



Title	ISS of a Clamped-Free Damped String for the Configurations Associated with the Loss of the Riesz-Spectral Properties
Authors(s)	Lhachemi, Hugo, Shorten, Robert
Publication date	2019-06-28
Publication information	Lhachemi, Hugo, and Robert Shorten. "ISS of a Clamped-Free Damped String for the Configurations Associated with the Loss of the Riesz-Spectral Properties." IEEE, June 28, 2019. https://doi.org/10.23919/ECC.2019.8795806 .
Conference details	The 2019 18th European Control Conference (ECC), Naples, Italy, 25-28 June 2019
Publisher	IEEE
Item record/more information	http://hdl.handle.net/10197/12741
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Publisher's version (DOI)	10.23919/ECC.2019.8795806

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ISS of a Clamped-Free Damped String for the Configurations Associated with the Loss of the Riesz-Spectral Properties

Hugo Lhachemi¹ and Robert Shorten¹

Abstract—This paper deals with the Input-to-State Stability (ISS) of a clamped-free damped string with respect to boundary disturbances for the configurations associated with the loss of the Riesz-spectral properties. Specifically, for most of the values of the physical parameters (namely the stiffness parameter and the damping coefficient), the ISS property of the clamped free damped string can be established based on the fact that the underlying disturbance-free operator is a Riesz-spectral operator. However, such a Riesz-spectral property does not hold true for certain configurations of the physical parameters of the damped string. This paper specifically investigates the establishment of an ISS estimate for these configurations. The proposed strategy relies on the projection of the original system trajectories in a Riesz basis obtained by adequately completing the set of eigenvectors of the disturbance-free operator.

I. INTRODUCTION

The study of the Input-to-State Stability (ISS) properties [1] of infinite dimensional systems is an active research topic [2]–[4]. For infinite dimensional systems such as described by PDEs, there exist two types of perturbations: the distributed (or in-domain) disturbances and the boundary disturbances. The second type of perturbation occurs in numerous practical applications including, e.g., robotics [5], [6], aerospace engineering [7], [8], and additive manufacturing [9].

While many results have been reported regarding the ISS properties with respect to distributed disturbances [10]–[16], the establishment of ISS estimates with respect to boundary disturbances remains challenging [17]–[21]. One of the most popular approaches relies on the search of an adequate Lyapunov function [17], [22]–[24]. Alternative approaches relying on functional analysis tools were investigated in [18], [25]. The ISS property for disturbances evaluated in the uniform norm is obtained in [25] for a class of analytic semigroups by embedding the problem into the extrapolation space while invoking admissible conditions for returning to the original state-space. A different approach that avoids the incursion into the extrapolation space was developed in [18] for the analysis of 1-D parabolic equations. Specifically, the ISS estimate was obtained by projecting the system trajectories onto a Hilbert basis formed by the eigenvectors of the underlying disturbance-free operator. By doing so, it was shown that the ISS estimate can be derived from the

study of a countably infinite number of ordinary differential equations (ODEs).

The latter approach was further investigated in [26], [27] to establish the ISS property of a clamped-free damped string [28]–[30] in the presence of both distributed and boundary disturbances evaluated in either the uniform or the L^2 norm. It was shown that for most of the numerical values of the string parameters, namely the stiffness parameter $\alpha > 0$ and the damping coefficient $\beta > 0$, the underlying disturbance-free operator is a Riesz-spectral operator. More precisely, the Riesz-spectral properties hold true if and only if the quantity defined by $\zeta(\alpha, \beta) \triangleq \frac{2\sqrt{\alpha}}{\pi\beta} - \frac{1}{2}$ is such that $\zeta(\alpha, \beta) \notin \mathbb{N}$. In this case, the ISS estimate was obtained by projecting the system trajectory onto a Riesz basis [31] formed by the eigenvectors of the disturbance-free operator. The study of the boundary-to-displacement asymptotic gains for a similar damped wave equation but with different boundary conditions was also reported in [32].

The objective of this paper is to investigate the ISS property of the clamped-free damped string (with perturbations evaluated in both uniform and L^2 norms) for the configurations associated with the loss of the above Riesz-spectral properties, i.e., for a stiffness parameters α and a damping coefficient β such that $\zeta(\alpha, \beta) \in \mathbb{N}$. In this case, the eigenvectors of the disturbance-free operator cannot be selected to form a Riesz basis. Nevertheless, it is shown that such a collection of eigenvectors can be adequately extended to form a Riesz basis by adding one appropriate vector of the state-space. The obtained Riesz basis is then used to project the system trajectories. By doing so, the study of the original system reduces to the analysis of a countably infinite number of ODEs describing the time evolution of the coefficients of the system trajectory when projected onto the considered Riesz basis. Contrary to the case $\zeta(\alpha, \beta) \notin \mathbb{N}$ investigated in [26] where the ODEs are totally uncoupled, the case $\zeta(\alpha, \beta) \in \mathbb{N}$ yields a coupling in the ODE associated with the additional vector introduced to form the Riesz basis. However, it is shown that such a coupling can be limited by selecting adequately the corresponding vector. Then, the desired ISS estimate is derived based on the study of the obtained ODEs.

The remainder of this paper is organized as follows. Notations are introduced in Section II. The model of the clamped-free damped string is presented in Section III. After studying the properties of the underlying disturbance-free operator in Section IV-A, the ISS estimate is established in Section V. Concluding remarks are given in Section VI.

This publication has emanated from research supported in part by a research grant from Science Foundation Ireland (SFI) under grant number 16/RC/3872 and is co-funded under the European Regional Development Fund and by I-Form industry partners.

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II. NOTATIONS

The sets of non-negative integers, real, non-negative real, positive real, and complex numbers are denoted by \mathbb{N} , \mathbb{R} , \mathbb{R}_+ , \mathbb{R}_+^* , and \mathbb{C} , respectively. For any $z \in \mathbb{C}$, $\text{Re } z$ denotes the real part of z . For any integer $k \in \mathbb{N}$, we define $\tilde{k} \triangleq k + 1/2$.

For an interval $I \subset \mathbb{R}$ and a normed space $(E, \|\cdot\|_E)$, $\mathcal{C}^n(I; E)$ denotes the set of functions $f : I \rightarrow E$ that are n times continuously differentiable. For any $a < b$, $\mathcal{C}^0([a, b]; E)$ is endowed with the usual norm $\|\cdot\|_{\mathcal{C}^0([a, b]; E)}$ defined for any $f \in \mathcal{C}^0([a, b]; E)$ by

$$\|f\|_{\mathcal{C}^0([a, b]; E)} = \sup_{t \in [a, b]} \|f(t)\|_E.$$

The Hilbert space of square-integrable functions (with respect to the Lebesgue measure) over an interval $(a, b) \subset \mathbb{R}$ is denoted by $L^2(a, b)$ with the associated inner product $\langle f, g \rangle_{L^2(a, b)} = \int_a^b f(\xi) \overline{g(\xi)} d\xi$. Denoting by f' , when it exists, the weak derivative of $f \in L^2(a, b)$, we introduce the Sobolev space $H^1(a, b) \triangleq \{f \in L^2(a, b) : f' \in L^2(a, b)\}$. Finally, $H_L^1(a, b) \triangleq \{f \in H^1(a, b) : f(a) = 0\}$ is a Hilbert space when endowed with the inner product $\langle f, g \rangle_{H_L^1(a, b)} \triangleq \langle f', g' \rangle_{L^2(a, b)}$.

For a given linear operator L , $\text{R}(L)$ and $\ker(L)$ denote its range and its kernel, respectively. The time derivative of a complex-valued differentiable function $f : I \rightarrow \mathbb{C}$ is denoted by \dot{f} . For a given Hilbert space \mathcal{H} , the time derivative of a \mathcal{H} -valued differentiable function $f : I \rightarrow \mathcal{H}$ is denoted by df/dt .

III. PROBLEM DESCRIPTION

A. Model of the clamped-free damped string

We consider the dynamics of a string with Kelvin-Voigt damping [28]–[30] and clamped-free boundary conditions described by:

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x} \left(\alpha \frac{\partial y}{\partial x} + \beta \frac{\partial^2 y}{\partial t \partial x} \right) = u, \quad \text{in } \mathbb{R}_+ \times (0, 1) \quad (1a)$$

$$y(t, 0) = 0, \quad t \in \mathbb{R}_+ \quad (1b)$$

$$\left(\alpha \frac{\partial y}{\partial x} + \beta \frac{\partial^2 y}{\partial t \partial x} \right) (t, 1) = d(t), \quad t \in \mathbb{R}_+ \quad (1c)$$

$$y(0, x) = y_0(x), \quad x \in (0, 1) \quad (1d)$$

$$\frac{\partial y}{\partial t}(0, x) = y_{t0}(x), \quad x \in (0, 1) \quad (1e)$$

where $\alpha, \beta \in \mathbb{R}_+^*$ are constant parameters. Functions $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0, 1))$ and $d \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{C})$ represent distributed and boundary disturbances, respectively. Functions $y_0 \in H_L^1(0, 1)$ and $y_{t0} \in L^2(0, 1)$ are the initial conditions.

To facilitate the upcoming developments, we introduce the following quantity:

$$\zeta(\alpha, \beta) \triangleq \frac{2\sqrt{\alpha}}{\pi\beta} - \frac{1}{2}.$$

Assuming that the coefficients $\alpha, \beta \in \mathbb{R}_+^*$ are such that $\zeta(\alpha, \beta) \notin \mathbb{N}$, the ISS property of the clamped-free damped string described by (1a-1e), for disturbances evaluated in

both uniform and L^2 norms, has been established in [26]. The objective of this paper is to study the configuration associated with parameters α and β such that $\zeta(\alpha, \beta) \in \mathbb{N}$.

B. Abstract form and well-posedness

The state-space associated with the studied evolution problem (1a-1e) is the following Hilbert space:

$$\mathcal{H} = H_L^1(0, 1) \times L^2(0, 1),$$

endowed with the inner product defined for all $(x_1, x_2), (\hat{x}_1, \hat{x}_2) \in \mathcal{H}$ by

$$\langle (x_1, x_2), (\hat{x}_1, \hat{x}_2) \rangle_{\mathcal{H}} = \int_0^1 \alpha x_1'(\xi) \overline{\hat{x}_1'(\xi)} + x_2(\xi) \overline{\hat{x}_2(\xi)} d\xi.$$

Then, (1a-1e) can be written under the following abstract form:

$$\begin{cases} \frac{dX}{dt}(t) = \mathcal{A}X(t) + U(t) & , t \geq 0 \\ \mathcal{B}X(t) = d(t) & , t \geq 0 \\ X(0) = X_0 \end{cases} \quad (2)$$

with the state $X(t) = (y(t, \cdot), y_t(t, \cdot)) \in \mathcal{H}$, the distributed disturbance $U = (0, u) \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$, and the initial condition $X_0 = (y_0, y_{t0})$. The operator $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$ is defined by

$$\mathcal{A}(x_1, x_2) = (x_2, (\alpha x_1' + \beta x_2')')$$

over the domain

$$D(\mathcal{A}) = \{(x_1, x_2) \in \mathcal{H} : x_2 \in H_L^1(0, 1), (\alpha x_1' + \beta x_2') \in H^1(0, 1)\},$$

while the boundary operator $\mathcal{B} : D(\mathcal{A}) \rightarrow \mathbb{C}$ is given by

$$\mathcal{B}(x_1, x_2) = (\alpha x_1' + \beta x_2')(1).$$

It is shown in [26] that (2) is a boundary control systems [33, Def. 3.3.2], yielding the existence and the uniqueness of the classical solutions $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ associated with $X_0 \in D(\mathcal{A})$, $d \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{C})$ such that $\mathcal{B}X_0 = d(0)$, and $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0, 1))$. In particular, the disturbance-free operator $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$ with $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ is the generator of a C_0 -semigroup T .

C. Main result

The objective of this paper is to demonstrate that the following theorem holds true when the parameters $\alpha, \beta \in \mathbb{R}_+^*$ are such that $\zeta(\alpha, \beta) \in \mathbb{N}$ (the case $\zeta(\alpha, \beta) \notin \mathbb{N}$ follows from [26]).

Theorem 3.1: For any initial condition $X_0 \in D(\mathcal{A})$ and any disturbances $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0, 1))$ and $d \in \mathcal{C}^2(\mathbb{R}_+)$ such that $\mathcal{B}X_0 = d(0)$, the abstract system (2) has a unique classic solution $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$. Furthermore, there exist constants $C_0, C_1, C_2, C_3, C_4 \in \mathbb{R}_+^*$, independent of X_0 , u , and d , such that for all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}} \leq C_0 e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}} + C_1 \|d\|_{\mathcal{C}^0([0, t])} + C_2 \|u\|_{\mathcal{C}^0([0, t]; L^2(0, 1))}, \quad (3)$$

and

$$\|X(t)\|_{\mathcal{H}} \leq C_0 e^{-\kappa_0 t} \|X_0\|_{\mathcal{H}} + C_3 \|d\|_{L^2(0,t)} + C_4 \|u\|_{L^2((0,t) \times (0,1))}, \quad (4)$$

where, denoting by $\lceil \cdot \rceil$ the ceiling function,

$$\kappa_0 = \begin{cases} \min\left(\frac{\beta\pi^2}{8}, \frac{\alpha}{\beta}\right) & \text{if } \lceil \zeta(\alpha, \beta) \rceil \geq 1; \\ \frac{\alpha}{\beta} & \text{if } \lceil \zeta(\alpha, \beta) \rceil = 0, \end{cases} \quad (5)$$

and is such that $\omega_0 = -\kappa_0 < 0$ is the growth bound of T .

IV. PROPERTIES OF \mathcal{A}_0

This section is devoted to the study of the spectral properties of both the disturbance-free operator \mathcal{A}_0 and its adjoint \mathcal{A}_0^* when constants α and β are such that $\zeta(\alpha, \beta) \in \mathbb{N}$. In the case $\zeta(\alpha, \beta) \notin \mathbb{N}$, it was shown in [26] that \mathcal{A}_0 is a Riesz-spectral operator. We show here that such a result does not hold true in the case $\zeta(\alpha, \beta) \in \mathbb{N}$. Throughout the remainder of the paper, we assume that $\zeta(\alpha, \beta) \in \mathbb{N}$ and we define the integer k_0 as follows:

$$k_0 \triangleq \zeta(\alpha, \beta) = \frac{2\sqrt{\alpha}}{\pi\beta} - \frac{1}{2} \in \mathbb{N}.$$

A. Spectral properties and Riesz basis

Straightforward computations show that the following lemma holds true.

Lemma 4.1: Assume that parameters $\alpha, \beta \in \mathbb{R}_+^*$ are such that $\zeta(\alpha, \beta) \in \mathbb{N}$. Then, the eigenvalues of \mathcal{A}_0 are given by $\sigma_p(\mathcal{A}_0) = \{\lambda_{k,\epsilon}, k \in \mathbb{N} \setminus \{k_0\}, \epsilon \in \{-1, +1\}\} \cup \{\lambda_{k_0}\}$ where

$$\lambda_{k,\epsilon} = \begin{cases} -\frac{\tilde{k}^2 \beta \pi^2}{2} + \epsilon i \frac{\tilde{k} \pi \sqrt{4\alpha - \tilde{k}^2 \beta^2 \pi^2}}{2}, & 0 \leq k \leq k_0 - 1; \\ -\frac{\tilde{k}^2 \beta \pi^2}{2} + \epsilon \frac{\tilde{k} \pi \sqrt{\tilde{k}^2 \beta^2 \pi^2 - 4\alpha}}{2}, & k \geq k_0 + 1, \end{cases} \quad (6)$$

with $\tilde{k} = k + 1/2$, and $\lambda_{k_0} = -2\alpha/\beta$. Furthermore, the associated eigenspaces are given by $\ker(\mathcal{A}_0 - \lambda_{k,\epsilon} I_{\mathcal{H}}) = \text{vect}_{\mathbb{C}}(\phi_{k,\epsilon})$ with

$$\phi_{k,\epsilon} = \frac{|\lambda_{k,\epsilon}|}{\lambda_{k,\epsilon}} \sqrt{\frac{2}{|\lambda_{k,\epsilon}|^2 + \tilde{k}^2 \alpha \pi^2}} \left(\sin(\tilde{k} \pi \cdot), \lambda_{k,\epsilon} \sin(\tilde{k} \pi \cdot) \right), \quad (7)$$

and $\ker(\mathcal{A}_0 - \lambda_{k_0} I_{\mathcal{H}}) = \text{vect}_{\mathbb{C}}(\phi_{k_0})$ with

$$\phi_{k_0} = \frac{1}{\lambda_{k_0}} \left(\sin(\tilde{k}_0 \pi \cdot), \lambda_{k_0} \sin(\tilde{k}_0 \pi \cdot) \right), \quad (8)$$

where $\|\phi_{k,\epsilon}\|_{\mathcal{H}} = 1$ and $\|\phi_{k_0}\|_{\mathcal{H}} = 1$.

Based on the description of the eigenvalues of \mathcal{A}_0 given by Lemma 4.1, the following properties hold true:

$$\forall k \in \mathbb{N} \setminus \{k_0\}, \quad \lambda_{k,+1} \lambda_{k,-1} = \tilde{k}^2 \alpha \pi^2, \quad (9)$$

$$\forall 0 \leq k \leq k_0 - 1, \quad \forall \epsilon \in \{-1, +1\}, \quad \text{Re } \lambda_{k,\epsilon} \leq -\frac{\beta \pi^2}{8}, \quad (10)$$

$$\forall k \geq k_0 + 1, \quad \lambda_{k,-1} \leq \lambda_{k,+1} \leq -\frac{\alpha}{\beta}, \quad (11)$$

$$\lambda_{k,-1} \underset{k \rightarrow +\infty}{\sim} -k^2 \beta \pi^2, \quad (12)$$

and

$$\lambda_{k,+1} \underset{k \rightarrow +\infty}{\longrightarrow} -\alpha/\beta. \quad (13)$$

In particular, we deduce that $\sup_{\lambda \in \sigma_p(\mathcal{A}_0)} \text{Re } \lambda = -\kappa_0$ with $\kappa_0 > 0$ given by (5).

Contrary to the configuration $\zeta(\alpha, \beta) \notin \mathbb{N}$ studied in [26], the family of eigenvectors $\Phi = \{\phi_{k,\epsilon}, k \in \mathbb{N} \setminus \{k_0\}, \epsilon \in \{-1, +1\}\} \cup \{\phi_{k_0}\}$ does not form a Riesz basis [31] of \mathcal{H} . Indeed, introducing the vector

$$\phi_0 = \frac{1}{\lambda_{k_0}} \left(-\sin(\tilde{k}_0 \pi \cdot), \lambda_{k_0} \sin(\tilde{k}_0 \pi \cdot) \right), \quad (14)$$

straightforward computations show that $\|\phi_0\|_{\mathcal{H}} = 1$,

$$\forall k \in \mathbb{N} \setminus \{k_0\}, \quad \forall \epsilon \in \{-1, +1\}, \quad \langle \phi_0, \phi_{k,\epsilon} \rangle_{\mathcal{H}} = 0, \quad (15)$$

and

$$\langle \phi_0, \phi_{k_0} \rangle_{\mathcal{H}} = 0. \quad (16)$$

Thus, we have $\phi_0 \in \overline{\text{span}_{\mathbb{C}}(\Phi)} \setminus \{0\}$, assessing the fact that $\overline{\text{span}_{\mathbb{C}}(\Phi)} \neq \mathcal{H}$. However, introducing $\tilde{\Phi} \triangleq \Phi \cup \{\phi_0\}$, the following result holds true.

Lemma 4.2: $\tilde{\Phi}$ is a Riesz basis, i.e., $\tilde{\Phi}$ is maximal and there exist $m, M \in \mathbb{R}_+^*$ such that for any sequence $(a_{k,\epsilon})_{k \in \mathbb{N} \setminus \{k_0\}, \epsilon \in \{-1, +1\}} \in \mathbb{C}^{\mathbb{N}}$ with finitely many non-zero terms and any $a_{k_0}, a_0 \in \mathbb{C}$,

$$\begin{aligned} & m \left\{ |a_0|^2 + |a_{k_0}|^2 + \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} |a_{k,\epsilon}|^2 \right\} \\ & \leq \left\| a_0 \phi_0 + a_{k_0} \phi_{k_0} + \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} a_{k,\epsilon} \phi_{k,\epsilon} \right\|_{\mathcal{H}}^2 \\ & \leq M \left\{ |a_0|^2 + |a_{k_0}|^2 + \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} |a_{k,\epsilon}|^2 \right\}. \end{aligned} \quad (17)$$

Proof. First, we show that $\tilde{\Phi}$ is maximal. Let $z = (z_1, z_2) \in \mathcal{H}$ such that $\langle \phi, z \rangle_{\mathcal{H}} = 0$ for all $\phi \in \tilde{\Phi}$. Then, for all $k \in \mathbb{N} \setminus \{k_0\}$ and all $\epsilon \in \{-1, +1\}$, $\langle \phi_{k,\epsilon}, z \rangle_{\mathcal{H}} = 0$ gives

$$\alpha \tilde{k} \pi \left\langle \cos(\tilde{k} \pi \cdot), z_1' \right\rangle_{L^2(0,1)} + \lambda_{k,\epsilon} \left\langle \sin(\tilde{k} \pi \cdot), z_2 \right\rangle_{L^2(0,1)} = 0,$$

which yields

$$\begin{bmatrix} \alpha \tilde{k} \pi & \lambda_{k,-1} \\ \alpha \tilde{k} \pi & \lambda_{k,+1} \end{bmatrix} \begin{bmatrix} \left\langle \cos(\tilde{k} \pi \cdot), z_1' \right\rangle_{L^2(0,1)} \\ \left\langle \sin(\tilde{k} \pi \cdot), z_2 \right\rangle_{L^2(0,1)} \end{bmatrix} = 0.$$

As $k \neq k_0$, $\alpha \tilde{k} \pi (\lambda_{k,+1} - \lambda_{k,-1}) \neq 0$, we obtain that for all $k \in \mathbb{N} \setminus \{k_0\}$, $\left\langle \cos(\tilde{k} \pi \cdot), z_1' \right\rangle_{L^2(0,1)} =$

$\langle \sin(\tilde{k}\pi\cdot), z_2 \rangle_{L^2(0,1)} = 0$. Furthermore, from $\langle \phi_{k_0}, z \rangle_{\mathcal{H}} = \langle \phi_0, z \rangle_{\mathcal{H}} = 0$, we deduce that

$$\begin{bmatrix} \alpha \tilde{k}_0 \pi & \lambda_{k_0} \\ -\alpha \tilde{k}_0 \pi & \lambda_{k_0} \end{bmatrix} \begin{bmatrix} \langle \cos(\tilde{k}_0 \pi \cdot), z'_1 \rangle_{L^2(0,1)} \\ \langle \sin(\tilde{k}_0 \pi \cdot), z_2 \rangle_{L^2(0,1)} \end{bmatrix} = 0.$$

Because $2\alpha \tilde{k}_0 \pi \lambda_{k_0} \neq 0$, we obtain that $\langle \cos(\tilde{k}_0 \pi \cdot), z'_1 \rangle_{L^2(0,1)} = \langle \sin(\tilde{k}_0 \pi \cdot), z_2 \rangle_{L^2(0,1)} = 0$. As both $\{\cos(\tilde{k}\pi\cdot), k \in \mathbb{N}\}$ and $\{\sin(\tilde{k}\pi\cdot), k \in \mathbb{N}\}$ are maximal in $L^2(0,1)$, we deduce that $z'_1 = z_2 = 0$. Based on $z_1(0) = 0$, we conclude that $z = 0$ and thus $\text{span}_{\mathbb{C}}(\tilde{\Phi}) = \mathcal{H}$.

It remains to show that there exist $m, M \in \mathbb{R}_+^*$ such that (17) holds true. From (15-16), the fact that $\langle \phi_{k_0}, \phi_{k,\epsilon} \rangle_{\mathcal{H}} = 0$ for all $k \in \mathbb{N} \setminus \{k_0\}$ and all $\epsilon \in \{-1, +1\}$, and $\|\phi_0\|_{\mathcal{H}} = \|\phi_{k_0}\|_{\mathcal{H}} = 1$, we deduce that

$$\begin{aligned} & \left\| a_0 \phi_0 + a_{k_0} \phi_{k_0} + \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} a_{k,\epsilon} \phi_{k,\epsilon} \right\|_{\mathcal{H}}^2 \\ &= |a_0|^2 + |a_{k_0}|^2 + \left\| \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} a_{k,\epsilon} \phi_{k,\epsilon} \right\|_{\mathcal{H}}^2. \end{aligned}$$

Following [26], there exist constants $m \in (0, 1)$ and $M > 1$, independent of the sequence $(a_{k,\epsilon})_{k \in \mathbb{N} \setminus \{k_0\}, \epsilon \in \{-1, +1\}} \in \mathbb{C}^{\mathbb{N}}$, such that

$$\begin{aligned} m \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} |a_{k,\epsilon}|^2 &\leq \left\| \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} a_{k,\epsilon} \phi_{k,\epsilon} \right\|_{\mathcal{H}}^2 \\ &\leq M \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} |a_{k,\epsilon}|^2. \end{aligned}$$

We conclude that (17) holds true. \square

B. Spectral properties of the adjoint operator \mathcal{A}_0^* and biorthogonal sequence of $\tilde{\Phi}$

In this section, we derive explicitly the biorthogonal sequence $\tilde{\Psi}$ associated with the Riesz basis $\tilde{\Phi}$. This will be useful in the next section for projecting the system trajectories onto the Riesz basis $\tilde{\Phi}$.

We recall that the adjoint operator \mathcal{A}_0^* of \mathcal{A}_0 is defined by $\mathcal{A}_0^*(x_1, x_2) = (-x_2, -(\alpha x'_1 - \beta x'_2)')$ over the domain

$$D(\mathcal{A}_0^*) = \{(x_1, x_2) \in \mathcal{H} : x_2 \in H_L^1(0, 1), (\alpha x'_1 - \beta x'_2) \in H^1(0, 1), (\alpha x'_1 - \beta x'_2)(1) = 0\}.$$

The spectral properties of \mathcal{A}_0^* are summarized in the following lemma.

Lemma 4.3: Assume that parameters $\alpha, \beta \in \mathbb{R}_+^*$ are such that $\zeta(\alpha, \beta) \in \mathbb{N}$. Then, the eigenvalues of \mathcal{A}_0^* are given by $\sigma_p(\mathcal{A}_0^*) = \{\mu_{k,\epsilon}, k \in \mathbb{N} \setminus \{k_0\}, \epsilon \in \{-1, +1\}\} \cup \{\mu_{k_0}\}$ where $\mu_{k,\epsilon} = \lambda_{k,\epsilon}$ and $\mu_{k_0} = \lambda_{k_0} = -2\alpha/\beta$. Furthermore,

the associated eigenspaces are given by $\ker(\mathcal{A}_0^* - \mu_{k,\epsilon} I_{\mathcal{H}}) = \text{vect}_{\mathbb{C}}(\psi_{k,\epsilon})$ with

$$\psi_{k,\epsilon} = \frac{\sqrt{2 \left(|\lambda_{k,\epsilon}|^2 + \tilde{k}^2 \alpha \pi^2 \right)}}{|\lambda_{k,\epsilon}| (\mu_{k,\epsilon} - \mu_{k,-\epsilon})} \left(-\sin(\tilde{k}\pi\cdot), \mu_{k,\epsilon} \sin(\tilde{k}\pi\cdot) \right), \quad (18)$$

and $\ker(\mathcal{A}_0^* - \lambda_{k_0} I_{\mathcal{H}}) = \text{vect}_{\mathbb{C}}(\psi_{k_0})$ with

$$\psi_{k_0} = \frac{1}{\mu_{k_0}} \left(-\sin(\tilde{k}_0 \pi \cdot), \mu_{k_0} \sin(\tilde{k}_0 \pi \cdot) \right) = \phi_0. \quad (19)$$

We also have, for all $k_1, k_2 \in \mathbb{N} \setminus \{k_0\}$ and all $\epsilon_1, \epsilon_2 \in \{-1, +1\}$,

$$\langle \phi_{k_1, \epsilon_1}, \psi_{k_2, \epsilon_2} \rangle_{\mathcal{H}} = \delta_{(k_1, \epsilon_1), (k_2, \epsilon_2)},$$

for all $k \in \mathbb{N} \setminus \{k_0\}$ and all $\epsilon \in \{-1, +1\}$,

$$\langle \phi_{k,\epsilon}, \psi_{k_0} \rangle_{\mathcal{H}} = \langle \phi_{k_0}, \psi_{k,\epsilon} \rangle_{\mathcal{H}} = \langle \phi_{k_0}, \psi_{k_0} \rangle_{\mathcal{H}} = \langle \phi_0, \psi_{k,\epsilon} \rangle_{\mathcal{H}} = 0,$$

and

$$\langle \phi_0, \psi_{k_0} \rangle_{\mathcal{H}} = 1.$$

We introduce the vector

$$\psi_0 = \frac{1}{\mu_{k_0}} \left(\sin(\tilde{k}_0 \pi \cdot), \mu_{k_0} \sin(\tilde{k}_0 \pi \cdot) \right) = \phi_{k_0}, \quad (20)$$

which is such that for all $k \in \mathbb{N} \setminus \{k_0\}$ and all $\epsilon \in \{-1, +1\}$,

$$\langle \phi_{k,\epsilon}, \psi_0 \rangle_{\mathcal{H}} = \langle \phi_0, \psi_0 \rangle_{\mathcal{H}} = 0,$$

and

$$\langle \phi_{k_0}, \psi_0 \rangle_{\mathcal{H}} = 1.$$

Then, we complete the family of eigenvectors $\Psi = \{\psi_{k,\epsilon}, k \in \mathbb{N} \setminus \{k_0\}, \epsilon \in \{-1, +1\}\} \cup \{\psi_{k_0}\}$ as follows $\tilde{\Psi} = \Psi \cup \{\psi_0\}$. Therefore, the developments above show that $\tilde{\Psi}$ is the biorthogonal sequence associated with the Riesz basis $\tilde{\Phi}$.

Finally, straightforward computations show that

$$\begin{aligned} \mathcal{A}_0^* \psi_0 &= \left(-\sin(\tilde{k}_0 \pi \cdot), 3\mu_{k_0} \sin(\tilde{k}_0 \pi \cdot) \right) \\ &= \mu_{k_0} \psi_0 + 2\mu_{k_0} \psi_{k_0}. \end{aligned} \quad (21)$$

V. ESTABLISHMENT OF THE ISS ESTIMATES

Let an initial condition $X_0 \in D(\mathcal{A})$, a boundary disturbance $d \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{C})$ such that $\mathcal{B}X_0 = d(0)$, and a distributed disturbance $u \in \mathcal{C}^1(\mathbb{R}_+; L^2(0, 1))$ be arbitrarily given. We denote by $X = (x_1, x_2) \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ the classical solution of (2). Because $\tilde{\Phi}$ is a Riesz basis with associated biorthogonal sequence $\tilde{\Psi}$, the projection of the system trajectory X onto this basis and the use of (17) yields for all $t \geq 0$,

$$\|X(t)\|_{\mathcal{H}}^2 \leq M \sum_{\psi \in \tilde{\Psi}} |c_{\psi}(t)|^2, \quad (22)$$

and

$$\sum_{\psi \in \tilde{\Psi}} |c_{\psi}(0)|^2 \leq \frac{1}{m} \|X_0\|_{\mathcal{H}}^2, \quad (23)$$

where $c_\psi \triangleq \langle X(\cdot), \psi \rangle_{\mathcal{H}} \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{C})$. We introduce the notation $\psi \triangleq (\psi^1, \psi^2)$. Integrations by parts can be used (see [26]) to show that we have for all $\psi \in \Psi$ and all $t \geq 0$,

$$\dot{c}_\psi(t) = \langle X(t), \mathcal{A}_0^* \psi \rangle_{\mathcal{H}} + d(t) \overline{\psi^2(1)} + \int_0^1 u(t) \overline{\psi^2} d\xi.$$

In particular, we have $\mathcal{A}_0^* \psi_{k,\epsilon} = \mu_{k,\epsilon} \psi_{k,\epsilon}$ with $\mu_{k,\epsilon} = \overline{\lambda_{k,\epsilon}}$. Thus, $c_{\psi_{k,\epsilon}}$ satisfies the following ODE for all $t \geq 0$

$$\dot{c}_{\psi_{k,\epsilon}}(t) = \lambda_{k,\epsilon} c_{\psi_{k,\epsilon}}(t) + d(t) \overline{\psi_{k,\epsilon}^2(1)} + \int_0^1 u(t) \overline{\psi_{k,\epsilon}^2} d\xi.$$

The integration of this ODE gives

$$\begin{aligned} c_{\psi_{k,\epsilon}}(t) &= e^{\lambda_{k,\epsilon} t} c_{\psi_{k,\epsilon}}(0) + \int_0^t e^{\lambda_{k,\epsilon}(t-\tau)} d(\tau) \overline{\psi_{k,\epsilon}^2(1)} d\tau \\ &\quad + \int_0^t e^{\lambda_{k,\epsilon}(t-\tau)} \int_0^1 u(\tau) \overline{\psi_{k,\epsilon}^2} d\xi d\tau, \end{aligned} \quad (24)$$

which can be bounded above as follows

$$\begin{aligned} |c_{\psi_{k,\epsilon}}(t)| &\leq e^{-\kappa_0 t} |c_{\psi_{k,\epsilon}}(0)| + \gamma_{k,\epsilon} \|d\|_{\mathcal{C}^0([0,t])} \\ &\quad + \frac{\sqrt{2}}{2} \gamma_{k,\epsilon} \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}, \end{aligned} \quad (25)$$

where $\gamma_{k,\epsilon} \triangleq |\psi_{k,\epsilon}^2(1) / \operatorname{Re} \lambda_{k,\epsilon}|$ and κ_0 is given by (5). Similarly, as $\mathcal{A}_0^* \psi_{k_0} = \mu_{k_0} \psi_{k_0}$, we have

$$\begin{aligned} |c_{\psi_{k_0}}(t)| &\leq e^{-\kappa_0 t} |c_{\psi_{k_0}}(0)| + \gamma_{k_0} \|d\|_{\mathcal{C}^0([0,t])} \\ &\quad + \frac{\sqrt{2}}{2} \gamma_{k_0} \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}, \end{aligned} \quad (26)$$

where $\gamma_{k_0} \triangleq 1 / |\operatorname{Re} \lambda_{k_0}|$. It remains to evaluate the coefficient $c_{\psi_0}(t)$. Note that the above approach does not apply because ψ_0 is not an eigenvector of \mathcal{A}_0^* . However, (21) can be used to obtain the following ODE:

$$\begin{aligned} \dot{c}_{\psi_0}(t) &= \lambda_{k_0} c_{\psi_0}(t) + 2\lambda_{k_0} c_{\psi_{k_0}}(t) \\ &\quad + d(t) \overline{\psi_0^2(1)} + \int_0^1 u(t) \overline{\psi_0^2} d\xi. \end{aligned} \quad (27)$$

Therefore, contrary to the case $\zeta(\alpha, \beta) \notin \mathbb{N}$ studied in [26] where all the ODEs are uncoupled, the case $\zeta(\alpha, \beta) \in \mathbb{N}$ yields a coupling in the ODE (27) due to the fact that the set of eigenvectors Φ of \mathcal{A}_0 is not maximal in the state-space \mathcal{H} . The integration of (27) gives for all $t \geq 0$,

$$\begin{aligned} c_{\psi_0}(t) &= e^{\lambda_{k_0} t} c_{\psi_0}(0) + 2\lambda_{k_0} \int_0^t e^{\lambda_{k_0}(t-\tau)} c_{\psi_{k_0}}(\tau) d\tau \\ &\quad + \int_0^t e^{\lambda_{k_0}(t-\tau)} d(\tau) \overline{\psi_0^2(1)} d\tau \\ &\quad + \int_0^t e^{\lambda_{k_0}(t-\tau)} \int_0^1 u(\tau) \overline{\psi_0^2} d\xi d\tau. \end{aligned} \quad (28)$$

Using (26) and noting that $\lambda_{k_0} = -2\alpha/\beta < -\alpha/\beta \leq -\kappa_0$, we have for all $t \geq 0$,

$$\begin{aligned} &\left| \lambda_{k_0} \int_0^t e^{\lambda_{k_0}(t-\tau)} c_{\psi_{k_0}}(\tau) d\tau \right| \\ &\leq |\lambda_{k_0}| e^{\lambda_{k_0} t} \int_0^t e^{-(\kappa_0 + \lambda_{k_0})\tau} d\tau |c_{\psi_{k_0}}(0)| \\ &\quad + \gamma_{k_0} \|d\|_{\mathcal{C}^0([0,t])} + \frac{\sqrt{2}}{2} \gamma_{k_0} \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))} \\ &\leq \left| \frac{\lambda_{k_0}}{\kappa_0 + \lambda_{k_0}} \right| e^{-\kappa_0 t} |c_{\psi_{k_0}}(0)| + \gamma_{k_0} \|d\|_{\mathcal{C}^0([0,t])} \\ &\quad + \frac{\sqrt{2}}{2} \gamma_{k_0} \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}. \end{aligned}$$

Furthermore, we have

$$\left| \int_0^t e^{\lambda_{k_0}(t-\tau)} d(\tau) \overline{\psi_0^2(1)} d\tau \right| \leq \gamma_0 \|d\|_{\mathcal{C}^0([0,t])}$$

and

$$\left| \int_0^t e^{\lambda_{k_0}(t-\tau)} \int_0^1 u(\tau) \overline{\psi_0^2} d\xi d\tau \right| \leq \frac{\sqrt{2}}{2} \gamma_0 \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))},$$

with $\gamma_0 \triangleq 1 / |\operatorname{Re} \lambda_{k_0}|$. Thus, we deduce from (28) that the following estimate holds true for all $t \geq 0$

$$\begin{aligned} |c_{\psi_0}(t)| &\leq e^{-\kappa_0 t} |c_{\psi_0}(0)| + 2 \left| \frac{\lambda_{k_0}}{\kappa_0 + \lambda_{k_0}} \right| e^{-\kappa_0 t} |c_{\psi_{k_0}}(0)| \\ &\quad + (\gamma_0 + 2\gamma_{k_0}) \|d\|_{\mathcal{C}^0([0,t])} \\ &\quad + \frac{\sqrt{2}}{2} (\gamma_0 + 2\gamma_{k_0}) \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}. \end{aligned} \quad (29)$$

We obtain from (25-26) that

$$\begin{aligned} |c_{\psi_{k,\epsilon}}(t)|^2 &\leq 3e^{-2\kappa_0 t} |c_{\psi_{k,\epsilon}}(0)|^2 + 3\gamma_{k,\epsilon}^2 \|d\|_{\mathcal{C}^0([0,t])}^2 \\ &\quad + \frac{3}{2} \gamma_{k,\epsilon}^2 \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}^2 \end{aligned} \quad (30)$$

and

$$\begin{aligned} |c_{\psi_{k_0}}(t)|^2 &\leq 3e^{-2\kappa_0 t} |c_{\psi_{k_0}}(0)|^2 + 3\gamma_{k_0}^2 \|d\|_{\mathcal{C}^0([0,t])}^2 \\ &\quad + \frac{3}{2} \gamma_{k_0}^2 \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}^2. \end{aligned} \quad (31)$$

From (29), we have that

$$\begin{aligned} |c_{\psi_0}(t)|^2 &\leq 6 \left(|c_{\psi_0}(0)|^2 + 4 \left| \frac{\lambda_{k_0}}{\kappa_0 + \lambda_{k_0}} \right|^2 |c_{\psi_{k_0}}(0)|^2 \right) e^{-2\kappa_0 t} \\ &\quad + 3(\gamma_0 + 2\gamma_{k_0})^2 \|d\|_{\mathcal{C}^0([0,t])}^2 \\ &\quad + \frac{3}{2} (\gamma_0 + 2\gamma_{k_0})^2 \|u\|_{\mathcal{C}^0([0,t]; L^2(0,1))}^2. \end{aligned} \quad (32)$$

For $k \geq k_0 + 1$, straightforward computations show that

$$\gamma_{k,\epsilon} = \frac{\sqrt{2 \left(|\lambda_{k,\epsilon}|^2 + \tilde{k}^2 \alpha \pi^2 \right)}}{\tilde{k} \pi |\lambda_{k,\epsilon}| \sqrt{\tilde{k}^2 \beta^2 \pi^2 - 4\alpha}}.$$

From, (12-13), we obtain that

$$\gamma_{k,+1} \underset{k \rightarrow +\infty}{\sim} \frac{1}{k\pi} \sqrt{\frac{2}{\alpha}}, \quad \gamma_{k,-1} \underset{k \rightarrow +\infty}{\sim} \frac{\sqrt{2}}{k^2 \beta \pi^2},$$

whence $\gamma_{k,\epsilon}$ is a square summable sequence. Thus, we can introduce $\gamma \in \mathbb{R}_+$ defined by

$$\gamma^2 \triangleq \gamma_{k_0}^2 + (\gamma_0 + 2\gamma_{k_0})^2 + \sum_{\substack{k \in \mathbb{N} \setminus \{k_0\} \\ \epsilon \in \{-1, +1\}}} \gamma_{k,\epsilon}^2.$$

We obtain from (22-23) and (30-32) that for all $t \geq 0$,

$$\begin{aligned} \|X(t)\|_{\mathcal{H}}^2 &\leq 6 \frac{M}{m} \left(1 + 4 \left| \frac{\lambda_{k_0}}{\kappa_0 + \lambda_{k_0}} \right|^2 \right) e^{-2\kappa_0 t} \|X_0\|_{\mathcal{H}}^2 \\ &\quad + 3M\gamma^2 \|d\|_{\mathcal{C}^0([0,t])}^2 + \frac{3}{2} M\gamma^2 \|u\|_{\mathcal{C}^0([0,t];L^2(0,1))}^2. \end{aligned}$$

It follows that the ISS estimate (3) holds true with

$$C_0 = \sqrt{6 \frac{M}{m} \left(1 + 4 \left| \frac{\lambda_{k_0}}{\kappa_0 + \lambda_{k_0}} \right|^2 \right)},$$

$C_1 = \gamma\sqrt{3M}$, and $C_2 = \gamma\sqrt{\frac{3M}{2}}$. The ISS estimate (4) can be similarly obtained by using the Cauchy-Schwartz inequality in the estimation of (24) and (28).

VI. CONCLUSIONS

This paper established the Input-to-State Stability (ISS) property of a clamped-free damped string for configurations of the physical parameters associated with the loss of the Riesz-spectral properties. Even if the set of eigenvectors of the underlying disturbance-free operator is not maximal, it has been shown that they can be completed with an adequately chosen vector to form a Riesz basis. Then, the ISS property of the system has been derived via the projection of the system trajectories onto this Riesz basis.

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