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# Exponential Input-to-State Stabilization of a Class of Diagonal Boundary Control Systems with Delay Boundary Control[⋆]

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ABSTRACT

This paper deals with the exponential input-to-state stabilization with respect to boundary disturbances of a class of diagonal infinite-dimensional systems via delay boundary control. The considered input delays are uncertain and time-varying. The proposed control strategy consists of a constant-delay predictor feedback controller designed on a truncated finite-dimensional model capturing the unstable modes of the original infinite-dimensional system. We show that the resulting closed-loop system is exponentially input-to-state stable with fading memory of both additive boundary input perturbations and disturbances in the computation of the predictor feedback.

## 1. Introduction

Feedback stabilization of finite-dimensional systems in the presence of input delays has been a very active research topic during the past decades [1, 31]. Motivated by the delay boundary control of Partial Differential Equations (PDEs), the opportunity of extending this topic to infinite-dimensional systems has recently attracted much attention [10, 34]. One of the early contributions on input delayed unstable PDEs, reported in [18], deals with a reaction-diffusion equation with a controller designed by resorting to the backstepping technique. More recently, the opportunity to use a predictor feedback for the stabilization of a reaction-diffusion equation was reported in [30]. The proposed control strategy, inspired by the early works [7, 8, 32] dealing with delay-free boundary feedback control, goes as follows. First, a finite-dimensional truncated model capturing the unstable modes of the infinite-dimensional system is obtained via spectral reduction. Then, using the Artstein transformation for handling the input delay, a predictor feedback is designed to stabilize the truncated model. Finally, the stability of the closed-loop infinite-dimensional system is assessed via a Lyapunov-based argument. This strategy was reused in [11] for the delay boundary feedback stabilization of a linear Kuramoto-Sivashinsky equation. This was then generalized to the boundary feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control for either a constant [19, 25] or a time-varying [20] input delay.

In this paper, we investigate the exponential input-to-state stabilization with respect to boundary disturbances of a class of diagonal infinite-dimensional systems via delay boundary control. In this setting, the considered input delay is uncertain and time-varying. The main motivation in achieving an input-to-state stabilization of the closed-loop system relies in the fact that the Input-to-State Stability (ISS) property, originally introduced by Sontag in [36], is one of the main tools for assessing the robustness of a system with respect to boundary disturbances. This property also plays a key role in the establishment of small gain conditions for the stability of interconnected systems [17]. Although the study of ISS properties of finite-dimensional systems has been intensively studied during the last three decades, its extension to infinite-dimensional systems, and in particular with respect to boundary disturbances, is more recent [4, 12, 13, 15, 16, 17, 21, 24, 26, 28, 29, 37, 38]. Moreover, most of these results deal with the establishment of ISS properties for open-loop stable distributed parameter systems. The literature regarding the input-to-state stabilization of open-loop unstable infinite-dimensional systems is less developed.

In the context of recent efforts about the establishment of ISS properties w.r.t. exogenous disturbances for predictor feedback control of finite-dimensional systems [5, 33], the present paper extends the results reported in [19, 20] regarding the use of a constant-delay predictor feedback for the delayed boundary stabilization of a class of diagonal infinite-dimensional systems. The validity of such an approach was first assessed in [19] for a constant, and known, input delay and then in [20] for an unknown and time-varying input delay via Lyapunov-based arguments. While such an approach allows the derivation of an ISS estimate with respect to distributed disturbances [19], it fails in the establishment of an ISS estimate, in strict form[1], with respect to boundary disturbances. It is worth noting that this increased difficulty regarding the establishment of ISS estimates w.r.t. boundary disturbances comparing to distributed ones seems

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[1] More precisely, this approach only allows the derivation of an ISS estimate with respect to both the boundary perturbation and its time derivative, but not an ISS estimate in strict form, i.e., with respect to the only magnitude of the boundary perturbation.

to be a global trend for infinite-dimensional systems [27]. In this paper, under the assumption of a sector condition on the eigenvalues corresponding to the modes which are not captured by the truncated model used for the design of the predictor feedback, we show that the resulting infinite-dimensional closed-loop system is exponentially ISS with fading memory [17] of the boundary disturbances for small variations of the time-varying delay around its nominal value. The adopted approach relies first on the extension of a small gain argument reported in [14] in order to establish the ISS property of the closed-loop truncated model, and then on the method reported in [24] for the establishment of ISS estimates with respect to boundary disturbances for diagonal infinite-dimensional systems.

This paper is organized as follows. The investigated control problem, the proposed control strategy, and the main result of this paper are introduced in Section 2. In Section 3 is reported the stability analysis of the finite-dimensional truncated model. Then, the proof of the main result of this paper, namely the ISS property of the resulting closed-loop infinite-dimensional system, is presented in Section 4. The relaxation of the assumed regularity assumptions for the boundary disturbances is discussed in Section 5. Finally, concluding remarks are formulated in Section 6.

## 2. Problem setting and main result

The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. Throughout the paper, the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. All the finite-dimensional spaces $\mathbb{K}^p$ are endowed with the usual euclidean inner product $\langle x, y\rangle = x^* y$ and the associated 2-norm $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^* x}$. For any matrix $M \in \mathbb{K}^{p\times q}$, $\|M\|$ stands for the induced norm of $M$ associated with the above 2-norms. For any $t_0 > 0$, we say that $\varphi \in C^0(\mathbb{R};\mathbb{R})$ is a *transition signal over* $[0, t_0]$ if $0 \leq \varphi \leq 1$, $\varphi|_{(-\infty,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$.

### 2.1. Preliminary definitions

Throughout the paper, $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ denotes a separable Hilbert space over the field $\mathbb{K}$.

**Definition 1 (Boundary control system [9]).** Consider the abstract system taking the form:

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t), & t \geq 0 \\ \mathcal{B}X(t) = v(t), & t \geq 0 \\ X(0) = X_0 \end{cases} \tag{1}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ an (unbounded) operator, $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathbb{K}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ the boundary operator, $v : \mathbb{R}_+ \to \mathbb{K}^m$ a boundary input, and $X_0 \in \mathcal{H}$ an initial condition. We say that $(\mathcal{A}, \mathcal{B})$ is a boundary control system if:

1. the disturbance-free operator $\mathcal{A}_0$, defined on the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$, is the generator of a $C_0$-semigroup $S$ on $\mathcal{H}$;
2. there exists a bounded operator $B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, called a lifting operator, such that $\mathrm{R}(B) \subset D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$ (i.e., is a bounded operator), and $\mathcal{B}B = I_{\mathbb{K}^m}$.

**Definition 2 (Riesz spectral operator [9]).** Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{H} \to \mathcal{H}$ be a linear and closed operator with simple eigenvalues $\lambda_n$ and corresponding eigenvectors $\phi_n \in D(\mathcal{A}_0)$, $n \in \mathbb{N}^*$. $\mathcal{A}_0$ is a Riesz-spectral operator if

1. $\{\phi_n,\ n \in \mathbb{N}^*\}$ is a Riesz basis [6]:
    (a) $\{\phi_n,\ n \in \mathbb{N}^*\}$ is maximal, i.e., $\overline{\operatorname{span}_{\mathbb{K}}}_{n\in\mathbb{N}^*} \phi_n = \mathcal{H}$;
    (b) there exist constants $m_R, M_R \in \mathbb{R}_+^*$ such that, for all $N \in \mathbb{N}^*$ and all $\alpha_1, \dots, \alpha_N \in \mathbb{K}$,
$$m_R \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|_{\mathcal{H}}^2 \leq M_R \sum_{n=1}^{N} |\alpha_n|^2; \tag{2}$$
2. the closure of $\{\lambda_n,\ n \in \mathbb{N}^*\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$, we have $[a, b] \triangleq \{xa + (1-x)b \,:\, x \in [0,1]\} \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$.

**Remark 1.** Let $\{\psi_n,\ n \in \mathbb{N}^*\}$ be the biorthogonal sequence associated with $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e., $\langle \phi_n, \psi_m\rangle_{\mathcal{H}} = \delta_{n,m}$. Then $\psi_n$ is an eigenvector of the adjoint operator $\mathcal{A}_0^*$ associated with $\overline{\lambda_n}$. Moreover, the following series expansion holds:

$$\forall z \in \mathcal{H}, \quad z = \sum_{n\geq 1} \langle z, \psi_n\rangle_{\mathcal{H}}\, \phi_n. \tag{3}$$

### 2.2. Problem and proposed control strategy

Let $D_0 > 0$ and $\delta \in (0, D_0)$ be given. We consider the abstract boundary control system (1) for which the boundary input $v$ takes the form:

$$v(t) = u(t - D(t)) + d_1(t) \tag{4}$$

for all $t \geq 0$ with $d_1 : \mathbb{R}_+ \to \mathbb{K}^m$ a boundary disturbance, $u : [-D_0 - \delta, +\infty) \to \mathbb{K}^m$ the boundary control with $u|_{[-D_0-\delta,0]} = 0$, and $D : \mathbb{R}_+ \to [D_0 - \delta, D_0 + \delta]$ a time-varying delay.

**Assumption 1.** *The disturbance-free operator $\mathcal{A}_0$ is a Riesz spectral operator.*

Then, the $C_0$-semigroup generated by $\mathcal{A}_0$ is given by

$$\forall z \in \mathcal{H}, \quad \forall t \geq 0, \quad S(t)z = \sum_{n\geq 1} e^{\lambda_n t} \langle z, \psi_n\rangle_{\mathcal{H}}\, \phi_n. \tag{5}$$

**Assumption 2.** *There exist $N_0 \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}_+^*$ such that*

1. *$\operatorname{Re}\lambda_n \leq -\alpha$ for all $n \geq N_0 + 1$ ;*
2. $\xi \triangleq \sup_{n\geq N_0+1} \left| \frac{\lambda_n}{\operatorname{Re}\lambda_n} \right| < \infty$.

**Remark 2.** If the first point of Assumption 2 holds, the second point $\xi < \infty$ is equivalent to the existence of a constant $\beta > 0$ such that $|\operatorname{Im}\lambda_n| \leq \beta |\operatorname{Re}\lambda_n|$ for all $n \geq N_0 + 1$.

The boundary feedback stabilization problem of the considered system was solved in [20] in the disturbance-free case by designing a constant-delay predictor feedback on a finite dimensional truncated model capturing the unstable modes of the infinite-dimensional system. In this paper, we go beyond the result reported in [20] by considering the impact of boundary disturbances while relaxing the assumed regularity properties and compatibility conditions. Specifically, assuming that the control input[2] $u$, the time-varying delay $D$, and the boundary disturbance $d_1$ are of class $C^1$, then, for any given initial condition $X_0 \in \mathcal{H}$, we can introduce $X \in C^0(\mathbb{R}_+; \mathcal{H})$ defined for all $t \geq 0$ by

$$\begin{aligned} X(t) = S(t)\{X_0 - Bv(0)\} + Bv(t) \\ + \int_0^t S(t-s)\{\mathcal{A}Bv(s) - B\dot{v}(s)\}\,\mathrm{d}s \end{aligned} \tag{6}$$

as the unique mild solution of (1), with control input $v$ given by (4), associated with $(D, X_0, d_1)$. We introduce the series expansion $X(t) = \sum_{n\geq 1} c_n(t)\phi_n$ with $c_n(t) \triangleq \langle X(t), \psi_n\rangle_{\mathcal{H}}$ the coefficients of projection of the system trajectory $X(t)$ into the Riesz basis $\left\{\phi_n,\ n \in \mathbb{N}^*\right\}$. The use of (6), combined with (3) and (5), and an integration by parts, show that $c_n$ statisfies

$$\begin{aligned} c_n(t) = e^{\lambda_n t} c_n(0) \\ + \int_0^t e^{\lambda_n(t-\tau)} \left\{\langle \mathcal{A}Bv(\tau), \psi_n\rangle_{\mathcal{H}} - \lambda_n \langle Bv(\tau), \psi_n\rangle_{\mathcal{H}}\right\} \mathrm{d}\tau \end{aligned} \tag{7}$$

for all $t \geq 0$. Thus $c_n \in C^1(\mathbb{R}_+; \mathbb{K})$ and satisfies for all $t \geq 0$ the following ODE (see also [24]):

$$\dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n \langle Bv(t), \psi_n\rangle_{\mathcal{H}} + \langle \mathcal{A}Bv(t), \psi_n\rangle_{\mathcal{H}}\,. \tag{8}$$

Let $\mathcal{E} = (e_1, e_2, \ldots, e_m)$ be the canonical basis of $\mathbb{K}^m$. Then, introducing[3] $b_{n,k} \triangleq -\lambda_n \langle Be_k, \psi_n\rangle_{\mathcal{H}} + \langle \mathcal{A}Be_k, \psi_n\rangle_{\mathcal{H}}$, we obtain that

$$\begin{aligned} \dot{Y}(t) &= A_{N_0} Y(t) + B_{N_0} v(t) \\ &= A_{N_0} Y(t) + B_{N_0}\{u(t - D(t)) + d_1(t)\}, && \text{(9a)} \\ Y(0) &= Y_0, && \text{(9b)} \end{aligned}$$

with

$$Y(t) = \begin{bmatrix} c_1(t) & \ldots & c_{N_0}(t) \end{bmatrix}^\top \in \mathbb{K}^{N_0}, \tag{10}$$

the matrices $A_{N_0} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{K}^{N_0\times N_0}$ and $B_{N_0} = (b_{n,k})_{1\leq n\leq N_0, 1\leq k\leq m} \in \mathbb{K}^{N_0\times m}$, and the initial condition

$$Y_0 = \begin{bmatrix} \langle X_0, \psi_1\rangle_{\mathcal{H}} & \ldots & \left\langle X_0, \psi_{N_0}\right\rangle_{\mathcal{H}} \end{bmatrix}^\top \in \mathbb{K}^{N_0}.$$

**Assumption 3.** *$(A_{N_0}, B_{N_0})$ is stabilizable.*

Under Assumption 3, one can design a predictor feedback achieving the stabilization of the truncated model (9). Then, following [20], such a predictor feedback can be successfully applied to the original infinite-dimensional system. Specifically, let $t_0, D_0 > 0$ and $\delta \in (0, D_0)$ be given. We consider a given transition signal[4] $\varphi \in C^1(\mathbb{R}; \mathbb{R})$ over $[0, t_0]$. We assume that $D \in C^1(\mathbb{R}_+; \mathbb{R})$ with $|D - D_0| \leq \delta$. The closed-loop system dynamics takes the following form:

$$\begin{aligned} &\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t), && \text{(11a)} \\ &\mathcal{B}X(t) = v(t) = u(t - D(t)) + d_1(t), && \text{(11b)} \\ &u(t) = \varphi(t)\Bigg\{ KY(t) + d_2(t) && \text{(11c)} \\ &\qquad + K\int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s \Bigg\}, \\ &X(0) = X_0 && \text{(11d)} \end{aligned}$$

for any $t \geq 0$. The adopted control strategy takes the form of a state-feedback in which the signal $Y(t)$ is computed based on the knowledge of the state $X(t)$ via (10). The feedback gain $K \in \mathbb{K}^{m\times N_0}$ is selected such that the matrix $A_{\mathrm{cl}} \triangleq A_{N_0} + e^{-D_0 A_{N_0}} B_{N_0} K$ is Hurwitz. Functions $d_1, d_2 : \mathbb{R}_+ \to \mathbb{K}^m$ represent boundary disturbances.

**Remark 3.** Examples of systems covered by Assumptions 1-3 and thus for which the proposed control strategy applies include reaction-diffusion equations [22, 30], linear Kuramoto-Sivashinsky equation [11], and certain damped flexible string or beam models [9, Ex. 2.23, p. 91][23]. For this type of system, the objective of the present paper is to establish a qualitative behavior regarding the closed-loop system dynamics (11), namely an ISS property with respect to boundary disturbances $d_1$ and $d_2$.

**Remark 4.** While disturbance $d_1$ represents an additive disturbance in the application of the delayed boundary control $u$, disturbance $d_2$ gathers uncertainties of either/both the output measurement $Y$ or/and the computation of the control law $u$ that is solution of a "fixed point implicit equality" involving an integral term [3]. The existence and uniqueness of solutions for such an implicit equation has been assessed in [3] in the case $\varphi = 1$. The proofs reported therein directly extend to the configuration studied in this paper by noting that $\varphi$ is a continuous function with $0 \leq \varphi \leq 1$. Moreover, as $Y$ is solution of the ODE (9), it can be shown that the closed-loop dynamics (11) with $Y$ given by (10) is actually equivalent to the dynamics (11) with $Y$ explicitly given by

$$Y(t) = e^{A_{N_0}t} Y_0 + \int_0^t e^{A_{N_0}(t-\tau)} B_{N_0} \left\{u(\tau - D(\tau)) + d_1(\tau)\right\} \mathrm{d}\tau.$$

Note however that this second form is not convenient for practical implementation as it requires the knowledge of the disturbance $d_1$ in real-time.

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[2] The construction of the control law must ensure this property.

[3] Note that the quantity $b_{n,k}$ is independent of the specifically selected lifting operator $B$ associated with $(\mathcal{A}, \mathcal{B})$, see [19].

[4] See the notation section at the beginning of Section 2.

## 2.3. Well-posedness in terms of mild solutions

In the first part of this paper, we consider the following concept of mild solutions for the closed-loop system dynamics.

**Definition 3.** Let $(\mathcal{A}, \mathcal{B})$ be an abstract boundary control system such that Assumption 1 holds. Let $t_0, D_0 > 0$, $\delta \in (0, D_0)$, a transition signal $\varphi \in C^1(\mathbb{R};\mathbb{R})$ over $[0, t_0]$, and $K \in \mathbb{K}^{m \times N_0}$ be arbitrary. For a time-varying delay $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, an initial condition $X_0 \in \mathcal{H}$, and boundary perturbations $d_1, d_2 \in C^1(\mathbb{R}_+;\mathbb{K}^m)$, we say that $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^1([-D_0 - \delta, +\infty);\mathbb{K}^m)$ is a mild solution of (11) associated with $(D, X_0, d_1, d_2)$ if 1) (6) holds for all $t \geq 0$ with $v$ given by (4); 2) $u$ satisfies (11c) for all $t \geq -D_0 - \delta$ with $Y$ defined by (10).

The following lemma, whose proof is placed in Appendix A, assesses the well-posedness of the closed-loop system (11) in terms of mild solutions.

**Lemma 1.** *For any* $D \in C^1(\mathbb{R}_+;\mathbb{R})$ *with* $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, *and* $d_1, d_2 \in C^1(\mathbb{R}_+;\mathbb{K}^m)$, *the closed-loop system (11) admits a unique mild solution* $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^1([-D_0 - \delta, +\infty);\mathbb{K}^m)$ *associated with* $(D, X_0, d_1, d_2)$.

**Remark 5.** If we assume the stronger regularity assumptions $\varphi \in C^2(\mathbb{R};\mathbb{R})$, $D \in C^2(\mathbb{R}_+;\mathbb{R})$, $d_1, d_2 \in C^2(\mathbb{R}_+;\mathbb{K}^m)$, as well as the compatibility condition $X_0 \in D(\mathcal{A})$ with $\mathcal{B}X_0 = d_1(0)$, it can be shown that the mild solution is actually a classical solution with a control input $u$ that is twice continuously differentiable.

## 2.4. Main stability result

The stability of the closed-loop system (11) in the disturbance free case (i.e., for $d_1 = d_2 = 0$) was assessed in [19, 25] for a constant delay $D(t) = D_0$ and in [20] for an uncertain and time-varying delay $D(t)$. The objective of this paper is to study the impact of the boundary disturbances $d_1$ and $d_2$ on the system trajectories. More precisely, we derive the following result.

**Theorem 2.** *Let* $(\mathcal{A}, \mathcal{B})$ *be an abstract boundary control system such that Assumptions 1, 2, and 3 hold. Let* $\varphi \in C^1(\mathbb{R};\mathbb{R})$ *be a transition signal over* $[0, t_0]$ *for some* $t_0 > 0$. *Let* $D_0 > 0$ *and* $K \in \mathbb{K}^{m \times N_0}$ *be such that* $A_{\mathrm{cl}} = A_{N_0} + e^{-D_0 A_{N_0}} B_{N_0} K$ *is Hurwitz. Let* $\delta \in (0, D_0)$ *be such that*[5]

$$M_\lambda \| B_{N_0} K \| \left[ e^{\|A_{\mathrm{cl}}\| \delta} - e^{-\lambda \delta} \right] < \lambda, \tag{12}$$

*where* $\lambda > 0$ *and* $M_\lambda \geq 1$ *are such that* $\| e^{A_{\mathrm{cl}} t} \| \leq M_\lambda e^{-\lambda t}$ *for all* $t \geq 0$. *Then, there exist* $\kappa \in (0, \alpha)$ *and* $\overline{C}_i > 0$, $1 \leq i \leq 6$, *such that, for any* $D \in C^1(\mathbb{R}_+;\mathbb{R})$ *with* $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, *and* $d_1, d_2 \in C^1(\mathbb{R}_+;\mathbb{K}^m)$, *the mild solution* $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^1([-D_0 - \delta, +\infty);\mathbb{K}^m)$ *of the closed-loop system (11) associated with* $(D, X_0, d_1, d_2)$ *satisfies*

$$\begin{aligned} \|X(t)\|_{\mathcal{H}} \leq \overline{C}_1 e^{-\kappa t} \|X_0\|_{\mathcal{H}} + \overline{C}_2 \sup_{\tau \in [0,t]} e^{-\kappa(t-\tau)} \|d_1(\tau)\| \\ + \overline{C}_3 \sup_{\tau \in [0, \max(t-(D_0-\delta), 0)]} e^{-\kappa(t-\tau)} \|d_2(\tau)\| \end{aligned} \tag{13}$$

*and*

$$\begin{aligned} \|u(t)\| \leq \overline{C}_4 e^{-\kappa t} \|X_0\|_{\mathcal{H}} + \overline{C}_5 \sup_{\tau \in [0,t]} e^{-\kappa(t-\tau)} \|d_1(\tau)\| \\ + \overline{C}_6 \sup_{\tau \in [0,t]} e^{-\kappa(t-\tau)} \|d_2(\tau)\|. \end{aligned} \tag{14}$$

*for all* $t \geq 0$.

The next two sections are devoted to the proof of Theorem 2. The extension of this result to continuous boundary disturbances $d_1, d_2$ is discussed in Section 5.

**Remark 6.** It is interesting to note that Theorem 2, involving the sector condtion $\xi < +\infty$ (see Assumption 2), does not introduce any constraint on the amplitude of variation of the time derivative $\dot{D}$ of the input delay $D$. This is in contrast with the result reported in [20] for the disturbance-free case (i.e., $d_1 = d_2 = 0$), which allows $\xi = +\infty$ but where the constant of the exponential stability property is a strictly increasing function, going to $+\infty$ at $+\infty$, of the supremum of $|\dot{D}|$. The occurence of a $\dot{D}$ term in the proof of the result reported in [20] is due to the use of a Lyapunov-based argument. As discussed in the sequel of this paper, the assumption $\xi < +\infty$ allows a proof of Theorem 2 that does not rely on such a Lyapunov-based argument.

# 3. Exponential ISS of the truncated model

In this section, we study the ISS property of the finite dimensional truncated model. We refer the reader to [35] for classical results about the establishment of ISS properties.

## 3.1. Preliminary lemma

We need the following preliminary lemma which is a disturbed version of the disturbance-free version ($p = 0$) reported in [14, Th. 2.5].

**Lemma 3.** *Let* $A \in \mathbb{K}^{n \times n}$ *be Hurwitz,* $C \in \mathbb{K}^{n \times n}$, *and* $r > 0$. *Let* $\epsilon \in (0, r)$ *be such that*

$$M_\lambda \|C\| \left[ e^{\|A\| \epsilon} - e^{-\lambda \epsilon} \right] < \lambda, \tag{15}$$

*where* $\lambda > 0$ *and* $M_\lambda \geq 1$ *are such that* $\|e^{At}\| \leq M_\lambda e^{-\lambda t}$ *for all* $t \geq 0$. *Then, there exist* $\sigma, N > 0$ *and* $M \geq 1$ *such that, for any* $d \in C^0(\mathbb{R}_+;\mathbb{R})$ *with* $|d| \leq 1$, *any* $p, q \in C^0(\mathbb{R}_+;\mathbb{K})$ *with* $|q| \leq 1$, *and any* $x_0 \in C^0([-r-\epsilon, 0];\mathbb{K}^n)$, *the trajectory of*

$$\dot{x}(t) = Ax(t) + q(t) C \left[ x(t - r - \epsilon d(t)) - x(t - r) \right] + p(t) \tag{16a}$$

$$x(\tau) = x_0(\tau), \quad -r - \epsilon \leq \tau \leq 0 \tag{16b}$$

*for* $t \geq 0$ *satisfies*

$$\|x(t)\| \leq M e^{-\sigma t} \sup_{\tau \in [-r-\epsilon, 0]} \|x_0(\tau)\| + N \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|p(\tau)\| \tag{17}$$

*for all* $t \geq 0$.

---

[5] Such a $\delta > 0$ always exists by a continuity argument in $\delta = 0$.

PROOF. As the case $C = 0$ is straightforward, we assume in the sequel that $C \neq 0$. The first part of the proof follows the one in [14] while considering the impact of the disturbing term $p$. We define, for all $t \geq 0$, $v(t) = x(t - r - \epsilon d(t)) - x(t-r)$. Let $\sigma \in (0, \lambda)$, which will be specified in the sequel, be arbitrary. The proof is divided into 4 main steps.

*Step 1: preliminary estimation of* $\sup_{\tau\in[r+\epsilon,t]} e^{\sigma\tau}\|v(\tau)\|$ *by an upper estimate involving* $\|x(\tau)\|$. As in [14], we consider the cases $d(t) \leq 0$ and $d(t) \geq 0$ separately. In the case $d(t) \leq 0$, we have by direct integration of (16a) that, for all $t \geq r$,

$$
\begin{aligned}
v(t) = &\left[e^{-\epsilon A d(t)} - I_n\right] x(t-r) \\
&+ \int_{t-r}^{t-r-\epsilon d(t)} e^{A(t-r-\epsilon d(t)-\tau)} \left[q(\tau)Cv(\tau) + p(\tau)\right] \mathrm{d}\tau
\end{aligned}
$$

because $t-r-\epsilon d(t) \geq t-r \geq 0$. Noting that $\|e^{-\epsilon A d(t)} - I_n\| \leq e^{\|A\|\epsilon} - 1$,

$$
\begin{aligned}
&\left\| \int_{t-r}^{t-r-\epsilon d(t)} e^{A(t-r-\epsilon d(t)-\tau)} q(\tau) C v(\tau)\, \mathrm{d}\tau \right\| \\
&\leq M_\lambda \|C\| \int_{t-r}^{t-r-\epsilon d(t)} e^{-\lambda(t-r-\epsilon d(t)-\tau)} \|v(\tau)\|\, \mathrm{d}\tau \\
&\leq M_\lambda \|C\| e^{-\lambda(t-r-\epsilon d(t))} \int_{t-r}^{t-r-\epsilon d(t)} e^{(\lambda-\sigma)\tau} \times e^{\sigma\tau} \|v(\tau)\|\, \mathrm{d}\tau \\
&\leq M_\lambda \|C\| e^{-\sigma(t-r)} e^{\sigma\epsilon d(t)} \frac{1 - e^{(\lambda-\sigma)\epsilon d(t)}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\
&\leq M_\lambda \|C\| e^{-\sigma(t-r)} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\|,
\end{aligned}
$$

where it has been used that $-1 \leq d(t) \leq 0$, and, similarly,

$$
\begin{aligned}
&\left\| \int_{t-r}^{t-r-\epsilon d(t)} e^{A(t-r-\epsilon d(t)-\tau)} p(\tau)\, \mathrm{d}\tau \right\| \\
&\leq M_\lambda e^{-\sigma(t-r)} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|,
\end{aligned}
$$

we obtain that, for all $t \geq r$ such that $d(t) \leq 0$,

$$
\begin{aligned}
e^{\sigma t}\|v(t)\| \leq &\left[e^{\|A\|\epsilon} - 1\right] e^{\sigma r} \times e^{\sigma(t-r)} \|x(t-r)\| \qquad (18) \\
&+ M_\lambda \|C\| e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\
&+ M_\lambda e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|.
\end{aligned}
$$

Now, in the case $d(t) \geq 0$, we have by direct integration of (16a) that, for all $t \geq r + \epsilon$,

$$
\begin{aligned}
v(t) = &-\left[e^{\epsilon A d(t)} - I_n\right] x(t - r - \epsilon d(t)) \\
&- \int_{t-r-\epsilon d(t)}^{t-r} e^{A(t-r-\tau)} \left[q(\tau)Cv(\tau) + p(\tau)\right] \mathrm{d}\tau
\end{aligned}
$$

because $t - r \geq t - r - \epsilon d(t) \geq t - r - \epsilon \geq 0$. Then, we deduce that, for all $t \geq r + \epsilon$ such that $d(t) \geq 0$,

$$
\begin{aligned}
&e^{\sigma t}\|v(t)\| \\
&\leq \left[e^{\|A\|\epsilon} - 1\right] e^{\sigma(r+\epsilon)} \times e^{\sigma(t-r-\epsilon d(t))} \|x(t-r-\epsilon d(t))\| \qquad (19) \\
&\quad + M_\lambda \|C\| e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r-\epsilon,t-r]} e^{\sigma\tau}\|v(\tau)\| \\
&\quad + M_\lambda e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[t-r-\epsilon,t-r]} e^{\sigma\tau}\|p(\tau)\|.
\end{aligned}
$$

Combining (18-19), we obtain that, for all $t \geq r + \epsilon$,

$$
\begin{aligned}
&\sup_{\tau\in[r+\epsilon,t]} e^{\sigma\tau}\|v(\tau)\| \\
&\leq \left[e^{\|A\|\epsilon} - 1\right] e^{\sigma(r+\epsilon)} \sup_{\tau\in[0,t-r]} e^{\sigma\tau}\|x(\tau)\| \qquad (20) \\
&\quad + M_\lambda \|C\| e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\
&\quad + M_\lambda e^{\sigma r} \frac{1 - e^{-(\lambda-\sigma)\epsilon}}{\lambda - \sigma} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|.
\end{aligned}
$$

*Step 2: preliminary estimation of* $\sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\|$ *by an upper estimate involving involving* $\|v(\tau)\|$. Integrating (16a) over $[0, t]$, we obtain for all $t \geq 0$,

$$
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} \left[q(\tau)Cv(\tau) + p(\tau)\right] \mathrm{d}\tau.
$$

As $x(0) = x_0(0)$, straightforward estimations show that, for all $t \geq 0$,

$$
\begin{aligned}
\sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\| \leq &M_\lambda \|x_0(0)\| + \frac{M_\lambda \|C\|}{\lambda - \sigma} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \\
&+ \frac{M_\lambda}{\lambda - \sigma} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|. \qquad (21)
\end{aligned}
$$

*Step 3: estimation of* $\sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\|$ *by an upper estimate involving only* $\|x_0(\tau)\|$ *and* $\|p(\tau)\|$ *via a small gain argument.* From (20-21), we deduce that, for all $t \geq r + \epsilon$,

$$
\begin{aligned}
\sup_{\tau\in[r+\epsilon,t]} e^{\sigma\tau}\|v(\tau)\| \leq &M_\lambda e^{\sigma(r+\epsilon)} \left[e^{\|A\|\epsilon} - 1\right] \|x_0(0)\| \\
&+ \delta \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\
&+ \frac{\delta}{\|C\|} \sup_{\tau\in[0,t-r+\epsilon]} e^{\sigma\tau}\|p(\tau)\|,
\end{aligned}
$$

where

$$
\delta \triangleq \frac{M_\lambda \|C\|}{\lambda - \sigma} e^{\sigma r} \left[e^{\sigma\epsilon}(e^{\|A\|\epsilon} - 1) + 1 - e^{-(\lambda-\sigma)\epsilon}\right].
$$

From the small gain assumption (15) and a continuity argument in $\sigma = 0$, we select $\sigma \in (0, \lambda)$ such that $\delta < 1$. Noting that the supremums appearing in the latter estimate are finite, we deduce that, for all $t \geq 0$,

$$
\begin{aligned}
\sup_{\tau\in[0,t]} e^{\sigma\tau}\|v(\tau)\| \leq &\frac{M_\lambda e^{\sigma(r+\epsilon)} \left[e^{\|A\|\epsilon} - 1\right]}{1 - \delta} \|x_0(0)\| \\
&+ \sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\|
\end{aligned}
$$

$$+ \frac{\delta}{\|C\|(1-\delta)} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|.$$

Using (21), we obtain that, for all $t \geq 0$,

$$\begin{aligned}
&\sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\| \\
&\leq M_\lambda \left\{ 1 + \frac{M_\lambda \|C\| e^{\sigma(r+\epsilon)} \left[ e^{\|A\|\epsilon} - 1 \right]}{(1-\delta)(\lambda-\sigma)} \right\} \|x_0(0)\| \qquad (22) \\
&\quad + \frac{M_\lambda \|C\|}{\lambda - \sigma} \sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\
&\quad + \frac{M_\lambda}{(1-\delta)(\lambda-\sigma)} \sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|.
\end{aligned}$$

It remains now to evaluate $\sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\|$. To do so, we note from the definition of $v$ that

$$\begin{aligned}
&\sup_{\tau\in[0,r+\epsilon]} e^{\sigma\tau}\|v(\tau)\| \\
&\leq 2e^{\sigma(r+\epsilon)} \left( \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + \sup_{\tau\in[0,2\epsilon]} \|x(\tau)\| \right). \qquad (23)
\end{aligned}$$

Based on (16), a standard application of Grönwall's inequality shows the existence of constants $\gamma_0, \gamma_1 > 0$, which only depend on $A, C, r, \epsilon$, such that

$$\sup_{\tau\in[0,2\epsilon]} \|x(\tau)\| \leq \gamma_0 \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + \gamma_1 \sup_{\tau\in[0,2\epsilon]} e^{\sigma\tau}\|p(\tau)\|. \qquad (24)$$

Combining (22-24), we deduce the existence of constants $M \geq 1$ and $N > 0$ such that, for all $t \geq 0$,

$$\begin{aligned}
&\sup_{\tau\in[0,t]} e^{\sigma\tau}\|x(\tau)\| \\
&\leq M \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,\max(t,2\epsilon)]} e^{\sigma\tau}\|p(\tau)\|. \qquad (25)
\end{aligned}$$

*Step 4: derivation of the claimed estimate (17).* To conclude, it remains to show that $\sup_{\tau\in[0,\max(t,2\epsilon)]} e^{\sigma\tau}\|p(\tau)\|$ can be replaced by $\sup_{\tau\in[0,t]} e^{\sigma\tau}\|p(\tau)\|$ in (25). This is obviously true for $t \geq 2\epsilon$, as well as $t = 0$ because $M \geq 1$. Thus, we focus on the case $0 < t < 2\epsilon$. Let $T \in (0, 2\epsilon)$ be arbitrary. Let $\zeta_n \in C^0(\mathbb{R}_+; \mathbb{R})$ with $0 \leq \zeta_n \leq 1$, $\zeta_n|_{[0,T]} = 1$ and $\zeta_n|_{[T+(2\epsilon-T)/n,+\infty)} = 0$ for $n \geq 1$. We define $p_n = \zeta_n p \in C^0(\mathbb{R}_+; \mathbb{K})$ and we denote by $x_n$ the solution of (16) associated with the initial condition $x_0$ and the disturbance $p_n$. As $p_n(t) = p(t)$ for all $0 \leq t \leq T$, we obtain that $x_n(t) = x(t)$ for all $0 \leq t \leq T$. Therefore, we obtain by applying (25) to $x_n$ at time $t = T$ that, for all $n \geq 1$,

$$\begin{aligned}
&\sup_{\tau\in[0,T]} e^{\sigma\tau}\|x(\tau)\| \\
&\leq M \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,2\epsilon]} e^{\sigma\tau}\|p_n(\tau)\| \\
&\underset{n\to+\infty}{\longrightarrow} M \sup_{\tau\in[-r-\epsilon,0]} \|x_0(\tau)\| + N \sup_{\tau\in[0,T]} e^{\sigma\tau}\|p(\tau)\|,
\end{aligned}$$

where the limit holds by a continuity argument. Thus, the claimed estimate (17) holds.

## 3.2. Study of the truncated model

We apply the result of Lemma 3 to the study of the finite-dimensional truncated model composed of (9) and (11c).

**Lemma 4.** *Under the assumptions of Theorem 2, there exist* $\sigma, C_1, C_2, C_3, \overline{C}_4, \overline{C}_5, \overline{C}_6 > 0$ *such that, for any* $D \in C^1(\mathbb{R}_+; \mathbb{R})$ *with* $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, *and* $d_1, d_2 \in C^1(\mathbb{R}_+; \mathbb{K}^m)$, $Y$ *defined by (10), where* $(X, u) \in C^0(\mathbb{R}_+; \mathcal{H}) \times C^1([-D_0 - \delta, +\infty); \mathbb{K}^m)$ *is the mild solution of the closed-loop system (11) associated with* $(D, X_0, d_1, d_2)$, *satisfies*

$$\begin{aligned}
\|Y(t)\| &\leq C_1 e^{-\sigma t}\|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau\in[0,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \qquad (26) \\
&\quad + C_3 \sup_{\tau\in[0,\max(t-(D_0-\delta),0)]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|
\end{aligned}$$

*for all* $t \geq 0$. *Furthermore, the control law* $u$ *satisfies (14) with* $\kappa = \sigma$ *for all* $t \geq 0$.

Proof. Let $\delta \in (0, D_0)$ satisfying the small gain condition (12) be given. Let $\sigma, N > 0$ and $M \geq 1$ be the constants provided by Lemma 3 for $A = A_{\mathrm{cl}}$, $C = B_{N_0}K$, $q = 1$, $r = D_0$, and $\epsilon = \delta$. We introduce the Artstein transformation [1, 31] by defining, for all $t \geq 0$,

$$Z(t) = Y(t) + \int_{t-D_0}^{t} e^{(t-D_0-s)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s. \qquad (27)$$

As $u(\tau) = 0$ for $\tau \leq 0$, we obtain that $u = \varphi K Z + \varphi d_2$. Taking the time derivative, (9) yields for all $t \geq 0$

$$\begin{aligned}
\dot{Z}(t) &= \left\{ A_{N_0} + \varphi(t) e^{-D_0 A_{N_0}} B_{N_0} K \right\} Z(t) \qquad (28) \\
&\quad + B_{N_0} K \left\{ [\varphi Z](t - D(t)) - [\varphi Z](t - D_0) \right\} \\
&\quad + B_{N_0} d_1(t) + \varphi(t) e^{-D_0 A_{N_0}} B_{N_0} d_2(t) \\
&\quad + B_{N_0} \left\{ [\varphi d_2](t - D(t)) - [\varphi d_2](t - D_0) \right\}.
\end{aligned}$$

We first study the case $t \geq t_1 \triangleq t_0 + D_0 + \delta$. We have for all $t \geq t_1$ that $\varphi(t) = \varphi(t - D_0) = \varphi(t - D(t)) = 1$ and thus

$$\begin{aligned}
\dot{Z}(t) &= A_{\mathrm{cl}} Z(t) + B_{N_0} K \left\{ Z(t - D(t)) - Z(t - D_0) \right\} \\
&\quad + B_{N_0} d_1(t) + e^{-D_0 A_{N_0}} B_{N_0} d_2(t) \\
&\quad + B_{N_0} \left\{ d_2(t - D(t)) - d_2(t - D_0) \right\}.
\end{aligned}$$

Consequently, by applying Lem. 3 to the above ODE with $x(t) = Z(t + t_1)$, it follows from (17) that, for all $t \geq t_1$,

$$\begin{aligned}
&\|Z(t)\| \qquad (29) \\
&\leq M e^{-\sigma(t-t_1)} \sup_{\tau\in[t_0,t_1]} \|Z(\tau)\| + \tilde{N}_1 \sup_{\tau\in[t_1,t]} e^{-\sigma(t-\tau)}\|d_1(\tau)\| \\
&\quad + \tilde{N}_2 \sup_{\tau\in[t_0,t]} e^{-\sigma(t-\tau)}\|d_2(\tau)\|,
\end{aligned}$$

with $\tilde{N}_1 = N\|B_{N_0}\|$ and

$$\tilde{N}_2 = N \left\{ \|e^{-D_0 A_{N_0}} B_{N_0}\| + \|B_{N_0}\| e^{\sigma D_0}(e^{\sigma\delta} + 1) \right\}.$$

In the case $0 \leq t \leq t_1$, based on (28), a standard application of Grönwall's inequality shows the existence of $\gamma_0, \gamma_1, \gamma_2 > 0$, which only depend of $A_{N_0}, B_{N_0}, K, D_0, \delta, t_0, \sigma$, such that

$$\begin{aligned} \|Z(t)\| \leq \gamma_0 e^{-\sigma t} \|Y(0)\| + \gamma_1 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_1(\tau)\| \\ + \gamma_2 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_2(\tau)\|, \end{aligned} \tag{30}$$

for all $0 \leq t \leq t_1$, where it has been used $Z(0) = Y(0)$. Thus, combining (29-30), we obtain that, for all $t \geq 0$,

$$\begin{aligned} \|Z(t)\| \leq \gamma_3 e^{-\sigma t} \|Y(0)\| + \gamma_4 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_1(\tau)\| \\ + \gamma_5 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_2(\tau)\| \end{aligned} \tag{31}$$

with $\gamma_3 = M e^{\sigma t_1} \gamma_0$, $\gamma_4 = M e^{\sigma t_1} \gamma_1 + \tilde{N}_1$, and $\gamma_5 = M e^{\sigma t_1} \gamma_2 + \tilde{N}_2$. From $u = \varphi K Z + \varphi d_2$ and (2), we deduce that (14) holds for $\kappa = \sigma$ with $\overline{C}_4 = \|K\| \gamma_3 / \sqrt{m_R}$, $\overline{C}_5 = \|K\| \gamma_4$, and $\overline{C}_6 = \|K\| \gamma_5 + 1$. Now, using estimates (2), (14) with $\kappa = \sigma$, and (31) into (27) and the fact that, for $i \in \{1, 2\}$,

$$\begin{aligned} \int_{\max(t-D_0,0)}^{t} \sup_{\tau \in [0,s]} e^{-\sigma(s-\tau)} \|d_i(\tau)\| \,\mathrm{d}s \\ \leq D_0 e^{\sigma D_0} \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_i(\tau)\|, \end{aligned}$$

we obtain that

$$\begin{aligned} \|Y(t)\| \leq C_1 e^{-\sigma t} \|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_1(\tau)\| \\ + C_3 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_2(\tau)\| \end{aligned} \tag{32}$$

for all $t \geq 0$, with

$$\begin{aligned} C_1 &= \gamma_3 / \sqrt{m_R} + e^{D_0(\sigma + \|A_{N_0}\|)} \|B_{N_0}\| \overline{C}_4 / \sigma, \\ C_2 &= \gamma_4 + D_0 e^{D_0(\sigma + \|A_{N_0}\|)} \|B_{N_0}\| \overline{C}_5, \\ C_3 &= \gamma_5 + D_0 e^{D_0(\sigma + \|A_{N_0}\|)} \|B_{N_0}\| \overline{C}_6. \end{aligned}$$

To conclude the proof, it remains to show that we can substitute in estimate (32) the term $\sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|d_2(\tau)\|$ by the term $\sup_{\tau \in [0, \max(t-(D_0-\delta),0)]} e^{-\sigma(t-\tau)} \|d_2(\tau)\|$. To do so, let $T \geq 0$ be arbitrary. Let $\zeta_n \in C^1(\mathbb{R}; \mathbb{R})$ with $0 \leq \zeta_n \leq 1$ be such that $\zeta_n|_{(-\infty, T-(D_0-\delta)]} = 1$ and $\zeta_n|_{[T-(D_0-\delta)+(D_0-\delta)/n, +\infty]} = 0$ for $n \geq 1$. Then we define $d_{2,n} = \zeta_n d_2 \in C^1(\mathbb{R}_+; \mathbb{K}^m)$. Thus we can introduce $(X_n, u_n)$ the mild solution of (11) associated with $(D, X_0, d_1, d_{2,n})$. From [19, Sec. III.C] (see also [3] in the case $\varphi = 1$), it can be seen that $X|_{[0,T]} = X_n|_{[0,T]}$ and $u|_{[-D_0-\delta, T-(D_0-\delta)]} = u_n|_{[-D_0-\delta, T-(D_0-\delta)]}$. Applying the ISS estimate (32) at time $t = T$ to $X_n$ for any $n \geq 1$, we obtain that

$$\begin{aligned} &\|Y(T)\| \\ &\leq C_1 e^{-\sigma T} \|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau \in [0,T]} e^{-\sigma(T-\tau)} \|d_1(\tau)\| \\ &\quad + C_3 \sup_{\tau \in [0,T]} e^{-\sigma(T-\tau)} \|\zeta_n(\tau) d_2(\tau)\|. \end{aligned}$$

By letting $n \to +\infty$, a continuity argument shows that (26) holds at time $t = T$. As $T \geq 0$ is arbitrary, this concludes the proof.

## 4. Exponential ISS of the infinite-dimensional system

This section is devoted to the proof of the main result of this paper: namely, Theorem 2. Let $\sigma > 0$ be provided by Lemma 4. Let $0 < \kappa < \min(\alpha, \sigma)$ be given and define $\epsilon = \kappa/\alpha \in (0, 1)$. First, we infer from $\xi = \sup_{n \geq N_0+1} \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n} \right| < \infty$ that the following estimate holds true for all $n \geq N_0 + 1$

$$\begin{aligned} \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n + \kappa} \right| &= \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n} \right| \times \left| \frac{\operatorname{Re} \lambda_n}{\operatorname{Re} \lambda_n + \kappa} \right| \\ &\leq \xi \left| \frac{-\alpha}{-\alpha + \kappa} \right| \leq \frac{\alpha \xi}{\alpha - \kappa}, \end{aligned} \tag{33}$$

where we have used the facts that $\operatorname{Re} \lambda_n \leq -\alpha < -\kappa < 0$ and the function $x \to x/(x+\kappa)$ is positive and strictly increasing for $x \in (-\infty, -\kappa)$. Now, from the integral expression (7) of the coefficient of projection $c_n$ and using the definition of the boundary input (4), we have for all $t \geq 0$

$$\begin{aligned} &|c_n(t)| \\ &\leq e^{\operatorname{Re} \lambda_n t} |c_n(0)| \\ &\quad + |\lambda_n| \int_0^t e^{\operatorname{Re} \lambda_n (t-\tau)} |\langle B u(\tau - D(\tau)), \psi_n \rangle_{\mathcal{H}}| \,\mathrm{d}\tau \\ &\quad + \int_0^t e^{\operatorname{Re} \lambda_n (t-\tau)} |\langle \mathcal{A} B u(\tau - D(\tau)), \psi_n \rangle_{\mathcal{H}}| \,\mathrm{d}\tau \\ &\quad + |\lambda_n| \int_0^t e^{\operatorname{Re} \lambda_n (t-\tau)} |\langle B d_1(\tau), \psi_n \rangle_{\mathcal{H}}| \,\mathrm{d}\tau \\ &\quad + \int_0^t e^{\operatorname{Re} \lambda_n (t-\tau)} |\langle \mathcal{A} B d_1(\tau), \psi_n \rangle_{\mathcal{H}}| \,\mathrm{d}\tau. \end{aligned} \tag{34}$$

We now evaluate the different terms on the right hand side of (34). Denoting by $e_1, \ldots, e_m$ the canonical basis of $\mathbb{K}^m$, we have $d_1(t) = \sum_{k=1}^{m} d_{1,k}(t) e_k$ with $d_{1,k}(t) \in \mathbb{K}$. Then, noting that $|d_{1,k}(t)| \leq \|d_1(t)\|$, we have for all $n \geq N_0 + 1$ and $t \geq 0$

$$\begin{aligned} &|\lambda_n| \int_0^t e^{\operatorname{Re} \lambda_n (t-\tau)} |\langle B d_1(\tau), \psi_n \rangle_{\mathcal{H}}| \,\mathrm{d}\tau \\ &\leq |\lambda_n| \sum_{k=1}^{m} |\langle B e_k, \psi_n \rangle_{\mathcal{H}}| \int_0^t e^{\operatorname{Re} \lambda_n (t-\tau)} \|d_1(\tau)\| \,\mathrm{d}\tau \\ &\leq |\lambda_n| \sum_{k=1}^{m} |\langle B e_k, \psi_n \rangle_{\mathcal{H}}| \int_0^t e^{(1-\epsilon) \operatorname{Re} \lambda_n (t-\tau)} \,\mathrm{d}\tau \\ &\quad \times \sup_{\tau \in [0,t]} e^{\epsilon \operatorname{Re} \lambda_n (t-\tau)} \|d_1(\tau)\| \\ &\leq \frac{1}{1-\epsilon} \left| \frac{\lambda_n}{\operatorname{Re} \lambda_n} \right| \sum_{k=1}^{m} |\langle B e_k, \psi_n \rangle_{\mathcal{H}}| \sup_{\tau \in [0,t]} e^{-\epsilon \alpha (t-\tau)} \|d_1(\tau)\| \end{aligned}$$

$$\leq \frac{\alpha\xi}{\alpha-\kappa}\sum_{k=1}^{m} |\langle Be_k,\psi_n\rangle_{\mathcal{H}}| \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_1(\tau)\|. \tag{35}$$

Similarly, we have for all $n \geq N_0+1$ and $t \geq 0$

$$\begin{aligned}&\int_0^t e^{\operatorname{Re}\lambda_n(t-\tau)}|\langle \mathcal{A}Bd_1(\tau),\psi_n\rangle_{\mathcal{H}}|\,\mathrm{d}\tau \\ &\leq \frac{1}{\alpha-\kappa}\sum_{k=1}^{m} |\langle \mathcal{A}Be_k,\psi_n\rangle_{\mathcal{H}}| \sup_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|d_1(\tau)\|. \end{aligned}\tag{36}$$

We now estimate the two remaining integral terms involving the control input $u$ on the right-hand side of (34). We note that these two integrals are null for $t \leq D_0-\delta$ because $u(\tau)=0$ for $\tau \leq 0$. Thus we focus on the case $t \geq D_0-\delta$. First, we evaluate the following integral:

$$\begin{aligned}\mathcal{I}_n(t) &= \int_0^t e^{-\operatorname{Re}\lambda_n\tau}\|u(\tau-D(\tau))\|\,\mathrm{d}\tau \\ &= \int_0^t e^{-\operatorname{Re}\lambda_n\tau}\chi_{\{s-D(s)\geq 0\}}(\tau)\|u(\tau-D(\tau))\|\,\mathrm{d}\tau\end{aligned}$$

for $t \geq D_0-\delta$. To do so, we note that

$$\begin{aligned}&\int_0^t e^{-\operatorname{Re}\lambda_n\tau}\chi_{\{s-D(s)\geq 0\}}(\tau) \sup_{s\in[0,\tau-D(\tau)]} e^{-\kappa((\tau-D(\tau))-s)}\|d_i(s)\|\,\mathrm{d}\tau \\ &\leq \frac{e^{\kappa(D_0+\delta)}}{|\operatorname{Re}\lambda_n+\kappa|}e^{-(\operatorname{Re}\lambda_n+\kappa)t} \sup_{s\in[0,t-(D_0-\delta)]} e^{\kappa s}\|d_i(s)\|.\end{aligned}$$

Therefore, recalling that $0<\kappa<\sigma$, the use of (14) provided by Lemma 4 yields

$$\mathcal{I}_n(t) \leq \frac{e^{\kappa(D_0+\delta)}}{|\operatorname{Re}\lambda_n+\kappa|}e^{-\operatorname{Re}\lambda_n t}\Delta(t)$$

with

$$\begin{aligned}\Delta(t) = {}&\overline{C}_4 e^{-\kappa t}\|X_0\|_{\mathcal{H}} + \overline{C}_5 \sup_{s\in[0,t-(D_0-\delta)]} e^{-\kappa(t-s)}\|d_1(s)\| \\ &+ \overline{C}_6 \sup_{s\in[0,t-(D_0-\delta)]} e^{-\kappa(t-s)}\|d_2(s)\|.\end{aligned}$$

With $u(t) = \sum_{k=1}^{m} u_k(t)e_k$ where $u_k(t) \in \mathbb{K}$, we have that $|u_k(t)| \leq \|u(t)\|$ for all $1 \leq k \leq m$. Recalling that $\operatorname{Re}\lambda_n \leq -\alpha < -\kappa < 0$ for any $n \geq N_0+1$, we obtain that, for all $n \geq N_0+1$ and $t \geq D_0-\delta$,

$$\begin{aligned}&|\lambda_n|\int_0^t e^{\operatorname{Re}\lambda_n(t-\tau)}|\langle Bu(\tau-D(\tau)),\psi_n\rangle_{\mathcal{H}}|\,\mathrm{d}\tau \\ &\leq |\lambda_n|\sum_{k=1}^{m}|\langle Be_k,\psi_n\rangle_{\mathcal{H}}|e^{\operatorname{Re}\lambda_n t}\mathcal{I}_n(t) \\ &\leq \frac{\alpha\xi e^{\kappa(D_0+\delta)}}{\alpha-\kappa}\sum_{k=1}^{m}|\langle Be_k,\psi_n\rangle_{\mathcal{H}}|\Delta(t)\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\int_0^t e^{\operatorname{Re}\lambda_n(t-\tau)}|\langle \mathcal{A}Bu(\tau-D(\tau)),\psi_n\rangle_{\mathcal{H}}|\,\mathrm{d}\tau \\ &\leq \sum_{k=1}^{m}|\langle \mathcal{A}Be_k,\psi_n\rangle_{\mathcal{H}}|e^{\operatorname{Re}\lambda_n t}\mathcal{I}_n(t) \\ &\leq \frac{e^{\kappa(D_0+\delta)}}{\alpha-\kappa}\sum_{k=1}^{m}|\langle \mathcal{A}Be_k,\psi_n\rangle_{\mathcal{H}}|\Delta(t).\end{aligned}\tag{38}$$

Based on (35-38), we deduce from (34), Young's inequality, estimates (2), and the fact $\|Y(0)\|^2 = \sum_{n=1}^{N_0}|c_n(0)|^2$ that, for all $t \geq 0$,

$$\begin{aligned}\sum_{n\geq N_0+1}|c_n(t)|^2 \leq {}&\tilde{C}_1 e^{-2\kappa t}\|X_0\|^2_{\mathcal{H}} + \tilde{C}_2 \sup_{\tau\in[0,t]} e^{-2\kappa(t-\tau)}\|d_1(\tau)\|^2 \\ &+ \tilde{C}_3 \sup_{\tau\in[0,\max(t-(D_0-\delta),0)]} e^{-2\kappa(t-\tau)}\|d_2(\tau)\|^2,\end{aligned}\tag{39}$$

where constants $\tilde{C}_i$ are given by

$$\begin{aligned}\tilde{C}_0 &= \alpha^2\xi^2\sum_{k=1}^{m}\|Be_k\|^2_{\mathcal{H}} + \sum_{k=1}^{m}\|\mathcal{A}Be_k\|^2_{\mathcal{H}}, \\ \tilde{C}_1 &= \frac{4}{m_R}\left(1+\frac{2m\overline{C}_4^2 e^{2\kappa(D_0+\delta)}}{(\alpha-\kappa)^2}\tilde{C}_0\right), \\ \tilde{C}_2 &= \frac{8m\left(1+\overline{C}_5 e^{\kappa(D_0+\delta)}\right)^2}{m_R(\alpha-\kappa)^2}\tilde{C}_0, \\ \tilde{C}_3 &= \frac{8m\overline{C}_6^2 e^{2\kappa(D_0+\delta)}}{m_R(\alpha-\kappa)^2}\tilde{C}_0.\end{aligned}$$

Consequently, as

$$\begin{aligned}\|X(t)\|_{\mathcal{H}} &\leq \sqrt{M_R\sum_{n\geq 1}|c_n(t)|^2} \\ &\leq \sqrt{M_R}\left(\|Y(t)\| + \sqrt{\sum_{n\geq N_0+1}|c_n(t)|^2}\right),\end{aligned}$$

we obtain from (26) and (39) that the ISS estimate (13) holds with $\overline{C}_i = \sqrt{M_R}\left(C_i+\sqrt{\tilde{C}_i}\right)$, $1 \leq i \leq 3$, which concludes the proof of Theorem 2.

## 5. Extension of the main result to continuous boundary perturbations

The result stated in Theorem 2 deals with mild solutions associated with continuously differentiable boundary disturbances. However, as shown in [24], the satisfaction of an ISS estimate, combined with the introduction of a proper concept of weak solution, can be employed to easily extend the obtained ISS estimate to boundary disturbances exhibiting relaxed regularity assumptions. Such a concept of weak solutions extends to abstract boundary control systems the concept of weak solutions originally introduced for infinite-dimensional nonhomogeneous Cauchy problems in [2] and further investigated in [9, Def. 3.1.6, Thm. 3.1.7, A.5.29] under a variational from. In this context and adopting the approach reported in [24], we introduce the following concept of weak solution for the closed-loop dynamics (11).

**Definition 4.** Let $(\mathcal{A}, \mathcal{B})$ be an abstract boundary control system such that Assumption 1 holds. Let $t_0, D_0 > 0$, $\delta \in (0, D_0)$, a transition signal $\varphi \in C^1(\mathbb{R};\mathbb{R})$ over $[0, t_0]$, and $K \in \mathbb{K}^{m \times N_0}$ be arbitrary. For a time-varying delay $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, an initial condition $X_0 \in \mathcal{H}$, and boundary perturbations $d_1, d_2 \in C^0(\mathbb{R}_+;\mathbb{K}^m)$, we say that $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ is a weak solution of (11) associated with $(D, X_0, d_1, d_2)$ if for any $T > 0$ and any $z \in C^0([0,T]; D(\mathcal{A}_0^*)) \cap C^1([0,T];\mathcal{H})$ with[6] $\mathcal{A}_0^* z \in C^0([0,T];\mathcal{H})$ and $z(T) = 0$, we have:

$$\begin{aligned}
&\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= -\langle X_0, z(0) \rangle_{\mathcal{H}} \qquad (41)\\
&\quad + \int_0^T \left\langle B(u(t - D(t)) + d_1(t)), \mathcal{A}_0^* z(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&\quad - \int_0^T \langle \mathcal{A} B(u(t - D(t)) + d_1(t)), z(t) \rangle_{\mathcal{H}} \mathrm{d}t,
\end{aligned}$$

with $B$ an arbitrarily given lifting operator associated with $(\mathcal{A}, \mathcal{B})$ and where the control input $u$ satisfies $u|_{[-D_0-\delta,0]} = 0$ and, for all $t \geq 0$,

$$\begin{aligned}
u(t) = \varphi(t) \Bigg\{ & KY(t) + d_2(t) \qquad (42)\\
& + K \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s \Bigg\}
\end{aligned}$$

with $Y$ defined by (10).

In particular, using the definition of the mild solutions (6) into the left hand side of (41), it is easy to show (see Appendix B for details) that any mild solution is also a weak solution.

**Remark 7.** Following [24, Sec. 4], we have the following facts.

- Definition 4 is independent of a specifically selected lifting operator in the sense that the right hand side of (41) is unchanged when switching between different lifting operators $B$ associated with $(\mathcal{A}, \mathcal{B})$.
- $X_0$ is the initial condition of the weak solution in the sense that if $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ is a weak solution of (11) associated with $(D, X_0, d_1, d_2)$, then we have $X(0) = X_0$.

We first state a preliminary result about the uniqueness of the weak solutions for the studied problem.

**Lemma 5.** *For any $D \in C^1(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in C^0(\mathbb{R}_+;\mathbb{K}^m)$, there exists at most one weak solution $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ of the closed-loop system (11) associated with $(D, X_0, d_1, d_2)$.*

PROOF. By linearity, it is sufficient to show that if $(X, u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^0(\mathbb{R}_+;\mathbb{K}^m)$ satisfies

$$\begin{aligned}
&\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= \int_0^T \left\langle Bu(t - D(t)), \mathcal{A}_0^* z(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&\quad - \int_0^T \langle \mathcal{A} Bu(t - D(t)), z(t) \rangle_{\mathcal{H}} \mathrm{d}t
\end{aligned}$$

for all $T > 0$ and for all test function $z$ over $[0, T]$ with $u|_{[-D_0-\delta,0]} = 0$ and

$$u(t) = \varphi(t) K \left\{ Y(t) + \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s \right\}$$

for all $t \geq 0$, then $X = 0$ and $u = 0$. We proceed by induction by showing that $X|_{[0,n(D_0-\delta)]=0}$ and $u|_{[-D_0-\delta,(n-1)(D_0-\delta)]=0}$ for all $n \geq 1$.

*Initialization:* From $u|_{[-D_0-\delta,0]} = 0$, we obtain for $T = D_0 - \delta$ that:

$$\int_0^{D_0-\delta} \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t = 0$$

for any test function $z$ over $[0, D_0 - \delta]$ because $t - D(t) \leq 0$ and thus $u(t - D(t)) = 0$ for all $t \leq D_0 - \delta$. Using the test function $z(t) = e^{-\overline{\lambda}_k t} \int_{D_0-\delta}^{t} \langle X(\tau), \psi_k \rangle_{\mathcal{H}} e^{\overline{\lambda}_k \tau}\,\mathrm{d}\tau\, \psi_k$, we obtain that $\mathcal{A}_0^* z(t) + \dfrac{\mathrm{d}z}{\mathrm{d}t}(t) = \langle X(t), \psi_k \rangle_{\mathcal{H}} \psi_k$ and thus $\int_0^{D_0-\delta} |\langle X(t), \psi_k \rangle_{\mathcal{H}}|^2\,\mathrm{d}t = 0$. By continuity of $X$, we infer that $\langle X(t), \psi_k \rangle_{\mathcal{H}} = 0$ for all $t \in [0, D_0 - \delta]$ and all $k \geq 1$. Then (3) yields $X(t) = 0$ for all $t \in [0, D_0 - \delta]$.

*Heredity:* Let $n \geq 1$ be such that $X|_{[0,n(D_0-\delta)]} = 0$ and $u|_{[-D_0-\delta,(n-1)(D_0-\delta)]=0}$. Then we have $Y|_{[0,n(D_0-\delta)]} = 0$ and thus

$$u(t) = \varphi(t) K \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A_{N_0}} B_{N_0} u(s)\,\mathrm{d}s$$

for all $t \in [0, n(D_0 - \delta)]$. Hence, using Grönwall's inequality (see also [3] and [19, Sec. III.C]), $u(t) = 0$ for all $t \in [0, n(D_0 - \delta)]$. Thus, $u(t - D(t)) = 0$ for all $t \in [0, (n+1)(D_0 - \delta)]$ and we obtain with $T = (n+1)(D_0 - \delta)$ that

$$\int_0^{(n+1)(D_0-\delta)} \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t = 0$$

for all test function $z$ over $[0, (n+1)(D_0 - \delta)]$. Using the test function $z(t) = e^{-\overline{\lambda}_k t} \int_{(n+1)(D_0-\delta)}^{t} \langle X(\tau), \psi_k \rangle_{\mathcal{H}} e^{\overline{\lambda}_k \tau}\,\mathrm{d}\tau\, \psi_k$, we obtain that $\mathcal{A}_0^* z(t) + \dfrac{\mathrm{d}z}{\mathrm{d}t}(t) = \langle X(t), \psi_k \rangle_{\mathcal{H}} \psi_k$ and thus we infer that $\int_0^{(n+1)(D_0-\delta)} |\langle X(t), \psi_k \rangle_{\mathcal{H}}|^2\,\mathrm{d}t = 0$. By continuity

---

[6] Such a function $z$ is called a test function over $[0, T]$.

of $X$, we infer that $\langle X(t), \psi_k \rangle_{\mathcal{H}} = 0$ for all $t \in [0, (n+1)(D_0 - \delta)]$ and all $k \geq 1$. We deduce from (3) that $X(t) = 0$ for all $t \in [0, (n+1)(D_0 - \delta)]$. This completes the proof by induction.

We can now state the main result of this section whose proof is an adapation of [24, Thm. 3].

**Theorem 6.** *In the context of both assumptions and conclusions of Theorem 2, for any $D \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$, there exists a unique weak solution $(X, u) \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H}) \times \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ associated with $(D, X_0, d_1, d_2)$ of the closed-loop system (11). Furthermore, this weak solution satisfies the ISS estimates (13-14) for all $t \geq 0$.*

PROOF. We consider $\delta \in (0, D_0)$, $\kappa \in (0, \alpha)$, and the constants $\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4, \overline{C}_5, \overline{C}_6 > 0$ as provided by Theorem 2. Let $D \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R})$ with $|D - D_0| \leq \delta$, $X_0 \in \mathcal{H}$, and $d_1, d_2 \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ be given. The uniqueness follows from Lemma 5. We prove the existence.

For a given $T > 0$, as $\mathcal{C}^1([0, T]; \mathbb{K}^m)$ is a dense subset of $\mathcal{C}^0([0, T]; \mathbb{K}^m)$, we introduce $d_{1,n}, d_{2,n} \in \mathcal{C}^1([0, T]; \mathbb{K}^m)$ such that $(d_{1,n})_n$ and $(d_{2,n})_n$ converge uniformly over $[0, T]$ to $d_1|_{[0,T]}$ and $d_2|_{[0,T]}$, respectively. We introduce $(X_n, u_n) \in \mathcal{C}^0([0, T]; \mathcal{H}) \times \mathcal{C}^1([-D_0 - \delta, T]; \mathbb{K}^m)$ the unique mild solution of the closed-loop system (11) over $[0, T]$ associated with $(D, X_0, d_{1,n}, d_{2,n})$. By linearity of (11), $(X_n - X_m, u_n - u_m)$ is the unique mild solution of the closed-loop system (11) over $[0, T]$ associated with $(D, 0, d_{1,n} - d_{1,m}, d_{2,n} - d_{2,m})$. Thus, we deduce from (13-14) that both $(X_n)_n$ and $(u_n)_n$ are Cauchy sequences of the Banach spaces $\mathcal{C}^0([0, T]; \mathcal{H})$ and $\mathcal{C}^0([0, T]; \mathbb{K}^m)$, respectively. Thus $X_n \underset{n \rightarrow +\infty}{\longrightarrow} X \in \mathcal{C}^0([0, T]; \mathcal{H})$ and $u_n \underset{n \rightarrow +\infty}{\longrightarrow} u \in \mathcal{C}^0([0, T]; \mathbb{K}^m)$ and $(X, u)$ satisfies the estimates (13-14) for any $t \in [0, T]$. It is easy to see from (13-14) that the obtained $X$ and $u$ are independent of the selected approximating sequences $(d_{1,n})_n$ and $(d_{2,n})_n$ but only depend on $D$, $X_0$, $d_1$, and $d_2$.

For any given $0 < T_1 < T_2$, let $(X_1, u_1) \in \mathcal{C}^0([0, T_1]; \mathcal{H}) \times \mathcal{C}^0([0, T_1]; \mathbb{K}^m)$ and $(X_2, u_2) \in \mathcal{C}^0([0, T_2]; \mathcal{H}) \times \mathcal{C}^0([0, T_2]; \mathbb{K}^m)$ be the result of the above construction over the time intervals $[0, T_1]$ and $[0, T_2]$, respectively. By restricting the approximating sequences of $d_1|_{[0,T_2]}$ and $d_2|_{[0,T_2]}$ to the interval $[0, T_1]$, we obtain approximating sequences of $d_1|_{[0,T_1]}$ and $d_2|_{[0,T_1]}$. Then, we infer that $X_2|_{[0,T_1]} = X_1$ and $u_2|_{[0,T_1]} = u_1$. Consequently, we can define $(X, u) \in \mathcal{C}^0(\mathbb{R}_+; \mathcal{H}) \times \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$ such that $(X|_{[0,T]}, u|_{[0,T]})$ is the result of the above construction for any $T > 0$. Then, $(X, u)$ satisfy the estimates (13-14) for all $t \geq 0$.

It remains to show that $(X, u)$ is actually the weak solution associated with $(D, X_0, d_1, d_2)$ of the closed-loop system (11). Let $T > 0$ be arbitrarily given. Let $(d_{1,n})_n \in \mathcal{C}^1([0, T]; \mathbb{K}^m)^{\mathbb{N}}$ and $(d_{2,n})_n \in \mathcal{C}^1([0, T]; \mathbb{K}^m)^{\mathbb{N}}$ be approximating sequences converging to $d_1|_{[0,T]}$ and $d_2|_{[0,T]}$, respectively. Thus, the corresponding sequence of mild solutions $(X_n, u_n)_n$ converges uniformly to $(X|_{[0,T]}, u|_{[0,T]})$. As mild solutions are weak solutions, we obtain that, for any test function $z$ over $[0, T]$ and any $n \geq 1$,

$$
\begin{aligned}
& \int_0^T \left\langle X_n(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= - \langle X_0, z(0) \rangle_{\mathcal{H}} \\
&\quad + \int_0^T \left\langle B(u_n(t - D(t)) + d_{1,n}(t)), \mathcal{A}_0^* z(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&\quad - \int_0^T \left\langle \mathcal{A}B(u_n(t - D(t)) + d_{1,n}(t)), z(t) \right\rangle_{\mathcal{H}} \mathrm{d}t
\end{aligned}
$$

with $u_n|_{[-D_0-\delta,0]} = 0$ and

$$
\begin{aligned}
u_n(t) = \varphi(t) \Bigg\{ & K Y_n(t) + d_{2,n}(t) \\
& + K \int_{\max(t-D_0,0)}^t e^{(t-s-D_0)A_{N_0}} B_{N_0} u_n(s) \,\mathrm{d}s \Bigg\}
\end{aligned}
$$

for all $t \in [0, T]$, where

$$
Y_n(t) = \begin{bmatrix} \langle X_n(t), \psi_1 \rangle_{\mathcal{H}} & \ldots & \left\langle X_n(t), \psi_{N_0} \right\rangle_{\mathcal{H}} \end{bmatrix}^\top .
$$

Recalling that $\mathcal{A}B$ and $B$ are bounded, we obtain that (41-42) hold for all $t \in [0, T]$ by letting $n \rightarrow +\infty$. As both $T > 0$ and the test function $z$ over $[0, T]$ have been arbitrarily selected, this concludes the proof.

## 6. Conclusion

This paper has investigated the input-to-state stabilization with respect to boundary disturbances of a class of diagonal infinite-dimensional systems via delay boundary control. First, a preliminary lemma regarding the robustness of a constant-delay predictor feedback with respect to uncertain and time-varying input delays has been derived under the form of an ISS estimate with fading memory of the disturbance input. This result was applied to a truncated model capturing the unstable modes of the studied infinite-dimensional system. Finally, this ISS property was extended to the closed-loop infinite-dimensional system, first considering mild solutions and then for weak solutions associated with disturbances exhibiting relaxed regularity assumptions.

## A. Proof of Lemma 1

As $u|_{[-D_0-\delta,0]} = 0$, (6) is equivalent over $[0, D_0 - \delta]$ to

$$
\begin{aligned}
X(t) = {} & S(t)\{X_0 - Bd_1(0)\} + Bd_1(t) \\
& + \int_0^t S(t-s)\{\mathcal{A}Bd_1(s) - B\dot{d}_1(s)\} \,\mathrm{d}s,
\end{aligned}
$$

which is well and uniquely defined as an element of $\mathcal{C}^0([0, D_0 - \delta]; \mathcal{H})$ with associated control input $u = 0 \in \mathcal{C}^1([-D_0 - \delta, 0]; \mathbb{K}^m)$.

We proceed by induction. Assume that, for a given $n \in \mathbb{N}^*$, there exists a unique pair $(X, u) \in \mathcal{C}^0([0, n(D_0 - \delta)]; \mathcal{H}) \times$

$C^1([-D_0-\delta,(n-1)(D_0-\delta)])$ such that (6) holds over $[0,n(D_0-\delta)]$ and (11c) holds over $[-D_0-\delta,(n-1)(D_0-\delta)]$. In particular, reproducing the developments reported in Subsection 2.2, $c_n$ is of class $C^1$ over $[0,n(D_0-\delta)]$, and thus $Y \in C^1([0,n(D_0-\delta)],\mathbb{K}^{N_0})$. We show that there exists a unique couple $(\tilde X,\tilde u) \in C^0([0,(n+1)(D_0-\delta)];\mathcal{H}) \times C^1([-D_0-\delta,n(D_0-\delta)];\mathbb{K}^m)$ such that

$$\tilde X(t) = S(t)\{X_0 - B\tilde v(0)\} + B\tilde v(t) + \int_0^t S(t-s)\{\mathcal{A}B\tilde v(s) - B\dot{\tilde v}(s)\}\,\mathrm{d}s \tag{43}$$

with $\tilde v(t) = \tilde u(t-D(t)) + d_1(t)$ for all $t \in [0,(n+1)(D_0-\delta)]$ and where the control law is characterized by $\tilde u|_{[-D_0-\delta,0]} = 0$ and, for all $t \in [0,n(D_0-\delta)]$,

$$\tilde u(t) = \varphi(t)\left\{ K\tilde Y(t) + d_2(t) + K\int_{\max(t-D_0,0)}^t e^{(t-s-D_0)A_{N_0}} B_{N_0}\tilde u(s)\,\mathrm{d}s \right\} \tag{44}$$

with

$$\tilde Y(t) = \left[ \left\langle \tilde X(t),\psi_1 \right\rangle_{\mathcal{H}} \quad \dots \quad \left\langle \tilde X(t),\psi_{N_0} \right\rangle_{\mathcal{H}} \right]^\top .$$

The induction assumption shows that $\tilde X\big|_{[0,n(D_0-\delta)]} = X$ and $\tilde u|_{[-D_0-\delta,(n-1)(D_0-\delta)]} = u$. In particular, we have $\tilde Y(t) = Y(t)$ for all $t \leq 0 \leq n(D_0-\delta)$. As $t - D(t) \leq n(D_0-\delta)$ for $t \leq (n+1)(D_0-\delta)$, we note that the control input $\tilde u$ over the time interval $[0,n(D_0-\delta)]$ is only defined by $X$ over the time interval $[0,n(D_0-\delta)]$ and does not depend on $\tilde X$ over $[n(D_0-\delta),(n+1)(D_0-\delta)]$. As $Y \in C^0([0,n(D_0-\delta)];\mathbb{K}^{N_0})$, we obtain from [19, Sec. III.C] (which is a direct extension of the result reported in [3], in the configuration $\varphi = 1$, to the case of a continuous function $\varphi$ satisfying $0 \leq \varphi \leq 1$) that the control $\tilde u$ given by the implicit equation (44) is well and uniquely defined on $[-D_0-\delta,n(D_0-\delta)]$ as an element of $C^0([-D_0-\delta,n(D_0-\delta)];\mathbb{K}^m)$ and is such that $\tilde u|_{[-D_0-\delta,(n-1)(D_0-\delta)]} = u$. Introducing

$$Z(t) = Y(t) + \int_{t-D_0}^t e^{(t-D_0-s)A_{N_0}} B_{N_0}\tilde u(s)\,\mathrm{d}s,$$

which is such that $Z \in C^1([0,n(D_0-\delta)];\mathbb{K}^{N_0})$, we can write $\tilde u(t) = \varphi(t)KZ(t) + \varphi(t)d_2(t)$ for all $t \in [0,n(D_0-\delta)]$. Thus $\tilde u \in C^1([-D_0-\delta,n(D_0-\delta)])$ and we obtain that (43) defines a unique $\tilde X \in C^0([0,(n+1)(D_0-\delta)];\mathcal{H})$. As the obtained $\tilde X$ and $\tilde u$ are extensions of $X$ and $u$, respectively, this completes the proof by induction.

## B. Mild solutions are weak solutions

Let $(X,u) \in C^0(\mathbb{R}_+;\mathcal{H}) \times C^1([-D_0-\delta,+\infty);\mathbb{K}^m)$ be a mild solution of (11). Then $X$ is given by (6) with $v$ the continuously differentiable function defined by (4). For a given $T > 0$, let $z$ be a test function over $[0,T]$, i.e., $z \in C^0([0,T];D(\mathcal{A}_0^*)) \cap C^1([0,T];\mathcal{H})$ with $\mathcal{A}_0^* z \in C^0([0,T];\mathcal{H})$ and $z(T) = 0$. We need to show that the system trajectory $X$ satisfies the identity (41). From the basic properties of $C_0$-semigroups and the fundamental theorem of calculus, we infer that

$$\begin{aligned}
&\int_0^T \left\langle S(t)\{X_0 - Bv(0)\}, \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= \int_0^T \left\langle X_0 - Bv(0), S^*(t)\left\{ \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\} \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= \int_0^T \frac{\mathrm{d}}{dt}\left( \langle X_0 - Bv(0), S^*(t)z(t) \rangle_{\mathcal{H}} \right) \mathrm{d}t \\
&= -\langle X_0 - Bv(0), z(0) \rangle_{\mathcal{H}} .
\end{aligned}$$

Using in addition the properties of the Bochner integral and Fubini theorem, we obtain that

$$\begin{aligned}
&\int_0^T \left\langle \int_0^t S(t-s)\mathcal{A}Bv(s)\,\mathrm{d}s, \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= \int_0^T \int_0^t \left\langle \mathcal{A}Bv(s), S^*(t-s)\left\{ \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\} \right\rangle_{\mathcal{H}} \mathrm{d}s\,\mathrm{d}t \\
&= \int_0^T \int_s^T \frac{\mathrm{d}}{dt}\left( \langle \mathcal{A}Bv(s), S^*(t-s)z(t) \rangle_{\mathcal{H}} \right) \mathrm{d}t\,\mathrm{d}s \\
&= -\int_0^T \langle \mathcal{A}Bv(s), z(s) \rangle_{\mathcal{H}}\,\mathrm{d}s.
\end{aligned}$$

Finally, the same approach yields

$$\begin{aligned}
&-\int_0^T \left\langle \int_0^t S(t-s)B\dot v(s)\,\mathrm{d}s, \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t \\
&= \int_0^T \langle B\dot v(s), z(s) \rangle_{\mathcal{H}}\,\mathrm{d}s \\
&= -\langle Bv(0), z(0) \rangle_{\mathcal{H}} - \int_0^T \left\langle Bv(s), \frac{\mathrm{d}z}{\mathrm{d}t}(s) \right\rangle_{\mathcal{H}} \mathrm{d}s,
\end{aligned}$$

where, recalling that $B$ is bounded, the last equality has been derived via an integration by parts. Now, the substitution of the definition of mild solutions (6) into the integral term $\int_0^T \left\langle X(t), \mathcal{A}_0^* z(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right\rangle_{\mathcal{H}} \mathrm{d}t$ and the use of the three latter identities show that the identity (41) is indeed satisfied.

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