49)$$

Now, if  $|\text{Re}(p^n u_0)| > n\zeta$  for  $0 \leq n < n_0$ , then  $\text{sgn}(\text{Re}(u_n)) = \text{sgn}(\text{Re}(p^n u_0))$ , because  $n\zeta > |\zeta p^{n-1} \sum_{i=0}^{n-1} p^{-i} b_{i-L}|$ . On the other hand, from equation (49), it is easy to see that if  $|\text{Im}(p^n u_0)| > n\zeta$ , for  $0 \leq n < n_0$ , then  $\text{sgn}(\text{Im}(pu_n)) = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$ .  $\square$

This last result, means that for an unbounded set of initial conditions of the resonator, regardless of the configuration of the oscillator, the bitstream is dominated at least for some time directly by the initial condition of the resonator. Under these circumstances it is possible to study the energy efficiency of the PDO actuation while actuating the resonator, i.e., the number of times the sign of the delta of force applied to the resonator coincided with the sign of its velocity.

*Definition 6:* The energy efficiency,  $\eta$ , of a given finite-time pulsed actuation with amplitudes  $b_i$ ,  $0 \leq i < n$  on a linear resonator with initial condition  $u_0$  is defined as:

$$\eta = \frac{-\sigma}{n} \sum_{i=0}^{n-1} b_i \text{sgn}(v_i) \quad (50)$$

where  $v_i$  is the velocity of the resonator at time  $\tau = iT_S$  when the resonator is actuated with  $n$  deltas with amplitudes  $b_i$ ,  $i = 0, \dots, n-1$ , and the initial condition of the resonator is  $u_0$ .

We will now analyze the energy efficiency of PDOs in anti-tuned frequencies.

*Theorem 3:* Given a PDO with a lossless resonator with an anti-tuned frequency,  $f = \frac{M}{N}$ , if the initial condition,  $u_0$ , is such that  $|\text{Re}(p^n u_0)| > n\zeta$ ,  $|\text{Im}(p^n u_0)| > n\zeta$ , for  $0 \leq n < n_0 N + L$ , then the energy efficiency of the actuation of the PDO from time  $n = L$  to  $n = L + n_0 N$  on the resonator is:

- for  $N$  even either  $\eta = -1$  or  $\eta = -\frac{N-2}{N}$ , and
- for  $N$  odd, either  $\eta = -1$  or  $\eta = -\frac{N-1}{N}$ .

*Proof.* The actuation the oscillator feeds into the resonator is  $\text{sgn}(\text{Re}(u_{1,n-L}))$ . From Lemma 2, it follows that  $\text{sgn}(\text{Re}(u_n)) = \text{sgn}(\text{Re}(p^n u_0))$  and  $q_n = \text{sgn}(\text{Im}(pu_n)) = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$ ,  $0 \leq n < n_0 N$ . If  $L \leq n < n_0 N + L$ , then  $b_{n-L} = \text{sgn}(\text{Re}(p^{n-L}u_0))$ .

We may distinguish three different cases:

a)  $N = 4K$ ,  $K \in \mathbb{N}$ : in this case  $p^{-L-1} = -j$ , and therefore  $b_{n-L} = \text{sgn}(\text{Re}(p^{n+1-L}u_0)) = \text{sgn}(\text{Re}(-jp^{n+1}u_0)) = \text{sgn}(\text{Im}(p^n u_0))$ . This means the the energy efficiency of this actuation is  $\eta = -1$ .

b)  $N = 4K + 2$ ,  $K \in \mathbb{N}$ : then  $p^{-L-1} = -je^{j\pi/N}$ . In this case,  $b_{n-L} = \text{sgn}(\text{Re}(p^{n+1-L}u_0)) = \text{sgn}(\text{Re}(-je^{j\pi/N}p^{n+1}u_0))$ . Now, if  $\langle \Theta(-ju_0)N \rangle \in [0.5, 1)$ , where  $\Theta(u)$  is defined as  $\Theta(u) = 1/4 - \arg(u)/2\pi$ , then  $\text{sgn}(\text{Re}(-je^{j\pi/N}p^{n+1}u_0)) = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$ , and therefore  $\eta = -1$ . On the other hand, if  $\langle \Theta(-ju_0)N \rangle \in [0, 0.5)$ , we will see that the difference between the sequences  $\text{sgn}(\text{Re}(-je^{j\pi/N}p^{n+1}u_0))$  and  $\text{sgn}(\text{Re}(-jp^{n+1}u_0))$  is of two bits each  $N$  bits, and therefore  $\eta = -\frac{N-2}{N}$ .

The  $S^3$  sequence  $q_n = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$  may be expressed in the following form:

$$\begin{aligned} \text{sgn}(\text{Re}(-jp^{n+1}u_0)) &= \langle (n+1)f - 1/2 + \Theta(-ju_0) \rangle - \\ &\quad - \langle (n+1)f + \Theta(-ju_0) \rangle = \\ &= \left\langle \frac{(n+1)M - N/2 + \Theta(-ju_0)N}{N} \right\rangle - \\ &\quad - \left\langle \frac{(n+1)M + \Theta(-ju_0)N}{N} \right\rangle \end{aligned} \quad (51)$$

Now, if the collection of sets  $A_n$  is defined as  $A_n = \{0, \dots, n-1\}$ , we may further define a function  $I_0 : \mathbb{Z} \rightarrow A_N$  as:

$$I_0(n) = \lfloor ((n+1)M + \Theta(-ju_0))N \rfloor \bmod N \quad (52)$$

which allows us to rewrite the sequence  $q_n$  as:

$$\begin{aligned} q_n &= \left\langle \frac{I_0(n) - N/2 + \langle \Theta(-ju_0)N \rangle}{N} \right\rangle - \left\langle \frac{I_0(n) + \langle \Theta(-ju_0)N \rangle}{N} \right\rangle = \\ &= \left\langle \frac{I_0(n) - N/2}{N} \right\rangle - \left\langle \frac{I_0(n)}{N} \right\rangle \end{aligned} \quad (53)$$

Then, for any  $n \in \mathbb{N}$ ,  $q_n = 1$  if  $I_0(n \bmod N) < N/2$ , and  $q_n = -1$  otherwise. On the other hand, the  $S^3$  sequence  $b_{n-L} = \text{sgn}(\text{Re}(-je^{j\pi/N}p^{n+1}u_0))$  may be written as:

$$\begin{aligned} b_{n-L} &= \left\langle \frac{I_0(n) - N/2 - 1/2 + \langle \Theta(-ju_0)N \rangle}{N} \right\rangle - \\ &\quad - \left\langle \frac{I_0(n) - 1/2 + \langle \Theta(-ju_0)N \rangle}{N} \right\rangle \end{aligned} \quad (54)$$

It is clear that if  $\langle \Theta(-ju_0)N \rangle \in [0.5, 1)$  then both bitstreams are identical. If  $\langle \Theta(-ju_0)N \rangle \in [0, 0.5)$  then:

$$\begin{aligned} b_{n-L} &= \left\langle \frac{(I_0(n) - 1) - N/2 + (1/2 + \langle \Theta(-ju_0)N \rangle)}{N} \right\rangle - \\ &\quad - \left\langle \frac{(I_0(n) - 1) + 1/2 + \langle \Theta(-ju_0)N \rangle}{N} \right\rangle = \\ &= \left\langle \frac{(I_0(n) - 1) \bmod N - N/2}{N} \right\rangle - \left\langle \frac{(I_0(n) - 1) \bmod N}{N} \right\rangle \end{aligned} \quad (55)$$

and therefore 2 in each  $N$  bits change from the sequence  $b_{n-L}$  with regard to  $q_n$ , which means that  $\eta = \frac{N-2}{N}$ .

c)  $N$  odd: then  $p^{-L-1} = -je^{j\pi/2N}$ , and therefore  $b_{n-L} = \text{sgn}(\text{Re}(-je^{j\pi/2N}p^{n+1}u_0))$ . Now, if  $\langle \Theta(-ju_0)2N \rangle \in [0.5, 1)$  then  $\text{sgn}(\text{Re}(-je^{j\pi/2N}p^{n+1}u_0)) = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$  and therefore the energy efficiency is  $\eta = -1$ .

Now, the  $S^3$  sequence  $q_n = \text{sgn}(\text{Re}(-jp^{n+1}u_0))$  may be expressed in the following form:

$$\begin{aligned} q_n &= \text{sgn}(\text{Re}(-jp^{n+1}u_0)) = \langle (n+1)f - 1/2 + \Theta(-ju_0) \rangle - \\ &\quad - \langle (n+1)f + \Theta(-ju_0) \rangle = \\ &= \left\langle \frac{(n+1)2M - N + \Theta(-ju_0)2N}{2N} \right\rangle - \\ &\quad - \left\langle \frac{(n+1)2M + \Theta(-ju_0)2N}{2N} \right\rangle \end{aligned} \quad (56)$$

Following a similar procedure as before, we may define  $I_1 : \mathbb{Z} \rightarrow A_{2N}$  as:

$$I_1(n) = \lfloor ((n+1)2M + \Theta(-ju_0))2N \rfloor \bmod 2N \quad (57)$$

and therefore:

$$q_n = \left\langle \frac{I_1(n) - N + \langle \Theta(-ju_0)2N \rangle}{2N} \right\rangle -$$

$$\begin{aligned} & - \left\langle \frac{I_1(n) + \langle \Theta(-ju_0)2N \rangle}{2N} \right\rangle = \\ & = \left\langle \frac{I_1(n) - N}{2N} \right\rangle - \left\langle \frac{I_1(n)}{2N} \right\rangle \end{aligned} \quad (58)$$

which implies that if  $I_1(n) < N$  then  $q_n = 1$ ,  $q_n = -1$  otherwise. Now, if we look at the sequence  $b_{n-L}$ :

$$\begin{aligned} b_{n-L} &= \left\langle \frac{I_1(n) - N - 1/2 + \langle \Theta(-ju_0)2N \rangle}{2N} \right\rangle - \\ & - \left\langle \frac{I_1(n) - 1/2 + \langle \Theta(-ju_0)2N \rangle}{2N} \right\rangle \end{aligned} \quad (59)$$

Once again it is clear that if  $\langle \Theta(-ju_0)2N \rangle \in [0.5, 1)$  then  $q_n = b_{n-L}$ , implying that  $\eta = -1$ . When  $\langle \Theta(-ju_0)2N \rangle \notin [0.5, 1)$ :

$$\begin{aligned} b_{n-L} &= \left\langle \frac{(I_1(n) - 1) - N + (1/2 + \langle \Theta(-ju_0)2N \rangle)}{2N} \right\rangle - \\ & - \left\langle \frac{(I_1(n) - 1) + 1/2 + \langle \Theta(-ju_0)2N \rangle}{2N} \right\rangle = \\ & = \left\langle \frac{(I_1(n) - 1) \bmod 2N - N}{2N} \right\rangle - \left\langle \frac{(I_1(n) - 1) \bmod 2N}{2N} \right\rangle \end{aligned} \quad (60)$$

and in this case it means that only one bit changes in  $N$  bits between  $q_n$  and  $b_{n-L}$ , because for any two  $n_1, n_2 \in \mathbb{Z}$  with  $n_1 - n_2 \bmod N \neq 0$  it is  $|I_1(n_1) - I_1(n_2)| \geq 2$ . Now the

energy efficiency is  $\eta = -\frac{N-1}{N}$ .  $\square$

*Corollary 1:* Given a PDO with a lossless resonator with a tuned frequency,  $f = \frac{M}{N}$ , if the initial condition,  $u_0$ , is such that  $|\text{Re}(p^n u_0)| > n\zeta$ ,  $|\text{Im}(p^n u_0)| > n\zeta$ , for  $0 \leq n < n_0 N + L$ , then the energy efficiency of the actuation of the PDO from time  $n = L$  to  $n = L + n_0 N$  on the resonator is:

- a) for  $N$  even either  $\eta = 1$  or  $\eta = \frac{N-2}{N}$ , and
- b) for  $N$  odd, either  $\eta = 1$  or  $\eta = \frac{N-1}{N}$ .

*Proof.* It follows step by step the proof of the previous result but taking into account that now the frequency is tuned, which means that either  $\eta = 1$ ,  $\eta = (N-2)/N$  or  $\eta = (N-1)/N$ , depending on whether  $N$  is even or not, and the initial condition of the resonator.

Now, the following result can be proved for leaky resonators:

*Theorem 4:* For any  $n_0 \in \mathbb{N}$  and any anti-tuned frequency,  $f = \frac{M}{N}$ , there are some maximum losses,  $0 < \alpha_0 < 1$ , such that for any PDO with a leaky resonator with  $\alpha_0 < |p| < 1$ , if the initial condition,  $u_0$  obeys  $|\text{Re}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ ,  $|\text{Im}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ , for  $0 \leq n < n_0 N + L$ , then the energy efficiency of the actuation of the PDO from time  $n = L$  to  $n = L + n_0 N$  on the resonator is:

- a) for  $N$  even either  $\eta = -1$  or  $\eta = -\frac{N-2}{N}$ , and
- b) for  $N$  odd, either  $\eta = -1$  or  $\eta = -\frac{N-1}{N}$ .

*Proof.* Due to the fact that  $n_0$  is finite, it is possible to find  $0 < \alpha_0 < 1$ , such that if  $\alpha_0 < |p| < 1$  and  $|\text{Im}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|} > 0$  it is for  $0 \leq n < n_0 N + L$ :

$$\text{sgn}(x'(nT_S^-)) = \text{sgn}(\text{Im}(p^n u_0) - \rho \text{Re}(p^n u_0)) \quad (61)$$

or, equivalently:

$$\text{sgn}(\text{Im}(p^n u_0) - \frac{\rho}{\sqrt{1-\rho^2}} \text{Re}(p^n u_0)) = \text{sgn}(\text{Im}(p^n u_0)) \quad (62)$$

Now, following a similar approach as in Lemma 2, if  $|\text{Re}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ ,  $|\text{Im}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ , for  $0 \leq n < n_0 N + L$ , then  $\text{sgn}(\text{Re}(u_n)) = \text{sgn}(p^n u_0)$  and  $\text{sgn}(\text{Im}(p^n u_0)) = \text{sgn}(\text{Re}(-jp^{n+1} u_0))$ ,  $0 \leq n < n_0 N$ . Finally following a similar procedure as before the result is

obtained.  $\square$

*Corollary 2:* For any  $n_0 \in \mathbb{N}$  and any tuned frequency,  $f = \frac{M}{N}$ , there are some maximum losses,  $0 < \alpha_0 < 1$ , such that for any PDO with a leaky resonator with  $\alpha_0 < |p| < 1$ , if the initial condition,  $u_0$  obeys  $|\text{Re}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ ,  $|\text{Im}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ , for  $0 \leq n < n_0 N + L$ , then the energy efficiency of the actuation of the PDO from time  $n = L$  to  $n = L + n_0 N$  on the resonator is:

- a) for  $N$  even either  $\eta = 1$  or  $\eta = \frac{N-2}{N}$ , and
- b) for  $N$  odd, either  $\eta = 1$  or  $\eta = \frac{N-1}{N}$ .

*Proof.* It follows step by step the proof of the previous result but taking into account that now the frequency is tuned.

#### D. Characterization of the anti-oscillation frequency segments

We have seen that if what governs the immediate evolution of the PDO is the initial condition of the oscillator, i.e., under the conditions of Lemma 2, the actuation of PDOs can produce extreme values of the energy efficiency, depending on whether the resonant frequency of the resonator is tuned or anti-tuned. The objective of this section is to establish a condition fully characterizing the Anti-oscillation frequency segments.

Let us assume an initial condition of the resonator that guarantees a “natural evolution” of the resonator for  $n_0 + L$  clock cycles. In the case of a lossless resonator the condition might be  $|\text{Re}(p^n u_0)| > n\zeta$ ,  $|\text{Im}(p^n u_0)| > n\zeta$ , for  $0 \leq n < n_0 + L$ . In the case of a leaky resonator,  $|\text{Re}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ ,  $|\text{Im}(p^n u_0)| > \zeta \frac{1-|p|^n}{1-|p|}$ , for  $0 \leq n < n_0 + L$ . Therefore, assuming the initial condition of the resonator is in these regions, we have that  $b_{n-L} = \text{sgn}(\text{Re}(p^{n-L} u_0))$  and  $q_n = \text{sgn}(\text{Im}(p u_n)) = \text{sgn}(\text{Re}(-jp^{n+1} u_0))$ ,  $L \leq n < n_0 + L$ . Therefore:

$$\begin{aligned} b_{n-L} &= \langle (n-L)f - \frac{1}{2} + \Theta(u_0) \rangle - \langle (n-L)f + \Theta(u_0) \rangle = \\ &= \langle (n+1)f - (L+1)f - \frac{1}{2} + \Theta(u_0) \rangle - \langle (n+1)f - (L+1)f + \Theta(u_0) \rangle \end{aligned} \quad (63)$$

and:

$$q_n = \langle (n+1)f - \frac{1}{2} + \frac{1}{4} + \Theta(u_0) \rangle - \langle (n+1)f + \frac{1}{4} + \Theta(u_0) \rangle \quad (64)$$

Now, considering that  $\sigma = 1$ , for  $n_0$  large, in order to have  $\eta \leq 0$ , on average for all initial conditions, it must be  $\langle \frac{1}{4} + (L+1)f \rangle \in [0, 1/4) \cup [3/4, 1)$ , which means that:

$$\langle (L+1)f \rangle \in [0.5, 1) \quad (65)$$

and these are precisely the frequency segments where the antioscillation behaviour is found (for  $\sigma > 0$ ). In case  $\sigma < 0$  then  $\langle (L+1)f \rangle \in [0, 0.5)$ .

## VI. CONCLUSIONS

The dynamics of PDOs working with MEMS resonators with more than one resonance were analyzed. We started with a lumped model of coupled ordinary differential equations obtained for clamped-free beams. Each ODE is related to one of the resonances of the beam, which result from the decomposition in eigenmodes of the solution to the partial differential equation governing the motion of the beam. From this lumped model a set of iterative maps has obtained that serve to describe the dynamics of a PDO working with resonator with more than one resonance. It has been proved that under some circumstances PDOs can activate one of the resonances of the MEMS structure, but that this fact can depend on the initial condition of the resonator. In order to achieve an active control on the activation of the resonances another mechanism is necessary: the anti-oscillation mode. This mode has been analyzed and it has been shown that the energy efficiency of the actuation in the anti-oscillation frequency segments is negative. This mechanism will be used later in part II to actuate different resonant modes on a MEMS cantilever.

## VII. ACKNOWLEDGEMENTS

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## VIII. APPENDIX I: PROOF OF THEOREM 1

In order to obtain the proof of Theorem 1 we need first the following result:

*Lemma 3:* Given  $M_1, M_2, N_1, N_2 \in \mathbb{N}$ , with  $g.c.d.(N_1, N_2) = 1$ , and  $g.c.d.(M_i, N_i) = 1$ ,  $i = 0, 1$ , then:

$$\begin{aligned} &\left\{ \left\langle \left( k \frac{M_1}{N_1}, k \frac{M_2}{N_2} \right) \right\rangle : k = 0, \dots, N_1 N_2 - 1 \right\} = \\ &= \left\{ \left\langle \left( \frac{k}{N_1}, \frac{l}{N_2} \right) \right\rangle : k = 0, \dots, N_1 - 1, l = 0, \dots, N_2 - 1 \right\} \end{aligned} \quad (66)$$

*Proof.* The result is evident for  $N_1 = N_2 = 1$ . Now, for  $N_1 N_2 > 1$ , let us assume that the first set does not have  $N_1 N_2$  elements, and therefore the above equality does not hold. This means that for a given  $0 < k_0 < N_1 N_2$  it is  $\langle k_0 \frac{M_1}{N_1} \rangle = \langle k_0 \frac{M_2}{N_2} \rangle = 0$ , or what is the same,  $k_0 M_1 \bmod N_1 = k_0 M_2 \bmod N_2 = 0$ . Now, taking into account that  $g.c.d.(M_i, N_i) = 1$ ,  $i = 0, 1$ , we have that  $k_0 = m_1 N_1$  and  $k_0 = m_2 N_2$ , for some  $m_1, m_2 \in \mathbb{Z}$ . On the other hand, because  $g.c.d.(N_1, N_2) = 1$ , then  $k_0 = m_0 N_1 N_2$ , with

$m_0 \in \mathbb{Z}$ , and therefore  $k_0 \notin \{1, \dots, N_1 N_2 - 1\}$ , which is a contradiction.

This only reflects the fact that  $\mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z}$  is a group isomorph to  $\mathbb{Z}/N_1 N_2\mathbb{Z}$  if and only if  $\text{g.c.d.}(N_1, N_2) = 1$ .  $\square$

This result allows us to bound the response of a given resonator, of resonant frequency  $f_2 = M_2/N_2$ , to an  $S^3$  sequence, of frequency  $f_1 = M_1/N_1$  of a different frequency, if their respective frequencies under the conditions of Lemma 3, at times multiple of  $N_1 N_2$ .

*Lemma 4:* For any binary sequence  $\mathbf{b}_{u_{1,0}} = (\text{sgn}(\text{Re}(p^n u_{1,0})))$ , with  $p = \exp(-j2\pi \frac{M_1}{N_1})$ ,  $u_{1,0} \in \mathbb{C}$ , if  $q = \exp(-j2\pi \frac{M_2}{N_2})$ , with  $\text{g.c.d.}(N_1, N_2) = 1$ , and  $\text{g.c.d.}(M_i, N_i) = 1, i = 0, 1$ , then  $|h_n(q, \mathbf{b}_{u_{1,0}})| \leq n \bmod N_1 N_2$ .

*Proof.* From Lemma 3, we have that:

$$\begin{aligned} & \sum_{i=0}^{N_1 N_2 - 1} e^{j2\pi \frac{M_2}{N_2} i} \text{sgn}(\text{Re}(e^{-j2\pi \frac{M_1}{N_1} i} u_{1,0})) = \\ & = \sum_{i=0}^{N_2 - 1} e^{j2\pi \frac{M_2}{N_2} i} \sum_{k=0}^{N_1 - 1} \text{sgn}(\text{Re}(e^{-j2\pi \frac{M_1}{N_1} k} u_{1,0})) = 0 \end{aligned} \quad (67)$$

and therefore  $h_{N_1 N_2}(q, \mathbf{b}_{u_{1,0}}) = 0$ . Now, due to the fact that:

$$\begin{aligned} h_n(q, \mathbf{b}_{u_{1,0}}) &= \left[ \frac{n}{N_1 N_2} \right] h_{N_1 N_2}(q, \mathbf{b}_{u_{1,0}}) \\ & \quad + h_{n \bmod N_1 N_2}(q, \mathbf{b}_{u_{1,0}}) \end{aligned} \quad (68)$$

which means that  $|h_n(q, \mathbf{b}_{u_{1,0}})| \leq n \bmod N_1 N_2$ .  $\square$

Now, we may proceed with proving Theorem 1:

*Proof of Theorem 1.* Following a similar approach to that used for proving Theorem 2 in [20], it can be shown that for a binary sequence  $\mathbf{b}_{u_{1,0}} = \text{sgn}(p^n u_{1,0})$  we have that:

$$h_n(p, \mathbf{b}_{u_{1,0}}) = h_{n \bmod N_1}(p, u_{1,0}) + \left[ \frac{n}{N_1} \right] p^n h_{N_1}(p, u_{1,0}) \quad (69)$$

Now, due to the fact that  $f_1$  is a tuned frequency, it is  $\text{sgn}(\text{Re}(p^n h_{N_1}(p, u_{1,0}))) = \text{sgn}(\text{Re}(p^n u_{1,0}))$  or  $\text{sgn}(\text{Re}(p^n h_{N_1}(p, u_{1,0}))) = 0, n \geq 0$ . On other hand, retaking Lemma 5, we have that  $|h_n(q, \mathbf{b}_{u_{1,0}})| \leq n \bmod N_1 N_2$ . Therefore for a sufficiently large  $n_0$ , it is for  $n > n_0$ ,  $\text{sgn}(\text{Re}(p^n u_{1,0})) = \text{sgn}(\text{Re}(\zeta_1 h_n(p, \mathbf{b}_{u_{1,0}})))$  and:

$$|\text{Re}(\zeta_1 h_n(p, \mathbf{b}_{u_{1,0}}))| > |\beta \text{Re}(q^n u_{2,0} + \zeta_2 h_n(q, \mathbf{b}_{u_{1,0}}))| \quad (70)$$

and therefore:

$$\begin{aligned} & \text{sgn}(\text{Re}(p^n u_{1,0} + \zeta_1 h_n(p, \mathbf{b}_{u_{1,0}}) + \\ & \quad + \beta(q^n u_{2,0} + \zeta_2 h_n(q, \mathbf{b}_{u_{1,0}})))) = \\ & = \text{sgn}(\text{Re}(p^n u_{1,0})), \quad n > n_0 \end{aligned} \quad (71)$$

Finally, if  $|\text{Re}(p^n u_{1,0})| > |\beta u_{2,0}| + |\zeta_1| n \bmod N_1 + |\beta \zeta_2| N_1 N_2, 0 \leq n < N_1$ , then equation (71) is also true for  $n \geq 0$ , and therefore Lemma 1 can be applied.  $\square$

## IX. APPENDIX II: PROOF OF THEOREM 2

We begin with an auxiliary result:

*Lemma 5:* Given a binary sequence  $\mathbf{b}_{u_{1,0}} = (\text{sgn}(\text{Re}(p^n u_{1,0})))$ , with  $p = \exp(-j2\pi \frac{M_1}{N_1})$ ,  $u_{1,0} \in \mathbb{C}$  and  $q = \exp(-j2\pi \frac{M_2}{N_2})$ , with  $\text{g.c.d.}(N_1, N_2) = 1$ , and  $\text{g.c.d.}(M_i, N_i) = 1, i = 0, 1$ , for any  $\epsilon > 0$  there is a  $\delta > 0$  such that for  $1 - \delta < \alpha < 1$ :

$$|h_n(\alpha q, \mathbf{b}_{u_{1,0}})| \leq \frac{1 - \alpha^{n \bmod N_1 N_2}}{1 - \alpha} + \epsilon \quad (72)$$

*Proof.* For any  $n \geq 0$ :

$$\begin{aligned} h_n(\alpha q, \mathbf{b}_{u_{1,0}}) &= h_{n \bmod N_1 N_2}(\alpha q, \mathbf{b}_{u_{1,0}}) + \\ & + \frac{1 - \alpha^{\lfloor \frac{n}{N_1 N_2} \rfloor N_1 N_2}}{1 - \alpha^{N_1 N_2}} \alpha^{n \bmod N_1 N_2} q^n h_{N_1 N_2}(\alpha q, \mathbf{b}_{u_{1,0}}) \end{aligned} \quad (73)$$

Now, due to the continuity of  $h_{N_1 N_2}(\alpha q, \mathbf{b}_{u_{1,0}})$  as a function of  $\alpha$ , and taking into account that  $h_{N_1 N_2}(q, \mathbf{b}_{u_{1,0}}) = 0$ , then for  $\alpha \rightarrow 1$  we have that

$h_{N_1 N_2}(\alpha q, \mathbf{b}_{u_{1,0}}) \rightarrow 0$ , and the result is easily obtained.  $\square$

*Corollary 3:* If to the conditions of Lemma 5 we add that  $\langle M_1/N_1 \rangle \neq 0$  or  $\langle M_2/N_2 \rangle \neq 0$  then there is a  $\delta > 0$  such that for  $1 - \delta < \alpha < 1$ :

$$|h_n(\alpha q, \mathbf{b}_{u_{1,0}})| < \frac{1 - \alpha^{n \bmod N_1 N_2}}{1 - \alpha} \quad (74)$$

With the above results we may proceed to prove the second theorem:

*Proof. of Theorem 2* For any  $n \geq 0$  we have that:

$$\begin{aligned} h_n(\alpha_p p, \mathbf{b}_{u_{1,0}}) &= h_{n \bmod N_1 N_2}(\alpha_p p, \mathbf{b}_{u_{1,0}}) + \\ & + a_n p^n h_{N_1 N_2}(\alpha_p p, \mathbf{b}_{u_{1,0}}) \end{aligned} \quad (75)$$

where  $a_n$  is:

$$a_n = \frac{1 - \alpha_p^{\lfloor \frac{n}{N_1 N_2} \rfloor N_1 N_2}}{1 - \alpha_p^{N_1 N_2}} \alpha_p^{n \bmod N_1 N_2} \quad (76)$$

which means that:

$$\lim_{n \rightarrow \infty, \alpha \rightarrow 1} \frac{1}{a_n} h_n(\alpha_p p, \mathbf{b}_{u_{1,0}}) = p^n h_{N_1 N_2}(p, \mathbf{b}_{u_{1,0}}) \quad (77)$$

Now, taking into account that for  $\langle f_1 \rangle = \frac{1}{4(L+1)}$ :

$$\left\langle \left( \frac{1}{4} - \arg(h_{N_1 N_2}(p, \mathbf{b}_{u_{1,0}})) / 2\pi \right) N_1 \right\rangle = \frac{1}{2N_1} \neq 0, \quad (78)$$

then, because  $h_{N_1 N_2}(\alpha_p p, \mathbf{b}_{u_{1,0}})$  is continuous as a function of  $\alpha_p$ , there is a  $\delta_0 > 0$  such that for  $\alpha_p \in (1 - \delta_0, 1 + \delta_0)$ , we have that  $\text{sgn}(\text{Re}(p^n h_{N_1 N_2}(\alpha_p p, \mathbf{b}_{u_{1,0}}))) = \text{sgn}(\text{Re}(p^n u_{1,0}))$ ,  $n \geq 0$ . Finally, taking into account expression (77) and Lemma 5, there exists a  $0 < \delta_1 \leq \delta_0$  such that for  $1 > \alpha_p > 1 - \delta_1$  and a sufficiently high  $n_0$  it is for  $n > n_0$ :

$$\left| \operatorname{Re} \left( \frac{1 - \alpha_p^{\lfloor \frac{n}{N_1 N_2} \rfloor N_1 N_2}}{1 - \alpha_p^{N_1 N_2}} \alpha_p^{n \bmod N_1 N_2} p^{n \bmod N_1 N_2} h_{N_1 N_2}(\alpha_p p, \mathbf{b}_{u_{1,0}}) \right) \right| > \frac{1 - \alpha_p^{n \bmod N_1 N_2}}{1 - \alpha_p} + |\beta| \frac{1 - \alpha_q^{n \bmod N_1 N_2}}{1 - \alpha_q} \geq \geq |h_{n \bmod N_1 N_2}(\alpha_p p, \mathbf{b}_{u_{1,0}})| + |\beta h_n(\alpha_q q, \mathbf{b}_{u_{1,0}})| \quad (79)$$

which means that  $\operatorname{sgn}(\operatorname{Re}(h_n(\alpha_p p, \mathbf{b}_{u_{1,0}}) + \beta \alpha_q^n q^n u_{2,0} + \beta h_n(\alpha_q q, \mathbf{b}_{u_{1,0}}))) = \operatorname{sgn}(\operatorname{Re}(p^n u_{1,0}))$  for  $n > n_0$ .

Finally, if  $|\operatorname{Re}(\alpha_p p^n u_{1,0})| > C_{\alpha_p} + |\beta u_{2,0}|$ , for  $0 \leq n \leq n_0$ , where:

$$C_{\alpha_p} = \frac{1 - \alpha_p^{N_1 N_2}}{1 - \alpha_p} + |\beta| \frac{1 - \alpha_q^{N_1 N_2}}{1 - \alpha_q} + \quad (80)$$

then  $\operatorname{sgn}(\operatorname{Re}(h_n(\alpha_p p, \mathbf{b}_{u_{1,0}}) + \beta \alpha_q^n q^n u_{2,0} + \beta h_n(\alpha_q q, \mathbf{b}_{u_{1,0}}))) = \operatorname{sgn}(\operatorname{Re}(p^n u_{1,0}))$  for  $n \geq 0$ , and

therefore Lemma 1 can be applied.  $\square$

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