| Title | Analysis of Rayleigh-Lamb wave scattering by a crack in an elastic plate |
|---|---|
| Authors(s) | Crane, L. J., Gilchrist, M. D., Miller, J. J. H. |
| Publication date | 1997-05-27 |
| Publication information | Crane, L. J., M. D. Gilchrist, and J. J. H. Miller. “Analysis of Rayleigh-Lamb Wave Scattering by a Crack in an Elastic Plate.” Springer-Verlag, May 27, 1997. https://doi.org/10.1007/s004660050205. |
| Publisher | Springer-Verlag |
| Item record/more information | http://hdl.handle.net/10197/5927 |
| Publisher's statement | The final publication is available at www.springerlink.com |
| Publisher's version (DOI) | 10.1007/s004660050205 |

# Analysis of Rayleigh-Lamb wave scattering by a crack in an elastic plate

**L. J. Crane, M. D. Gilchrist, J. J. H. Miller**

**Abstract** This paper considers the scattering of low-frequency elastic waves by a crack in a plate. A simple formula is derived for the reflection coefficient which serves as a lower bound to the reflection coefficient at higher frequencies.

**Notation**

| | |
|---|---|
| $a$ | half crack length |
| $c$ | speed of $S$-waves in plate |
| $c_L$ | speed of longitudinal waves in an infinite medium |
| $c_T$ | speed of transverse waves in an infinite medium |
| $d$ | half thickness of plate |
| $f$ | real function defined in (41) |
| $g$ | real function defined in (42) |
| $r$ | constant defined by (27) |
| $u, v$ | displacements in $x$, $y$ directions |
| $u_N$ | displacement due to normal stress defined by (40) |
| $u_S$ | displacement due to shear stress defined by (40) |
| $\hat{u}$ | integrated displacement defined by (16) |
| $x, y$ | cartesian coordinates |
| $A$ | function defined by (26) |
| $B$ | function defined by (26) |
| $K$ | dimensionless wave number defined by (29) |
| $R$ | reflection coefficient |
| $S_0$ | zero-order symmetric Lamb wave |
| $T_{xx}$ | stress component |
| $T_{xy}$ | stress component |
| $X$ | dimensionless coordinate defined by (24) |
| $\alpha$ | constant defined by (4) |
| $\beta$ | constant defined by (4) |
| $\kappa$ | wave number defined by (3) |
| $\lambda$ | wavelength |
| $\lambda_1$, $\mu_1$ | Lamé constants |
| $\nu$ | Poisson ratio |
| $\rho$ | density of plate |
| $\phi$, $\psi$ | potentials defined in (17) |
| $\Phi$ | function defined by (47) |
| $\omega$ | angular frequency |
| $\Omega$ | nondimensional frequency defined by (6) |

*Communicated by P. E. O'Donoghue, M. D. Gilchrist, K. B. Broberg, 6 January 1997*

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This work was supported in part by the BRITE-EURAM SISCO project no. BRE2, CT94-0990 and by the Institute for Numerical Computation and Analysis, Dublin. The authors are grateful to Professor B. Broberg for useful discussions.

## 1
## Introduction

Evaluation of engineering structures with the aid of nondestructive tests has increased in importance now that high-performance materials such as composites and high-strength alloys are used more and more in safety-critical applications and the resulting structures are retained in use for increasingly longer service lives. Conventional nondestructive evaluation (NDE) by ultrasonic techniques, for example C-scanning, commonly uses waves propagating normal to the surface. Although such methods are accurate, they are generally very slow in scanning large areas of a plane surface. There would accordingly be considerable practical benefit from the development of a NDE technique that can rapidly detect and locate the presence of damage in a large component such as an aircraft primary structure. Once a defect was found, conventional methods could be used to assess it in greater detail.

A wide range of physical defects can exist in metallic and composite material systems, and many engineering fractures tend to initiate at the site of naturally-occurring material imperfections such as voids, inclusions and crack-like defects. Within metals such defects could be porosities, impurities, imperfections, slag etc., while delaminations, matrix cracks, fibre fracture and splitting can often be the source of failure of composite systems. Defects such as delaminations along ply interfaces, impurities along a rolling direction or surface machining scratches are oriented in particular directions. Other defects can be randomly aligned and distributed throughout a material. Many such defects are often microscopically small and will propagate due to applied forces, for example in-flight fatigue loads on a rotorcraft. It is only when a defect has grown to a certain size that it will be detected by means of some NDE technique. Then maintenance scheduling is used to monitor the development of defects before they reach the critical size at which catastrophic failure of the component can occur.

A simple example of such a defect, which is amenable to mathematical analysis, is that of a two-dimensional crack embedded in an infinite plate. This paper is concerned with the scattering of long waves by such a crack normal to the plane of the plate. We obtain a simple formula (50) for the reflection coefficient, a formula valid at small frequencies defined by $\Omega \ll 1$, where $\Omega$ is a nondimensional frequency defined by Eq. (6). The formula shows that the reflection coefficient increases with the square of the crack size and is inversely proportional to the wavelength of the incident wave.

Previous work on scattering of long waves by cracks in plates will be found in Koshiba et al. (1984). This work is limited to a single frequency, whereas the present paper is applicable to a range of frequencies. Like Koshiba et al., we restrict ourselves to cracks which are symmetric about the central plane of the plate.

## 2 Rayleigh-Lamb waves

We consider an isotropic plate of thickness $2d$, with Lamé constants $\lambda_1, \mu_1$, in which elastic waves propagate in the $x$-direction, where the $x$-axis is taken along the central plane of the plate; the $y$-axis is normal to the plane of the plate, and the frequency of the waves if $\omega/2\pi$, where $\omega$ is the angular frequency. Two types of Rayleigh-Lamb wave can be propagated in the plate, namely those which are antisymmetric about the central plane of the plate and those which are symmetric about it. In this paper we are concerned only with symmetric waves for which the displacements are confined to the $(x, y)$ plane.

We consider, then, symmetric waves which are incident upon a symmetric vertical crack of length 2a:

$$x = 0, \quad -a \le y \le a \tag{1}$$

Since the incident wave and the crack are both symmetric about the central plane, it follows that the reflected wave is also a symmetric wave.

The properties of both types of waves have been extensively studied by Rayleigh (1889) and by Lamb (1917); an overall survey is given by Viktorov (1967). In the above works it is shown that when a symmetric wave is propagated in a plate whose faces are free of stress, the dispersion relation is

$$\frac{\tanh \beta d}{\tanh \alpha d} = \frac{4\kappa^2 \alpha\beta}{(\kappa^2 + \beta^2)^2} \tag{2}$$

where $\kappa$ is the wave number, which may be defined in terms of the wavelength $\lambda$ by

$$\kappa = \frac{2\pi}{\lambda}\ , \tag{3}$$

and where

$$\begin{aligned} \alpha^2 &= \kappa^2 - \frac{\omega^2}{(c_L)^2}\ , \\ \beta^2 &= \kappa^2 - \frac{\omega^2}{(c_T)^2}\ , \end{aligned} \tag{4}$$

where $c_L$ and $c_T$ are the speeds of propagation in an infinite medium of longitudinal and transverse elastic waves respectively. These speeds can be expressed in terms of the Lamé constants $\lambda_1, \mu_1$ and the density $\rho$ by

$$\begin{aligned} (c_L)^2 &= \frac{\lambda_1 + 2\mu_1}{\rho} \\ (c_T)^2 &= \frac{\mu_1}{\rho} \end{aligned} \tag{5}$$

The properties of Rayleigh-Lamb waves may be conveniently expressed in terms of the dimensionless wave frequency $\Omega$, defined as

$$\Omega = \frac{\pi \times (\text{plate thickness}) \times (\text{frequency})}{(\text{speed of transverse waves})} = \frac{\omega d}{c_T} \tag{6}$$

Furthermore it can be shown from Eq. (2) that the speed of propagation of symmetric Rayleigh-Lamb waves, namely $c$, tends to the following limit as $\Omega$ tends to zero:

$$2c_T \sqrt{\frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1}}\ . \tag{7}$$

In the case when $\lambda_1 = 2\mu_1$, which corresponds to a Poisson ratio $v = 1/3$, it can be shown that

$$\frac{c}{c_T} = \sqrt{3}\left[1 - \frac{1}{72}\Omega^2 + O(\Omega^4)\right]\ ; \tag{8}$$

the relation (8) clearly indicates that symmetric waves in a plate are nondispersive when $\Omega \ll 1$. Even in the interval $0 < \Omega \le 1$, the maximum variation in wave speed is only about 1%.

In this paper we restrict ourselves to symmetric waves for which $\Omega \ll 1$ and consequently $d \ll \lambda$. In this case, as is shown in for example Victorov (1967), only one wave is possible; this wave will be denoted by $S_0$. For an $S_0$ wave, Rayleigh (1889) obtained an approximation, valid to order $\Omega^2$, in which the complex displacements in the $x, y$ directions, namely $u, v$ respectively, are given by

$$u = \exp|i(\omega t + \kappa x)] \tag{9}$$

$$v = -i\kappa y\left[\frac{\lambda_1}{\lambda_1 + 2\mu_1}\right]\exp[i(\omega t + \kappa x)] \tag{10}$$

Since $\kappa y$ is small, this wave, which is travelling towards the origin from $x = +\infty$, is essentially a longitudinal wave whose complex displacement in the $x$ direction is constant at any given instant of time on each cross-section of the plate.

## 3 Reflection of $S_0$ wave from the crack

In this section we consider the $S_0$ wave given by (9), (10) striking the face $x = O+$ of the crack (1). The crack is assumed to be an open crack whose faces do not touch and on which there is zero stress. Thus when the above $S_0$ wave strikes the surface $x = O+$ of the crack it produces a stress in that surface. The stress is reduced to zero by a reflected wave moving in the direction of increasing $x$, which generates on the surface $x = O+$ an equal but opposite stress to that produced by the incident wave (9), (10).

This reflected wave has the same frequency, namely $\omega/2\pi$, as the incident wave, and becomes an $S_0$ wave at large distances from the crack.

## 4 Average motion of the reflected wave

The $x$-component of the displacement generated by the reflected wave, namely $u$, satisfies the dynamic equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(T_{xx}) + \frac{\partial}{\partial y}(T_{xy}) \ , \tag{11}$$

where $T_{xx}, T_{xy}$ are components of stress and $t$ is the time. Since the reflected wave has frequency $\omega/2\pi$, Eq. (11) becomes

$$\rho\omega^2 u = \frac{\partial}{\partial x}(T_{xx}) + \frac{\partial}{\partial y}(T_{xy}) \tag{12}$$

Integrating (10) over a section of the plate, and recalling that $x$ is constant there, we get

$$-\rho\omega^2 \int_{-d}^{d} u \mathrm{d}y = \frac{\partial}{\partial x}\int_{-d}^{d} T_{xx}\mathrm{d}y + [T_{xy}]_{-d}^{d} \ . \tag{13}$$

The term in square brackets in (13) vanishes because the surfaces of the plate are free from stress.

Using the relation

$$T_{xx} = \lambda_1\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu_1\frac{\partial u}{\partial x} \tag{14}$$

Eq. (13) becomes

$$-\rho\omega^2 \hat{u} = \lambda_1 \frac{\partial}{\partial x}\int_{-d}^{d}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\mathrm{d}y + 2\mu_1\frac{\partial^2 \hat{u}}{\partial x^2} \ , \tag{15}$$

where

$$\hat{u} = \int_{-d}^{d} u \ \mathrm{d}y \ . \tag{16}$$

We now estimate the magnitude of the term $\partial u/\partial x + \partial v/\partial y$ in (15). Rayleigh (1889) and Lamb (1917) showed that when the displacements $u, v$ are replaced by equivalent expressions involving potentials $\phi, \psi$, defined by

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y} \tag{17}$$

$$v = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \ , \tag{18}$$

then the dynamic equation for $\phi$ is

$$\rho\frac{\partial^2\phi}{\partial t^2} = (\lambda_1 + 2\mu_1)\nabla^2\phi \ . \tag{19}$$

Now

$$\nabla^2\phi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{20}$$

and since $\phi$ has frequency $\omega/2\pi$,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-\rho\omega^2\phi}{\lambda_1 + 2\mu_1} \tag{21}$$

Hence Eq. (15) becomes

$$-\rho\omega^2\hat{u} = \frac{-\rho\lambda_1\omega^2}{\lambda_1 + 2\mu_1}\int_{-d}^{d}\frac{\partial\phi}{\partial x}\mathrm{d}y + 2\mu_1\frac{\partial^2\hat{u}}{\partial x^2} \ . \tag{22}$$

Since $\partial\phi/\partial x$ has the same order of magnitude as $u$ (indeed, for a fully-developed $S_0$ wave in a material whose Poisson ratio is 1/3, $\partial\phi/\partial x$ is one-half of the value of $u$), it follows that (22) reduces to

$$\frac{\partial^2\hat{u}}{\partial x^2} = O\left(\frac{\omega^2}{c_T^2}\hat{u}\right) \ . \tag{23}$$

Let us now consider equation (23) in a neighbourhood of the crack. We introduce the coordinate transformation

$$X = \frac{x}{d} \ , \tag{24}$$

so that in the neighbourhood of the crack where $X = O(1)$, (23) becomes

$$\frac{\partial^2\hat{u}}{\partial X^2} = O\left(\frac{\omega^2 d^2}{c_T^2}\hat{u}\right) = O(\Omega^2)\hat{u} \ . \tag{25}$$

Thus, neglecting powers of $\Omega^2$,

$$\hat{u} = [A + BX + O(\Omega^2)] \ , \tag{26}$$

where $A, \ B$ are unknown functions of $t$.

We now evaluate the functions $A, \ B$ by using the theory of matched asymptotic expansions (see for example Van Dyke 1975).

When $x \gg d$, the reflected wave becomes an $S_0$ wave similar to (9), (10) for which when $\Omega \ll 1$

$$u = r\ \exp[i(\omega t - \kappa x)] \ , \tag{27}$$

where $r$ is a complex constant. When (27) is expressed in terms of the inner variable $X$ it becomes

$$u = r\ \exp[-iKX]\mathrm{e}^{i\omega t} \ , \tag{28}$$

where $K$ is the dimensionless wave number defined by

$$K = \kappa d \ . \tag{29}$$

Now $K$ is related to $\Omega$ by

$$K = \Omega\frac{c_T}{c} = \frac{\Omega}{2}\sqrt{\frac{\lambda_1 + 2\mu_1}{\lambda_1 + \mu_1}}[1 + O(\Omega^2)] \tag{30}$$

(see (5)). It follows that

$$K = O(\Omega) \ . \tag{31}$$

When $X$ is small (28) becomes

$$u = r\left[1 - iKX - \frac{K^2X^2}{2} + \ldots\right]\mathrm{e}^{i\omega t} \ . \tag{32}$$

Since $K^2 = O(\Omega^2)$ it follows that for $X = O(1)$

$$u = r[1 - iKX + O(\Omega^2)]\mathrm{e}^{i\omega t} \tag{33}$$

and that

$$\hat{u} = 2rd\ [1 - iKX + O(\Omega^2)]\mathrm{e}^{i\omega t} \ . \tag{34}$$

We now match (26) with (34). It follows that

$$A = 2rd\mathrm{e}^{i\omega t} \tag{35}$$

$$B = -2rdiKe^{i\omega t} \ . \quad (36)$$

Thus the complex constant $r$ is given by

$$r = \frac{A}{2d} = \left( \frac{1}{2d} \int_{-a}^{a} u \, dy \right)_{x=O+} \ . \quad (37)$$

Now the reflection coefficient $R$ is defined by

$$R = \frac{\text{amplitude of the } x\text{-component of the reflected wave}}{\text{amplitude of the } x\text{-component of the incident wave}} \ .$$

Since the amplitude of the $x$-component of the incident wave is unity, it follows that the reflection coefficient is equal to the modulus of the complex constant $r$, given by (37); it then follows that $R = |r|$.

## 5 Effect of the normal and shear stresses at the crack

It is clear from the analysis in the last section that the acceleration term in the dynamical Eq. (25) can be neglected, to within an error of order $\Omega^2$, at distances from the crack which are small compared with the plate thickness. This implies that the displacement close to the crack is quasi-static: that is to say, the displacement of the crack at any instant is equal to the static displacement produced by the instantaneous load on the crack.

The stresses produced by the incident wave (see (9), (10)) in front of the crack, namely $x = O+$, are

$$\begin{aligned} T_{xx} &\equiv \lambda_1 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu_1 \frac{\partial u}{\partial x} \\ &\doteq 4\mu_1 i\kappa \left[ \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} \right] e^{i\omega t} \ , \end{aligned} \quad (38)$$

$$\begin{aligned} T_{xy} &\equiv \mu_1 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\ &\doteq \frac{\mu_1 \kappa^2 y \lambda_1}{\lambda_1 + 2\mu_1} e^{i\omega t} \ , \end{aligned} \quad (39)$$

Let the $x$-component of the deflection of the surface $x = O+$ of the crack due to the reflected wave be written in the form

$$u_N + u_S \ , \quad (40)$$

where $u_N, \ u_S$ are displacements due to normal and shear stresses on the crack respectively.

It is clear from (38), (39) that $u_N, \ u_S$ are of the form

$$u_N = ie^{i\omega t} f(y) \quad (41)$$

$$u_S = e^{i\omega t} g(y) \ , \quad (42)$$

the functions $f, \ g$ being purely real.

It follows that

$$\hat{u}(x = O+) = e^{i\omega t} \left[ i \int_{-a}^{a} f(y) \, dy + \int_{-a}^{a} g(y) \, dy \right] \ , \quad (43)$$

and hence (37) that

$$|r| = \frac{1}{2d} \sqrt{\left[ \int_{-a}^{a} f(y) \, dy \right]^2 + \left[ \int_{-a}^{a} g(y) \, dy \right]^2} \ . \quad (44)$$

Now for $\Omega \ll 1$, it is readily shown that the order of magnitude of $T_{xy}$ is

$$\Omega T_{xx} \ .$$

Consequently the normal displacement of the crack due to the shear stress is at most of order $\Omega$ times that due to the normal stress. This implies that (44) may be written as

$$\begin{aligned} |r| &= \frac{1}{2d} \int_{-a}^{a} f(y) \, dy \sqrt{1 + O(\Omega^2)} \\ &= \frac{1}{2d} \int_{-a}^{a} f(y) \, dy [1 + O(\Omega^2)] \ . \end{aligned} \quad (45)$$

Thus to within an error of order $\Omega^2$, the reflection coefficient $R$ (whose value is $|r|$) depends only on the normal stress generated by the reflected wave on the surface $x = O+$ of the crack.

## 6 Estimate for the normal static deflection of the crack

Wu and Carlsson (1991) give the static displacement normal to a crack under a uniform normal stress $T_{xx}$; from their analysis it can be shown that when $0 \le a/d \le 0.4$, the static normal displacement of the surface of the crack is given by

$$u \doteq \frac{T_{xx}}{\mu_1} (1 - \nu) \sqrt{a^2 - y^2} \ \Phi \ , \quad (46)$$

where $\Phi$ is a correction factor which is defined by

$$\Phi \doteq 1 + \frac{5}{8} \frac{a^2}{d^2} \ . \quad (47)$$

Integrating (47) gives the value of $\hat{u}$ on $x = O+$ as

$$\begin{aligned} \hat{u} &= \int_{-a}^{a} u \ dy \\ &= \frac{\pi}{2} a^2 \cdot \frac{T_{xx}}{\mu_1} (1 - \nu) \ \Phi \end{aligned} \quad (48)$$

## 7 Evaluation of the reflection coefficient

The value of $\hat{u}(x = O+)$ may now be determined by inserting (38) (with sign reversed) into (48). This gives

$$\hat{u}(x = O+) = -2\pi i\kappa a^2 (1 - \nu) \left( \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} \right) \Phi \ . \quad (49)$$

Putting

$$\kappa = \frac{2\pi}{\lambda}$$

and using (37), it follows that the reflection coefficient is

$$R = \pi^2 \frac{a^2}{\lambda d} \ \Phi \ . \quad (50)$$

This formula is valid for $a/d \le 0.4$ and for $\Omega \ll 1$.

## 8 Discussion and conclusions

It is remarkable that the reflection coefficient depends on three lengths, namely

$a$, the half-length of the crack;
$d$, the half-thickness of the plate, and
$\lambda$, the wavelength of the incident $S_o$ wave.

At first sight it would appear that the reflection coefficient is independent of the material properties of the plate; however, these material properties determine the relation between the imposed frequency and the resulting wavelength by way of the dispersion relation (2).

So far the discussion has been limited to small frequencies in the sense that $\Omega \ll 1$. Recently, however, one of the authors (M.D.G) has calculated reflection coefficients for an $S_0$ wave incident on a symmetric crack in an aluminium plate (for which the Poisson ratio is approximately 1/3) using a finite element method. Full details of this work will be presented in a forthcoming paper (Gilchrist 1997). Some of the results given in this paper for the reflection coefficient when $\Omega = 1$ are summarised in the table below.

**Table 1.** Reflection Coefficients when $\Omega = 1$

| a/d | R. Gilchrist | R(50) |
|---|---|---|
| 0.5 | 0.25 | 0.27 |
| 0.4 | 0.15 | 0.16 |
| 0.3 | 0.07 | 0.09 |
| 0.2 | 0.03 | 0.04 |

In this table $R_{50}$ is the reflection coefficient evaluated from Eq. (50) with the value

$$\lambda = 10.6\ d\ ;$$

this value of $\lambda$ is calculated using Eq. (8) with $\Omega = 1$.

As this table shows, Eq. (50), which has been proved for small values of $\lambda$, is approximately valid even up to $\Omega = 1$, for cracks of up to half the plate thickness.

For values of $\Omega$ in excess of unity we refer again to the work of Koshiba et al. (1984) on the reflection of an $S_0$ wave by a symmetric crack in an aluminium plate. Koshiba considers the case when the nondimensional frequency $\Omega$ is equal to $\pi/2$, and gives reflection coefficients for a range of crack lengths for which $a/d$ varies between zero and unity. When $a/d$ is less than 0.4, the reflection coefficient of Koshiba may be shown to fit the relation

$$R = 14\frac{a^2}{\lambda d}\ \Phi\ .$$

This formula is not valid when $a/d$ is greater than 0.4.

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